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ON THE TOTAL VERTEX IRREGULARITY STRENGTH OF SERIES PARALLEL GRAPH $sp(m, r, 4)$

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ABSTRACT

This study aims to determine the total vertex irregularity strength on a series parallel graph $sp(m, r, 4)$ for $m \geq 5$ and $r \geq 1$. Total labeling is said to be vertex irregular if the weights for each vertex are different. Determination of the total vertex irregularity of series parallel graph is done by obtaining the largest lower bound and the smallest upper bound. The lower bound is obtained by analyzing the structure of the graph to obtain the largest minimum label of k, and the upper bound is analyzed by labeling the vertices and edges of the graph, where the largest label is k and the values for each vertices weight is different. The result obtained for the total vertex irregularity strength of a series parallel graph $sp(m, r, 4)$ is $tvs(sp(m, r, 4)) = \left\lceil \frac{4mr+2}{2} \right\rceil$.

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1. INTRODUCTION

Many real-world scenarios can be illustrated using diagrams, which are made up of several vertices and a series of lines connecting them. A vertex can represent a person, a line can represent a pair of friends in a group of people, a vertex can represent a communication center, and a line can represent a communication network in a telecommunication company. It's important to note that the diagram considers whether a pair of vertices is connected by a line or not. The definition of graphs arises from the mathematical abstraction of such conditions [1].

Graph G is defined as a set pair, written with the notation $G = (V, E)$, in which case V is a non-empty set of points (vertices or nodes), and E is the set of lines (edges or arcs) that connects a pair of vertices. Graphs are used to represent discrete objects as well as the relationships between those objects [2]. In the mathematical discipline of graph theory where the assignment of labels, traditionally represented by integers, to edges and/or vertices of a graph is called graph labeling [3].

The object of study for graph labeling is in the form of a graph, which is generally represented by vertices and edges and a subset of numbers called labels. Graph labeling is a function with a set domain of vertices and a set of edges or both with a range of real numbers [4]**.** Labeled graphs serve as useful models for a wide variety of applications, such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, database management, secret sharing schemes, and models for constraint programming over finite domains [5].

The concept of irregular labeling on a graph was first introduced in [6], where k-labeling is irregular on the graph G defined as a mapping of a set of edge e of G to an integer $\{1,2,\ldots,k\}$ such that each vertex v has a different weight. The sum of the labels v and the label of edges connected with v is the weighted of the vertex v, represented by $w(v)$. The sum of labels e and labels for all vertices linked to e is the weighted of the edge e, represented by $w(e)$.

In [7] the idea of studying irregular total k-labeling was introduced. A total k-labelling is defined to be an edge irregular total k-labelling of the graph G, if for every two different edges e and f of graph G there is $w(e) \neq w(f)$, and to be a vertex irregular total k-labelling of G, if for every two distinct vertices x and y of G there is $w(x) \neq w(y)$. The minimum k for which the graph G has an edge irregular total k-labelling is called the total edge irregularity strength of the graph G,denoted by $tes(G)$. Analogously, we define the total vertex irregularity strength of G, $tvs(G)$, as the minimum k for which there exists a vertex irregular total k-labelling of G.

Research related to the irregular total labeling in a graph continues to grow. One of the studies [8] looked at the irregularity strength of Diamond graph Br_n, where $tes(Br_4) = \left\lceil \frac{5n-3}{3} \right\rceil$ and $tvs(Br_4) = \left\lceil \frac{n+1}{3} \right\rceil$, for $n \geq 3$. Then in [9] were given the total edge irregularity strength of the Kite graph (n,t), and showed $tes(n,t) = \left\lceil \frac{n+t+2}{3} \right\rceil$ for $n \geq 3$ and $t \geq 1$. Later in [10], the total edge irregularity strength of centralized uniform theta graphs $\theta * (n;m;p)$ were presented and resulted $tes\big(\theta * (n;m;p)\big) = \left\lceil \frac{(n(m+1)p+2)}{3} \right\rceil$ for $n \geq 3, m \geq 1$ and $p \geq 3$.

Then based on research in [11], the total irregularity strength of the comb product of a two-cycle graph C_m and C_n is $tvs(C_m \rhd_o C_n) = \left\lceil \frac{m(n-1)+2}{3} \right\rceil$ for $m \geq 3$ and $n \geq 3$, whereas the total irregularity strength of a two-star graph S_m and S_n is $tvs(S_m \rhd_o S_n) = \left\lceil \frac{n(m+1)+1}{2} \right\rceil$ for $m \geq 2$ and $n \geq 2$. The total irregularity edge strength of the m-copy of the path graph P_n is $tes(mP_n) = \left\lceil \frac{(n-1)m+2}{3} \right\rceil$, for $m \geq 2$ and $n \geq 6$, were presented in [12].

The total irregularity strength of the tadpole chain graph $T_r(4,1)$ was shown in [13] that one of the result was $tvs\big(T_r(4,1)\big) = \left\lceil \frac{4r+2}{5} \right\rceil$ for $r \geq 3$. While research [14] gave the total edge irregularity strength for ladder graph SC_n with $tes(SC_n) = \left\lceil \frac{n(n+3)+2}{3} \right\rceil$, double ladder graph DSC_n with $tes(DSC_n) = \left\lceil \frac{2n^2+3n+1}{3} \right\rceil$ and mirror ladder graph MSC_n with $tes(MSC_n) = \left\lceil \frac{n(2n+5)+2}{3} \right\rceil$. Hinding in his research [15] examines the total vertex irregularity strength of a hexagon cluster graph $HC(n)$ and gets the results for $tvs\big(HC(n)\big) = \left(\frac{3n^2+1}{2}\right) n \geq 2$.

The total edge irregularity strength of parallel series graphs garnered some attention as well. Among them is written by Rajasingh [16] which has determined the total edge irregularity strength on a series parallel graph $sp(m,r,l)$ and obtained $tes\big(sp(m,r,l)\big) = \left\lceil \frac{lm(r+1)+2}{3} \right\rceil$ for $r \geq 1$. Yuliarti [17] and Riskawati [18], in both papers has determined the total irregularity vertex on a series parallel graph, where the graph $sp(m,1,3)$ is $tvs\big(sp(m,1,3)\big) = \left\lceil \frac{3m+2}{3} \right\rceil$ for $m \geq 4$ and on the graph $sp(m,r,2)$ is $tvs\big(sp(m,r,2)\big) = \left\lceil \frac{2mr+2}{3} \right\rceil$ for $m \geq 3$ and $r \geq 3$. In this study, we want to determine the total irregularity vertex on a series parallel graph $sp(m, r, l)$ for a larger l, that is $l = 4$. So, this research determined the total vertex irregularity strength of a parallel series graph $sp(m, r, 4)$ for $m \geq 5$ and $r \geq 1$.

2. RESEARCH METHODS

This research is a literature study, where information is obtained from books and journals related to research. The following steps are used to determine the total vertex irregularity strength of parallel series graph $sp(m, r, 4)$:

1. Given a parallel series graph $sp(m, r, 4)$, where m is the longitude of each theta graph, r is the number of vertices of degree 2 spanning each longitude, on 4 uniform theta graph. For example, for $sp(10,3,4)$, then there are 10 longitudes on each theta graph, 3 vertices on each longitude, on 4 uniform theta graph.
2. Determine the lower bound of $tvs\big(sp(m, r, 4)\big)$ by analyzing the structure of the graph $sp(m, r, 4)$.
3. Determine the upper bound of $tvs\big(sp(m, r, 4)\big)$ by indicating the existence of an irregular total k-labeling on the graph, and k is the lower bound obtained in step 2.
4. Determine the vertex labeling and edge labeling formulas for the $sp(m, r, 4)$ graph, with reference to the labeling obtained in step 3.
5. Determine the formula of vertices weight of $sp(m, r, 4)$, with reference to the formula obtained in step 4.
6. Proving that the labeling obtained is the total vertex irregular labeling of graph $sp(m, r, 4)$, by proving that each vertices weight on the graph is different.
7. Determine total vertex irregularity strength of graph $sp(m, r, 4)$, that is the minimum largest label k so that graph $sp(m, r, 4)$ has a total vertex irregular k-labelling.
8. Applying the formula of the total vertex irregularity strength of graph $sp(m, r, 4)$, as an example, we will give the total vertex irregular k-labeling for graph $sp(15,1,4)$.

3. RESULTS AND DISCUSSION

Before we decide the total vertex irregularity strength of the graph $sp(m, r, 4)$, we will first look at the illustration of graph $sp(m, r, 4)$ is given in Figure 1 below.

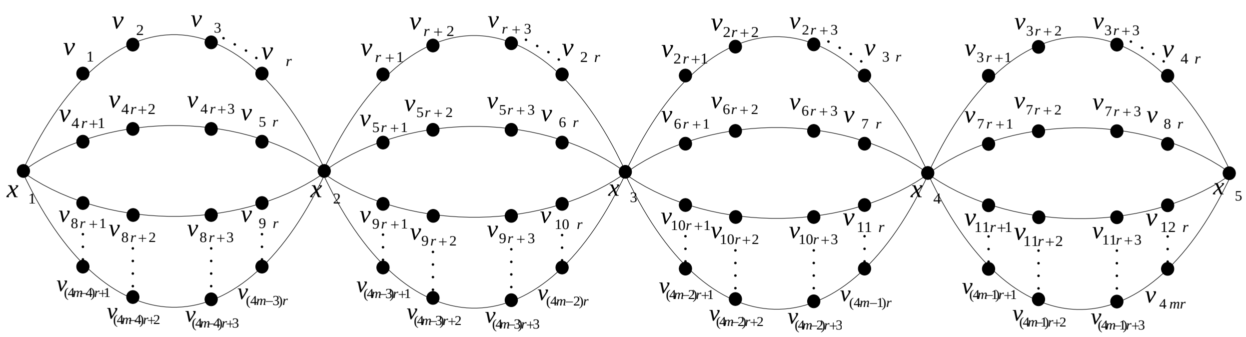

Figure 1. Graph Illustration $sp(m, r, 4)$

Suppose that $G = (V, E)$ is a graph where V is a non-empty set of vertices and E is a set of edges connecting a pair of vertices. A graph is called a parallel series graph $(m, r, 4)$ if it is formed from a series composition of 4 uniform theta graphs, where m is the longitude on each theta graph and r is the number of vertices of degree 2 spanning in each longitude. We will find the total vertex irregularity strength of a parallel series graph $sp(m, r, 4)$ for $m \geq 5$ and $r \geq 1$. The set of vertices V of the graph $sp(m, r, 4)$ is defined, where $V = \{v_i : i = 1,2,3, \dots, 4mr\} \cup \{x_i : i =$

1,2,3,4,5}. To simplify the process of formulating edge labeling and calculating vertex weight, a set of vertices is grouped from the graph, into:

a. set of vertices with v_i ; $i = 1, 4r+1, 8r+1, 12r+1, \dots, (4m-4)r+1$
b. set of vertices with v_i ; $i = r, 5r, 9r, 13r, \dots, (4m-3)r$
c. set of vertices with v_i ; $i = r+1, 5r+1, 9r+1, 13r+1, \dots, (4m-3)r+1$
d. set of vertices with v_i ; $i = 2r, 6r, 10r, 14r, \dots, (4m-2)r$
e. set of vertices with v_i ; $i = 2r+1, 6r+1, 10r+1, 14r+1, \dots, (4m-2)r+1$
f. set of vertices with v_i ; $i = 3r, 7r, 11r, 15r, \dots, (4m-1)r$
g. set of vertices with v_i ; $i = 3r+1, 7r+1, 11r+1, 15r+1, \dots, (4m-1)r+1$
h. set of vertices with and v_i ; $i = 4jr+2, 4jr+3, 4jr+4, \dots, 4jr+(r-1) j = 0, 1, 2, 3, \dots, m-1$
i. set of vertices with and v_i ; $i = 4jr+(r+2), 4jr+(r+3), 4jr+(r+4), \dots, 4jr+(2r-1);\ j = 0, 1, 2, 3, \dots, m-1$
j. set of vertices wSith and v_i ; $i = 4jr+(2r+2), 4jr+(2r+3), 4jr+(2r+4), \dots, 4jr+(3r-1); j = 0, 1, 2, 3, \dots, m-1$
k. set of vertices with and.v_i ; $i = 4jr+(3r+2), 4jr+(3r+3), 4jr+(3r+4), \dots, 4jr+(4r-1);\ j = 0, 1, 2, 3, \dots, m-1$

The set of edges E of the graph is defined, where $sp(m, r, 4)$

$$\begin{aligned} E = &\{x_1v_i : i = 1, 4r+1, 8r+1, 12r+1, \dots, (4m-4)r+1\} \cup \\ &\{x_2v_i : i = r+1, 5r+1, 9r+1, 13r+1, \dots, (4m-3)r+1\} \cup \\ &\{x_3v_i : i = 2r+1, 6r+1, 10r+1, 14r+1, \dots, (4m-2)r+1\} \cup \\ &\{x_4v_i : i = 3r+1, 7r+1, 11r+1, 15r+1, \dots, (4m-1)r+1\} \cup \\ &\{x_2v_i : i = r, 5r, 9r, 13r, \dots, (4m-3)r\} \cup \\ &\{x_3v_i : i = 2r, 6r, 10r, 14r, \dots, (4m-2)r\} \cup \\ &\{x_4v_i : i = 3r, 7r, 11r, 15r, \dots, (4m-1)r\} \cup \\ &\{x_5v_i : i = 4r, 8r, 12r, 16r, \dots, 4mr\} \cup \\ &\{v_iv_{i+1} : i = 1, 2, 3, \dots, 4mr, i \neq r, 2r, 3r, \dots, 4mr\}. \end{aligned}$$

For example, graph $sp(15,3,4)$ can be seen in **Figure 2**.

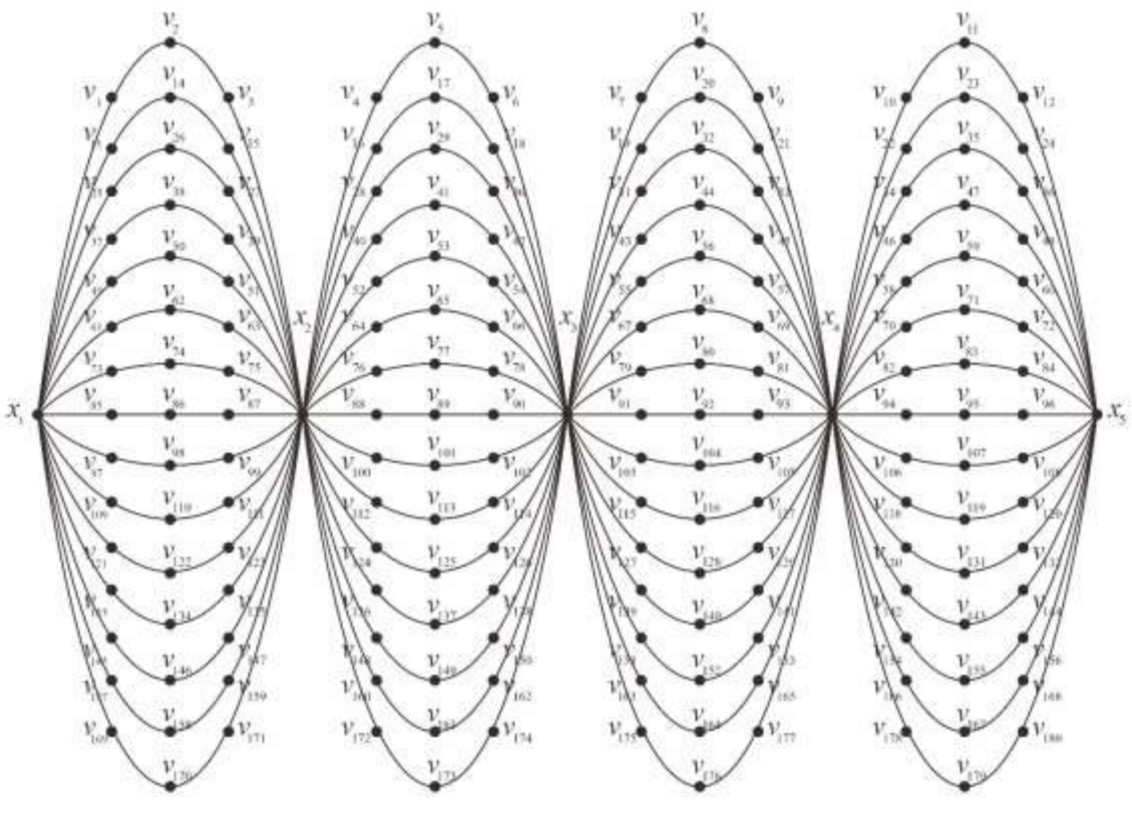

Figure 2. Naming Vertices and Edges on a Graph $sp(15, 3, 4)$

The result of this research is about the total vertex irregularity strength of the graph $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$ given in the following theorem. In the theorem's proof will be explained that the lower bound and the upper bound of $tvs(sp(m,r,4))$ are $\left\lceil \frac{4mr+2}{3} \right\rceil$.

Theorem 1. *Total vertex irregularity strength of the graph $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$ is*

$$tvs(sp(m,r,4)) = \left\lceil \frac{4mr+2}{3} \right\rceil \quad (1)$$

Proof. Note that the degree of the smallest vertex of the graph $sp(m,r,4)$ is 2 and the number of vertices with the smallest degree, which is degree 2 on the graph $sp(m,r,4)$, is $4mr$. To obtain optimal labeling, the weight of each vertex with degree 2 are labeled as $3,4,5,\dots,4mr+2$. Since the vertex weight is the sum of labels of 1 vertex and 2 edges which associated with that vertex, the largest label is more or equal to $\left\lceil \frac{4mr+2}{3} \right\rceil$. The ceiling function is used because in irregular total labeling of vertices, it is only allowed to label the graph with an integer. To guarantee this, the lower bound is rounded up. Then it is evident that $tvs(sp(m,r,4)) \geq \left\lceil \frac{4mr+2}{3} \right\rceil$.

Next, it will be proved that $tvs(sp(m,r,4)) \leq \left\lceil \frac{4mr+2}{3} \right\rceil$ by showing the vertex irregular total *k*-labeling of the graph $sp(m,r,4)$ for m,r natural numbers, $m \geq 5$ and $r \geq 1$. It is defined $\lambda: V \cup E \to \left\{1,2,3,\dots,\left\lceil \frac{4mr+2}{3} \right\rceil\right\}$, where the vertex labeling and edge labeling of the graph $sp(m,r,4)$ are as follows:

1) The vertex labeling of the graphs $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$

   a. $\lambda(v_i) = \left\lceil \frac{i}{3} \right\rceil$; $for\ i = 1,2,3,\dots,4mr$

   b. $$\lambda(x_1) = \begin{cases} 5 + \frac{20}{3}(r-1) & ; if\ r \equiv 1 (mod\ 3)\ and\ m = 5 \\ 11 + \frac{20}{3}(r-2) & ; if\ r \equiv 2 (mod\ 3)\ and\ m = 5 \\ 18 + \frac{20}{3}(r-3) & ; if\ r \equiv 0 (mod\ 3)\ and\ m = 5 \\ 1 + 4(r-1) & ; if\ r \equiv 1 (mod\ 3)\ and\ m = 6 \\ 5 + 4(r-2) & ; if\ r \equiv 2 (mod\ 3)\ and\ m = 6 \\ 9 + 4(r-3) & ; if\ r \equiv 0 (mod\ 3)\ and\ m = 6 \\ 1 & ; if\ m \geq 7 \end{cases}$$

   c. $\lambda(x_2) = 1$

   d. $\lambda(x_3) = 1$

   e. $$\lambda(x_4) = \lambda(x_5) = \begin{cases} 2 & ; if\ m = 5 \\ 1 & ; if\ m \geq 6 \end{cases}$$

2) The edge labels of the graphs $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$

   a. For $i = 1, 4r+1, 8r+1, 12r+1, \dots, (4m-4)r+1$

   $\lambda(x_1 v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor$

   b. For $i = r, 5r, 9r, 13r, \dots, (4m-3)r$

   $$\lambda(x_2 v_i) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ (mod\ 3) \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ (mod\ 3) \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

   c. For $i = r+1, 5r+1, 9r+1, 13r+1, \dots, (4m-3)r+1$

   $\lambda(x_2 v_i) = \left\lfloor \frac{i+3}{3} \right\rfloor$

   d. For $i = 2r, 6r, 10r, 14r, \dots, (4m-2)r$

   $$\lambda(x_3 v_i) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ (mod\ 3) \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ (mod\ 3) \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

   e. For $i = 2r+1, 6r+1, 10r+1, 14r+1, \dots, (4m-2)r+1$

$\lambda(x_3v_i) = \left\lfloor\frac{i+3}{3}\right\rfloor$

f. For $i = 3r, 7r, 11r, 15r, \dots, (4m-1)r$

$$\lambda(x_4v_i) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ (mod\ 3) \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ (mod\ 3) \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

g. For $i = 3r+1, 7r+1, 11r+1, 15r+1, \dots, (4m-1)r+1$

$\lambda(x_4v_i) = \left\lfloor\frac{i+3}{3}\right\rfloor$

h. For $i = 4r, 8r, 12r, 16r, \dots, 4mr$

$$\lambda(x_5v_i) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ (mod\ 3) \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ (mod\ 3) \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

i. For $i = r, 5r, 9r, 13r, \dots, (4m-3)r, 2r, 6r, 10r, 14r, \dots, (4m-2)r, 3r, 7r, 11r, 15r, \dots, (4m-1)r, 4r, 8r, 12r, 16r, \dots, 4mr$ applies

$\lambda(v_{i-1}v_i) = \left\lfloor\frac{i+3}{3}\right\rfloor$

j. For $i = 1, 4r+1, 8r+1, 12r+1, \dots, (4m-4)r+1, r+1, 5r+1, 9r+1, 13r+1, \dots, (4m-3)r+1, 2r+1, 6r+1, 10r+1, 14r+1, \dots, (4m-2)r+1, 3r+1, 7r+1, 11r+1, 15r+1, \dots, (4m-1)r+1$ applies

$$\lambda(v_iv_{i+1}) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ (mod\ 3) \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ (mod\ 3) \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ (mod\ 3) \end{cases}$$

k. For $i = 4jr+2, 4jr+3, 4jr+4, \dots, 4jr+(r-1), 4jr+(r+2), 4jr+(r+3), 4jr+(r+4), \dots, 4jr+(2r-1), 4jr+(2r+2), 4jr+(2r+3), 4jr+(2r+4), \dots, 4jr+(3r-1), 4jr+(3r+2), 4jr+(3r+3), 4jr+(3r+4), \dots, 4jr+(4r-1)$ applies

$\lambda(v_{i-1}v_i) = \left\lfloor\frac{i+3}{3}\right\rfloor$

$$\lambda(v_iv_{i+1}) = \begin{cases} \frac{i+2}{3} & ;\ if\ i \equiv 1\ mod\ 3 \\ \frac{i+4}{3} & ;\ if\ i \equiv 2\ mod\ 3 \\ \frac{i+3}{3} & ;\ if\ i \equiv 0\ mod\ 3 \end{cases}$$

Based on the above labeling, the vertex weights v_i from graph to $sp(m,r,4)$ for $m \geq 5$ and, $r \geq 1$ notated by $w(v_i)$ is $w(v_i) = i+2$. The weight of the vertex v_i with is a consecutive integer $3,4,5,\dots,4mr+2$, so it is proved that each vertex weight v_i is different in the graphs $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$.

Next will be calculated the vertex weight x_i, with $i = 1,2,3,4,5$, from the graph $sp(m,r,4)$, for $m \geq 5$ and $r \geq 1$, and obtained the following results.

1. For $m = 5$ and $r \geq 1$

$w(x_1) = 20r + 3$

$w(x_2) = 30r + 11$

$$w(x_3) = \begin{cases} \frac{100r+35}{3} & ;\ for\ r \equiv 1(mod\ 3) \\ \frac{100r+31}{3} & ;\ for\ r \equiv 2(mod\ 3) \\ \frac{100r+33}{3} & ;\ for\ r \equiv 0(mod\ 3) \end{cases}$$

$$w(x_4) = \begin{cases} \dfrac{110r + 34}{3} & ;\ for\ r \equiv 1(mod\ 3) \\ \dfrac{110r + 38}{3} & ;\ for\ r \equiv 2(mod\ 3) \\ \dfrac{110r + 36}{3} & ;\ for\ r \equiv 0(mod\ 3) \end{cases}$$

$$w(x_5) = 20r + 7$$

2. For $m = 6$ and $r \geq 1$

$w(x_1) = 24r + 3w(x_2) = 44r + 13$

$w(x_3) = 48r + 13$

$w(x_4) = 52r + 13$

$w(x_5) = 28r + 7$

3. For $m \geq 7$ and $r \geq 1$

$$w(x_1) = \begin{cases} \left\lfloor \dfrac{2m^2r - 2mr + 3m + 3}{3} \right\rfloor & ; for\ r \equiv 1(mod\ 3) and\ m \equiv 2(mod\ 3) \\ \left\lceil \dfrac{2m^2r - 2mr + 3m + 3}{3} \right\rceil & ; for\ others \end{cases}$$

$$w(x_2) = \begin{cases} \left\lfloor \dfrac{4m^2r - 2mr + 6m + 1}{3} \right\rfloor & ;\ for\ r \equiv 1(mod\ 3) and\ m \equiv 1(mod\ 3) \\ \left\lceil \dfrac{4m^2r - 2mr + 6m + 1}{3} \right\rceil & ; for\ others \end{cases}$$

$$w(x_3) = \begin{cases} \left\lfloor \dfrac{4m^2r + 6m + 5}{3} \right\rfloor & ;\ for\ r \equiv 2(mod\ 3) and\ m \equiv 1(mod\ 3)\ or\ m \equiv 2(mod\ 3) \\ \left\lceil \dfrac{4m^2r + 6m + 5}{3} \right\rceil & ; for\ others \end{cases}$$

$$w(x_4) = \begin{cases} \left\lfloor \dfrac{4m^2r + 2mr + 6m + 3}{3} \right\rfloor & ; for\ r \equiv 1(mod\ 3) and\ m \equiv 2(mod\ 3) \\ \left\lceil \dfrac{4m^2r + 2mr + 6m + 3}{3} \right\rceil & ; for\ others \end{cases}$$

$$w(x_5) = \begin{cases} \left\lfloor \dfrac{2m^2r + 2mr + 3m + 2}{3} \right\rfloor & ;\ for\ r \equiv 1(mod\ 3) and\ m \equiv 1(mod\ 3) \\ \left\lceil \dfrac{2m^2r + 2mr + 3m + 2}{3} \right\rceil & ; for\ others \end{cases}$$

Based on calculation of the vertex weight above, obtained $w(v_{4mr}) < w(x_1) < w(x_5) < w(x_2) < w(x_3) < w(x_4)$. Whereas $w(v_i) < w(v_{i+1})$ for all $i = 1,2, \dots, 4mr - 1$. This shows that each vertex of graph $sp(m, r, 4)$ have different weight. It can be concluded that each vertex in the vertex irregular total labeling on the graph $sp(m, r, 4)$ has a different weight and $tvs\big(sp(m, r, 4)\big) \leq \left\lceil \frac{4mr+2}{3} \right\rceil$. Based on the above explanation, it is found that $tvs\big(sp(m, r, 4)\big) \geq \left\lceil \frac{4mr+2}{3} \right\rceil$ and $tvs\big(sp(m, r, 4)\big) \leq \left\lceil \frac{4mr+2}{3} \right\rceil$, so it is proven that $tvs\big(sp(m, r, 4)\big) = \left\lceil \frac{4mr+2}{3} \right\rceil$. ∎

As an illustration of the **Theorem 1**, an example is given for labeling the irregular totals of vertices for the graphs $sp(m, r, 4)$ for $m = 15$ and $r = 1$.

Example 2. Label vertices and edges of graph $sp(m, r, 4)$ for $m = 15$ and $r = 1$ with the labeling of vertices and edges in **Theorem 1**.

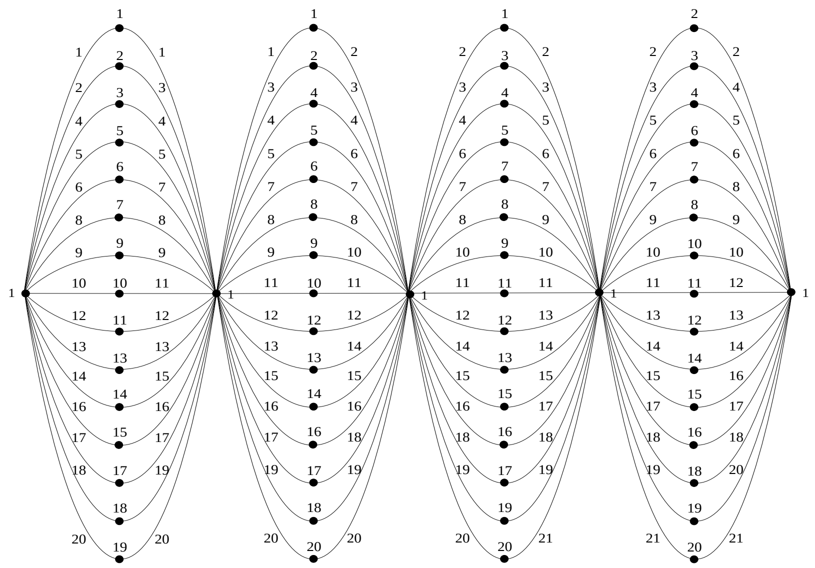

Figure 3. Labeling-21 Total Irregular Vertices on a Graph $sp(15,1,4)$

Based on **Theorem 1**, the total vertex irregularity strength of the graph is $tvs(sp(15,1,4)) = \left\lceil \frac{4(15)(1)+2}{3} \right\rceil = 21$.

4. CONCLUSIONS

The total vertex irregularity strength on the graphs $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$, is $tvs(sp(m,r,4)) = \left\lceil \frac{4mr+2}{3} \right\rceil$. This is proven through two steps. The first, showing that $tvs(sp(m,r,4)) \leq \left\lceil \frac{4mr+2}{3} \right\rceil$ based on structure of graph $sp(m,r,4)$. The second, proven that $tvs(sp(m,r,4)) \geq \left\lceil \frac{4mr+2}{3} \right\rceil$ by showing there is an irregular total $\left\lceil \frac{4mr+2}{3} \right\rceil$-labeling of vertices on a parallel series graph $sp(m,r,4)$ for $m \geq 5$ and $r \geq 1$.

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