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RUIN PROBABILITY IN THE CLASSICAL RISK PROCESS WITH WEIBULL CLAIMS DISTRIBUTION

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ABSTRACT

In the classical risk process, ruin is the situation when the surplus falls below zero. Ruin probability is a tool used to predict bankruptcy in the insurance company. The ruin probability can be determined by solving the Integral-Differential equation that arises from the classical risk process. In this paper, we are interested in calculating the ruin probability when the claim distribution follows the Weibull distribution. Based on the Weibull parameter, the calculation is divided into two cases: when alpha equals 1 and when $\alpha > 1$. The Laplace transform gives the analytical solution of the Integral-Differential equation. However, when $\alpha > 1$ the analytical solution cannot be determined since the Laplace transform is no longer applicable due to the presence of an improper integral that is not possible to solve analytically. Therefore, for the case alpha greater than 1, Euler's method is applied to determine its numerical solution. The accuracy of the numerical solution is validated by comparing it with the analytical solution for the case $\alpha = 1$ Then, using the accuracy determined from the first case, we apply the Euler method to determine the numerical solution for the case $\alpha > 1$. The numerical method gives good accuracy to the analytical solution with the order of 10^{-3} calculated from $u = 0$ until $u = 100$.

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1. INTRODUCTION

According to Dickson [1] in the classical risk process, there are three components that contribute to the insurer's surplus at a fixed time $t > 0$. Those components are the amount of surplus at the time $t = 0$, the amount of premium income received until the time t, and the amount paid for claims up to the time t. From these three components, the variable that is random is the amount that is paid for claims. Let $\{N(t)\}_{t\geq 0}$ be a counting process which denotes the number of claims that occur in the fixed time interval $[0, t]$. In the classical risk process, the counting process $\{N(t)\}_{t\geq 0}$ is assumed to be a Poisson process. The individual claims amounts are modeled as a sequence of independent and identically distributed random variables $\{X_i\}_{i=1}^{\infty}$, where X_i denotes the amount of the ith claim. The aggregate claim amount up to time t is defined as

$$S(t) = \sum_{i=1}^{N(t)} X_i \tag{1}$$

with the assumption that $S(t) = 0$ when $N(t) = 0$, that is no the aggregate claim is 0 when there is no claim occurring. The aggregate claims process $\{S(t)\}_{t\geq 0}$ is known as a compound Poisson process. The surplus process $\{U(t)\}_{t\geq 0}$ is defined as

$$U(t) = u + ct - S(t) \tag{2}$$

Where u is the insurer's initial surplus and c is the insurer's rate of premium income per unit of time assumed to be received continuously. Let F be the distribution function of X_1 and f be its density function with the assumption $F(0) = 0$, so that all claim amounts are positive, and the density function is continuous. Then, the kth moment of X_1 is denoted by, and the insurer's rate of premium income $c > \lambda m_1$, where λ is the Poisson parameter. It is convenient to express $c = (1 + \theta)\lambda m_1$, where θ is a non-negative real number known as the premium loading factor. The probability of ruin in infinite time, also known as the ultimate ruin probability, is defined as

$$\psi(u) = \Pr(U(t) < 0 \; for\; some\; t > 0) \tag{3}$$

Next, let us define $\phi(u) = 1 - \psi(u)$, that is the probability the ruin never occurs starting from initial surplus u known as the survival probability. The survival probability can be determined by solving the Integral-Differential equation

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u f(x)\phi(u-x)dx \tag{4}$$

with initial condition $\phi(0) = 1 - \frac{\lambda m_1}{c} = \frac{\theta}{1+\theta}$. The value of the initial condition is between 0 and 1 since this indicates the probability.

Many approaches have been developed to study the ruin probability. Hamzah and Febrianti in [2] study the solvability conditions of Laplace transform to solve **Equation (4)** where the claims distribution is Exponential. In that paper, the conditions of loading factor θ are discussed which guarantee the success of the Laplace transform method. Goovaerts and De Vylder in [3] develop a recursive algorithm to calculate the ultimate ruin probabilities. That method determines the lower and upper bound of the ruin probability. Boots and Shahbuddin [4] studied the ruin probabilities where the distribution of the claims is subexponential. Constantinescu et. al. [5] studied the ruin probabilities for the Gamma claims distribution. Goffard et. al. [6] developed a polynomial expansion method to approximate the ultimate ruin probability. Sanchez and Baltazar [7] employed Banach's fixed point theorem to approximate the ruin probability. Santana and Rincon [8] studied the ruin probability for the discrete-time risk model. Dufresne and Gerber [9] studied three methods to calculate the ruin probability. One of the methods is called the upper and lower bound method which determines the upper estimate and the lower estimate of the ruin probability. Chau et. al. [10] applied the Fourier-cosine method for calculating ruin probabilities. Ignatov and Kaishev [11] study the ruin probability in the finite time domain where the calim is continuous. You et. al. [12] estimate the interval for ruin probability in the classical compound Poisson risk model. Dickson and Waters [13] study the probability and severity of ruin in finite and infinite time. Diasparra and Romera [14] study the bounds for ruin probability of discrete-time risk process. Finally, Das and Nath [15] studied the ruin probability where the claim is Weibull using the Fast Fourier Transform method and The Forth Moment Gamma De Vylder approximation. Based on the previous research, we are interested to solve the Integral-Differential **Equation (4)** using the simplest numerical method which is Euler's method. The integral term in **Equation (4)** will be approximated using the trapezoid rule.

The Weibull distribution is a continuous random variable that is often used to analyze life data, model failure time, and access reliability. This distribution was first introduced by Wallodi Weibull in 1951 and has been widely used in reliability engineering, survival analysis, and other fields. Weibull distribution is often applied in insurance companies to model the distribution of claims due to its flexibility. The probability density function of the Weibull random variable is

$$f(x;\beta,\alpha)=\begin{cases}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)\exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\} & ,x\geq 0\\ 0 & ,x<0\end{cases} \tag{5}$$

where $\alpha>0$ is known as the shape parameter and $\beta>0$ is called the scale parameter. When $\alpha=1$, the Weibull distribution **Equation (5)** becomes the Exponential distribution with parameter $\frac{1}{\beta}$. In this paper, the discussion of ruin probability with Weibull claims is divided based on the shape parameter α. When the shape parameter $\alpha=1$, the ruin probability can be determined analytically using the Laplace transform method. However, when $\alpha>1$ the Laplace transform method is no longer applicable since there is a step that requires calculating an improper integral which is not possible to solve analytically. Then, for the case $\alpha>1$, we use Euler's method to solve the **Equation (4)**. The error estimate of Euler's method is determined by comparing it with the analytical solution determined from the Laplace transform. This result will be used as the error estimate for the case of $\alpha>1$.

2. RESEARCH METHODS

Consider the **Equation (4)** with claim distribution f is Weibull distribution as in **Equation (5)**. Applying the Laplace transform to both sides of **Equation (4)** we determine the Laplace transform of $\phi^*(s)$ as

$$\phi^*(s)=\frac{c\phi(0)}{cs-\lambda\big(1-f^*(s)\big)} \tag{6}$$

where $f^*(s)$ is the Laplace transform of claim probability function $f(x)$. The survival probability is determined by taking the inverse of Laplace transform that is $\phi(u)=\mathcal{L}^{-1}\{\phi^*(s)\}$. Now consider the **Equation (5)** for the case $\alpha>1$, for simplicity let $\alpha=2$ and $\beta=1$. Now the **Equation (5)** become

$$f(x)=2x\exp\{-x^2\} \tag{7}$$

Taking the Laplace transform of **Equation (7)**, we have to solve the integral

$$\int_0^{\infty}2x\exp\{-(x^2+sx)\}dx \tag{8}$$

The integral **Equation (8)** cannot be solved analytically, thus the inverse of **Equation (6)** is not available. Therefore, to solve the **Equation (4)** for f as in **Equation (5)** with $\alpha>1$ numerical approach is preferred. Consider the partition of interval $u\in[0,l]$ that is $u_0=0<u_1<u_2<\cdots<u_N=l$ with step size $\Delta u=\frac{l}{N}$ such that $u_n=n\Delta u, n=0,1,\dots,N$. The first derivative on the right-hand side of **Equation (4)** can be approximated by the first-order numerical differentiation that is

$$\frac{d}{du}\phi(u)\approx\frac{\phi(u+\Delta u)-\phi(u)}{\Delta u} \tag{9}$$

Let $\phi(u)=\phi(u_n)$ and $\phi(u+\Delta u)=\phi(u_{n+1})$. Then the **Equation (9)** become

$$\frac{d}{du}\phi(u_n)\approx\frac{\phi(u_{n+1})-\phi(u_n)}{\Delta u} \tag{10}$$

Let $x_0=0<x_1<x_2<\cdots<x_N=u$ be the partition of $x\in[0,u]$ with step size $\Delta x=\frac{u}{N}$ such that $x_i=i\Delta x, i=0,1,\dots,N$. The integral term on the right-hand side of **Equation (4)** can be approximated by the Trapezoid rule

$$\int_0^u f(x)\phi(u-x)dx \approx \frac{\Delta x}{2}\sum_{i=0}^{N-1} f(x_i)\phi(u-x_i) + f(x_{i+1})\phi(u-x_{i+1}) \tag{11}$$

Substituting **Equation (10)** and **Equation (11)** to **Equation (4)**, the numerical scheme for solving **Equation (4)** is

$$\phi(u_{n+1}) = \phi(u_n) + \Delta u\left\{\frac{\lambda}{c}\phi(u_n) - \frac{\lambda}{c}\frac{\Delta x}{2}\sum_{i=0}^{N-1} f(x_i)\phi(u-x_i) + f(x_{i+1})\phi(u-x_{i+1})\right\} \tag{12}$$

The computation start from $n = 0$ until $n = N$ with $\Delta x = \Delta u$

3. RESULTS AND DISCUSSION

In this section, the computation of ruin probability with Weibull claims distribution will be discussed. The calculation of ruin and survival probability will be divided into two cases based on the Weibull parameter α that is the case where $\alpha = 1$ and the case $\alpha > 1$.

Case I: $\alpha = 1$

Let us consider **Equation (5)** with $\alpha = 1$, that is

$$f(x;\beta,1) = \begin{cases} \frac{1}{\beta}\exp\left\{-\frac{x}{\beta}\right\} & ,x \geq 0 \\ 0 & ,x < 0 \end{cases} \tag{13}$$

The **Equation (13)** is the probability function of exponential distribution with parameter $\frac{1}{\beta}$. Applying the Laplace transform by using **Equation (6)** we get

$$\phi(u) = 1 - \psi(0)\exp\left\{-\frac{\phi(0)}{\beta}u\right\} \tag{14}$$

where $\psi(0) = 1 - \phi(0) = \frac{1}{1+\theta}$. Using $\theta = 0.1$ and $\beta = 10$ we get $\phi(0) = 0.09091$, and $\psi(0) = 0.909091$. Then, with $\Delta u = \Delta x = 0.01$ the exact and the numerical solution of survival probability $\phi(0)$ for $u = 0$ until $u = 100$ can be viewed in **Table 1**.

Table 1. Exact Solution, Numerical Solution, And Error Values Of Survival Probability.

u	ϕ_{Exact}	$\phi_{Numeric}$	Error
0	0.0909	0.0909	0
10	0.1699	0.1698	7.78e-05
20	0.2420	0.2417	0.00025
30	0.3079	0.3073	0.00054
40	0.3680	0.3670	0.00096
50	0.4229	0.4214	0.00153
60	0.4731	0.4708	0.00226
70	0.5188	0.5157	0.00314
80	0.5607	0.5565	0.00419
90	0.5988	0.5934	0.00540
100	0.6337	0.6269	0.00678

The error is determined by calculating the absolute values between the Exact and the Numerical values which is

$$Error = |\phi_{Exact}(u) - \phi_{Numeric}(u)|$$

Graphically, the exact and numerical solution of survival probability can be viewed in **Figure 1**. It can be seen that for $u = 0$ until $u = 100$ the exact and the numerical solution looks to agree with each other, however, the error is continuously increasing as u increases. In this case, the maximum error is at the order of 10^{-3}. Therefore, from this observation, the error can be kept at the order of 10^{-3} if the survival probability is calculated until $u = 100$. The graphic of the error can be seen in **Figure 2.**

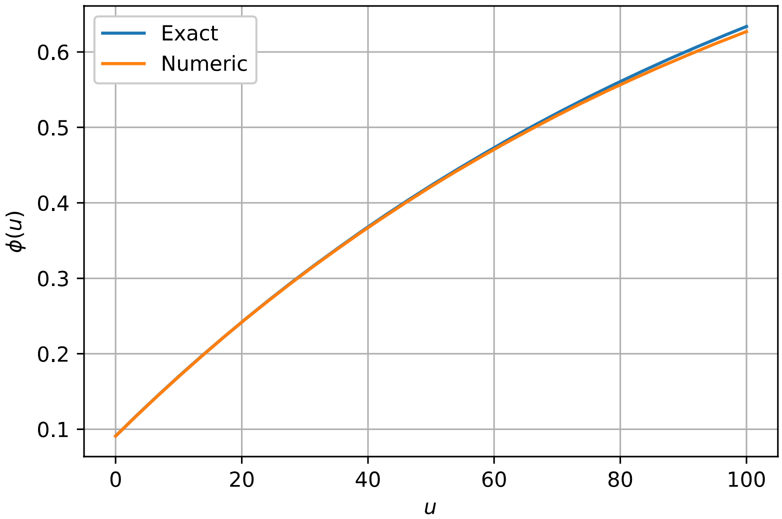

Figure 1. Exact Solution and Numerical Solution of Survival Probability.

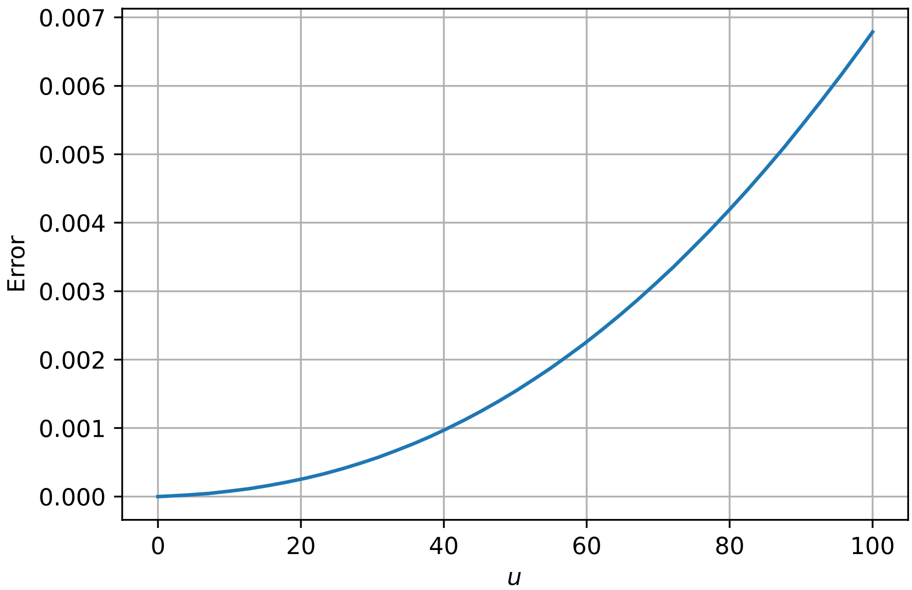

Figure 2. The Graphic Of Error Values Between The Numerical And Exact Solution Of Survival Probability.

Case II: $\alpha > 1$

Let us consider the **Equation (5)** when $\alpha = 2$, then the **Equation (5)** become

$$f(x) = \begin{cases} \frac{2x}{\beta^2} \exp\left\{-\left(\frac{x}{\beta}\right)^2\right\} & ,x \geq 0 \\ 0 & ,x < 0 \end{cases} \tag{15}$$

When solving **Equation (4)** with claim distribution **Equation (15)** using the Laplace transform **Equation (6)** we need to find the Laplace transform of **Equation (15)** that is

$$f^*(x) = \int_0^\infty \frac{2x}{\beta^2} \exp\left\{-\left(\frac{x}{\beta}\right)^2 - sx\right\} dx \tag{16}$$

However, the integral Equation (16) cannot be solved analytically then the numerical method is selected. By applying the Euler method in Equation (12) the results can be seen in Table 2.

Table 2: Numerical Solution of Equation (4) With Weibull Claim Equation (15) With $\alpha = 2$ and $\beta = 10$.

u	$\phi(u)$	$\psi(u)$
0	0.0909	0.9090
10	0.4631	0.5368
20	0.6900	0.3099
30	0.8214	0.1785
40	0.8975	0.1024
50	0.9416	0.0583
60	0.9671	0.0328
70	0.9819	0.0180
80	0.9905	0.0094
90	0.9956	0.0043
100	0.9985	0.0014

The results show that the survival probability $\phi(u)$ is increasing as the initial investment is increasing. In contrast, the ruin probability $\psi(u)$ is decreasing as the initial investment is increasing. This condition agrees with the actual situation that is, as the initial investment increases the insurance company will have more money to cover the claims that occurred. Therefore the company will be more sustainable to continue its operations.

4. CONCLUSIONS

Euler's method offers a direct approach for solving the Integral-Differential Equation (4) without considering the solvability conditions of the integral that appeared in the Laplace transform. From case I and case II, we have seen that the Euler method can produce a good approximation to the Integral-Differential Equation (4) with accuracy at 10^{-3} for $u = 0$ until $u = 100$. However, the accuracy decreases as the value u increases more than 100. The numerical solution produced from the Euler method is reliable as long as the numerical result does not exceed 1 or below 0 since the value represents the probability of an event. In this case, the survival probability value determined from the Euler method exceeds 1 if u is greater than 200. Another numerical method such as Heun or Runge-Kutta method can be considered for future research.

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