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TRAFFIC CONGESTION ANALYSIS USING SIR EPIDEMIC MODEL

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ABSTRACT

In this work, we propose a mathematical model to represent traffic congestion in the street under some consideration. A congestion problem in a city highway becomes a critical issue since congestion at one point affected congestion propagation on the other points. We focus on the propagation of traffic propagation by adopting the concept of disease spread using the SIR model. We consider that the disease in traffic problems is congestion. Meanwhile, vehicles that enter the highway are susceptible to congestion. In contrast, vehicles free from traffic jams represent individuals free from disease. The SIR model can explain the spread of congestion by looking at the congestion variable as an infected variable. We discuss and analyze the existence and stability of the equilibrium points. The local stability equilibrium point is verified using the Routh-Hurwitz criteria. At the same time, the global stability is analyzed using Lyapunov function. The numerical simulation is provided in the last section to validate the discussion results.

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1. INTRODUCTION

Traffic conditions in the 20th century experienced a very significant improvement, especially in urban traffic with a high level of activity. Road conditions and various types of vehicles affect traffic density. The research conducted by [1] explained the flow of traffic on the freeway with a lack of visibility due to a foggy environment resulting in a stack of vehicles on the road body. Economically, congestion causes losses. According to [2] the impact of traffic congestion in Jakarta causes a financial loss of 65 trillion annually. This remains a problem that needs to be solved in big cities. Controlling the number of vehicles and road traffic is an attempt to solve the congestion problem.

In practice, congestion on one road section can lead to prolonged congestion. Furthermore, this will cause congestion on other roads. This means congestion propagation on the roads around the congestion point. Looking at this phenomenon, the process of congestion propagation is similar to the process of spreading disease where congestion will infect vehicles around the congestion to get involved in the congestion due to contact with vehicles that are in congestion. This is in line with the spread of disease where populations that are susceptible to the disease will be infected with the disease when they come into contact with infected populations. Research conducted by [3] and [4] investigates disease transmission using the SIR disease spread model. From a disease-spreading point of view, congestion is a disease that is the object of study. The process of spreading congestion shows that the population in congestion is an infected population. In contrast, the population likely to be involved in congestion and come into contact with the congestion population is called as susceptible. The population that manages to get out of congestion is a population that recovers from the disease. From this perspective, we analyze traffic congestion using the mathematical approach of the Suspected-Infected-Recovery (SIR) disease spread model. A study conducted by Hong-Xia [5] introduced a dynamical model of infectious disease for cyclist on urban city traffic which characterize the cyclist behaviour. While Indrakanti [6] delivered a study on modelling the communication process between vehicle using SIR model to model the spreads of congestion endemic. This also showed the traffic with six different conditions.

The research conducted by [7] also investigated the modeling and analysis of traffic congestion using a disease spread model approach motivated by a myopic model of the density load of computer communication network deployment. The congestion SIR model was built with microscopic network flow. A simple congestion process in an urban city was also investigated by [8] using dynamic network traffic jams adopted from SIR model. The other research about traffic was also conducted by [5] which study about the congestion propagation of vehicles on the freeway road with a foggy environment using the combination of SIR model and the Cellular-Automata (CA) model. The other cellular networks analysis using SIR model also showed by Mirahsam on [9] with Gaussian distribution and parameter estimation for the SIR statistical distribution of the traffic. We also take notes for the research on the transportation network. It has a similar behavior to the traffic on city roads. In [10] discussing epidemic spread model to the transportation spread when the infected people of Covid-19 are transported to the hospital or other health facility using SEIRS model. In contrast, the traffic congestion for network links was also investigated by [11] to analyze the node that the black virus might infect. While [12] investigates the epidemic spreading in a network-driven based on a simple suspected-infected-recovery model with a homogenous distribution for the network.

The dynamical analysis for the traffic model using SIR were given to study the behaviour of the system. In [13], Tian were investigate the global stability of connected SIR model by using the Lyapunov function. The results showed the global stability for both free desease and equilibrium endemic. A further research for the dynamical analysis is the bifurcation analysis. In the [14] the bifurcation theory were given to show the dynamic of the system using a double period transitions.

In this research, we constructed a simple congestion process using SIR model with some consideration based on the assumption and the existing research on urban traffic. It is assumed that the population is homogeneous, i.e., the population involved in traffic is the automobile population. In comparison, other vehicles are equated with cars. For example, one bus is equivalent to two cars, one container vehicle is equivalent to three cars, and four motorcycles are equivalent to one car. The investigation is constructed at the urban crossroad with only one road lane. The study’s contribution is to construct the model and analyze the distribution of congestion. Thus, we provide the dynamic of the system with four populations namely potential congested, congested, stuck in the traffic light, and retrieved from congestion. A theoretical performance shown in this paper is the existence solution of the dynamical model, the reproduction number, and the stability analysis. The analysis described in this research was carried out from [15], [16], [17] and [18].

The rest of this paper is organized as follows. Section 2 gives the formulation of the mathematical model to represent the congestion problem. In section 3, we give the existance and stability of the equilibrium points. The numerical simulation is given in section 4 to verify the analytical results.

2. RESEARCH METHOD

In this section, we construct a mathematical model using a simple SIR model to represent urban traffic congestion. Thus, we divided the population into four subpopulations: susceptible denoted by $\boldsymbol{S(t)}$ describe the number of cars that enter the system (roads) and are potentially involved in congestion, infected denoted by $\boldsymbol{I(t)}$ which represent the number of cars that infected to the congestion, infected with red light denoted with $\boldsymbol{Re(t)}$ describe the number of cars congested affected by the red light in the system, and the recovered denoted by $\boldsymbol{R(t)}$ describe the number of cars free form congestion. The relationship between variables can be explained through the following diagram.

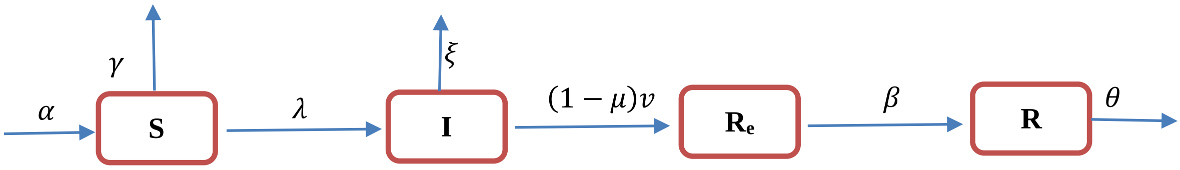

Figure 1. Compartment Diagram of Congestion

It is assumed that the susceptible cars $\boldsymbol{S(t)}$ are recruited at a rate $\boldsymbol{\alpha}$. The population decreases when the cars get off the main road at rate of $\boldsymbol{\gamma}$. The potentially congested cars can be transferred to the congestion $\boldsymbol{I(t)}$ with transmission rate at $\boldsymbol{\lambda}$. The birth rate $\boldsymbol{\lambda}$ were taken from congested links framework conducted by [5] where congestion propagation is the fraction of cars speed and the maximum speed on the link defined as

$$\boldsymbol{\lambda = \frac{v}{v_{\max}}}$$

The number of cars stuck in traffic can still get off from congestion with rate of $\boldsymbol{\xi}$. In this case, the parameter $\boldsymbol{\gamma}$ and $\boldsymbol{\xi}$ works as natural death rate to the system. The congested cars continue to get stuck in traffic if they are involved in a red light situation. The fraction of $\boldsymbol{(1-\mu)v}$ is the rate of additional cars to the red light queuing. Further, the rate of cars passing the red light is stated as $\boldsymbol{\beta}$. Last, the number of cars exit of the system (free from congestion) has a rate of $\boldsymbol{\theta}$. In this problem, there is no treatment for the system. The extension of SIR model can be described follows:

$$\begin{aligned}
\frac{dS}{dt} &= \alpha - \gamma S - \lambda SI \\
\frac{dI}{dt} &= \lambda SI - (1-\mu)v\, I\, Re - \xi I \\
\frac{dRe}{dt} &= (1-\mu)v\, Re\, I - \beta\, Re \\
\frac{dR}{dt} &= \beta\, Re - \theta\, R
\end{aligned} \tag{1}$$

with the initial conditions $\boldsymbol{S(0) \geq 0, I(0) \geq 0, Re(0) \geq 0, R(0) \geq 0}$.

The non-negativity rates of each population are:

$$\frac{dS}{dt} = \alpha \geq 0,$$

$$\frac{dI}{dt} = 0 \geq 0\ ,$$

$$\frac{dRe}{dt} = 0 \geq 0,$$

$\frac{dR}{dt} = \beta\, Re \geq 0.$

The total population for the system (1) at $t \geq 0$ is $N(t) = S(t) + I(t) + Re(t) + R(t)$. So the population dynamic

$$\frac{dN(t)}{dt} = \alpha - \theta N$$

with the solution $N(t) = \frac{\alpha}{\theta} + \left(N(0) - \frac{\alpha}{\theta}\right) e^{-\theta t}$ where $N(0) \geq 0$. The solution of $N(t)$ yields a non-negative value that imply $N(t) \leq \frac{\alpha}{\theta}$ if $N(0) \leq \frac{\alpha}{\theta}$ for $t \longrightarrow \infty$.

Hance, the feasible region for $N(t)$ is

$$\Omega = \left\{(S, I, Re, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\alpha}{\theta}\right\}.$$

The above results can be stated as the following theorem.

Theorem 1. Let (S, I, Re, R) be the solution of congestion model Equation (1) with the initial condition $S(0), I(0), Re(0), R(0) \geq 0$. The set of feasible solution

$$\Omega = \left\{(S(t), I, (t)Re(t), R(t)) \in \mathbb{R}_+^4 : N(t) = S(t) + I(t) + Re(t) + R(t) \leq \frac{\alpha}{\theta}\right\}$$

Is positively invariant and attracting set for the system Equation (1).

Since the total population of the system Equation (1) assumed to be constant, than the total population dynamic $\alpha - \theta N(t) = 0$ that imply to $N(t) = \frac{\alpha}{\theta}$ for $t \geq 0$.

3. RESULTS AND DISCUSSION

3.1 Stability Analysis

The existence solution of system Equation (1) and its stability analysis will be discussed on this section. Regarding to the Equation (1), we can investigate the disease-free equilibrium [19], [20] where there are no cars infected by congestion, i.e. $I = 0$. Thus, the equilibrium point of a disease-free is $E_0 = \left(\frac{\alpha}{\gamma}, 0, 0, 0\right)$. Than, we can define the basic reproduction number R_0 [21] and [4] that characterises the ability of the spreads of congestion epidemic. Using the Next Generation Matrix (NGM) method [22] with $K = TU^{-1}$, we derive the R_0 by choosing new infection vector and transmission vector respectively as follows,

$$F = \begin{bmatrix} \lambda SI \\ (1-\mu)\nu I\, Re \end{bmatrix} \text{ and } V = \begin{bmatrix} -(\mu\nu + \xi)I \\ -\beta\, Re \end{bmatrix}$$

Further, the Jacobian matrices are as follows,

$$T = \begin{bmatrix} \lambda S & 0 \\ (1-\mu)\nu\, Re & (1-\mu)\nu\, I \end{bmatrix} \text{ and } U = \begin{bmatrix} -(\mu\nu + \xi) & 0 \\ 0 & -\beta \end{bmatrix}. \quad (2)$$

The basic reproduction number R_0 is the spectral radius of K matrix which obtain

$$R_0 = \frac{\lambda}{\mu\nu + \xi} \quad (3)$$

It is clear that $R_0 < 1$, thus there is no congestion strain survive in the population and the infected cars tend to zero. For the disease-free equilibrium point $E_0 = \left(\frac{\alpha}{\gamma}, 0, 0, 0\right)$, we give the stability analysis using the eigen values of the Jacobian matrix which states in the following theorem.

Theorem 2. Let $E_0 = \left(\frac{\alpha}{\gamma}, 0, 0, 0\right)$ be the disease-free equilibrium point of Equation (1). If $R_0 < 1$ than the disease-free equilibrium point E_0 is locally asymptotically stable. If $R_0 > 1$ than the disease-free equilibrium point E_0 is unstable.

Proof. Taking into account for Equation (1) - Equation (3), as stated on [23] and [24] the Jacobian matrix for Equation (1) at E_0 is

$$J(E_0) = \begin{bmatrix} -\gamma & 0 & 0 & 0 \\ -\frac{\lambda\gamma}{\alpha} & \frac{\lambda\gamma}{\alpha} - \xi & 0 & 0 \\ 0 & 0 & -\beta & \beta \\ 0 & 0 & 0 & -\theta \end{bmatrix}$$

The eigenvalues of Jacobian matrix $J(E_0)$ are $\lambda_1 = -\gamma$, $\lambda_2 = -\theta$, $\lambda_3 = -\beta$ and $\lambda_4 = \frac{\lambda\gamma - \alpha\xi}{\alpha}$. Hence, the equilibrium point E_0 is locally asymptotically stable if $\gamma\lambda < \alpha\xi$. In contrast, there at least one eigen value of the Jacobi matrix that has a positive real part which makes equilibrium point E_0 is unstable. ∎

Further, the discussion for the global stability analysis of Equation (1) described as follows [19].

Theorem 3. Let $E_0 = \left(\frac{\alpha}{\gamma}, 0, 0, 0\right)$ be the disease-free equilibrium point of Equation (1). If $R_0 \leq 1$ than the disease-free equilibrium is globally asymptotical stable.

Proof. In the following, we consider a Lyapunov function.

$$L = \lambda I.$$

Then, taking the derivative of the Lyapunov function we obtain

$$\frac{dL}{dt} = \lambda \frac{dI}{dt}$$

$$= \lambda I(\lambda S - (1-\mu)v\, Re - \xi) \leq 0$$

Since $\frac{dL}{dt} = 0$ if only if $I = 0$ than the maximum compact invariant $I = 0$. Consequently, the solution will tend to E_0 for $t \to \infty$. Thus, E_0 is globally asymptotically stable. ∎

The system Equation (1) has an equilibrium point that given by $E^* = (S^*, I^*, Re^*, R^*)$ where the population stated as

$$S^* = \frac{(1-\mu)\alpha v}{(1-\mu)\gamma v + \lambda\beta}$$

$$I^* = \frac{\beta}{(1-\mu)v}$$

$$Re^* = \frac{\xi}{(1-\mu)v} - \frac{\alpha\lambda}{(1-\mu)\gamma v + \beta\lambda}$$

$$R^* = \frac{\beta\xi}{(1-\mu)\theta v} - \frac{\alpha\beta\lambda}{(1-\mu)\theta\gamma v + \beta\lambda}.$$

3.2 Numerical Simulation

In this section, a numerical result is given to analyze the behavior of the solution system Equation (1). The simulation is begun by choosing a set of feasible parameter values.

Table 1. Parameter Simulation Set

Parameter	Value
α	0.1
β	0.2
γ	0.05
θ	0.2
μ	0.6
v	0.2
λ	0.1
ξ	0.01

Using the parameter on **Table 1** We have the reproduction number $R_0 = 0.09 < 1$. Taking the initial as a set of $\big(S(0), I(0), Re(0), R(0)\big) = (20, 8, 3, 6)$ in the normal road situation which means there is no rush hour, so the total population $N(0) = 0.5$. in this simulation, the experiment time is chosen to be $t \in [0,20]$ to capture the synamic ans the stability until the equilibrium-free points is achieved at $E_0 = \left(\frac{\alpha}{\gamma}, 0,0,0\right)$ as $t \to \infty$. The simulation results shows on the following graph.

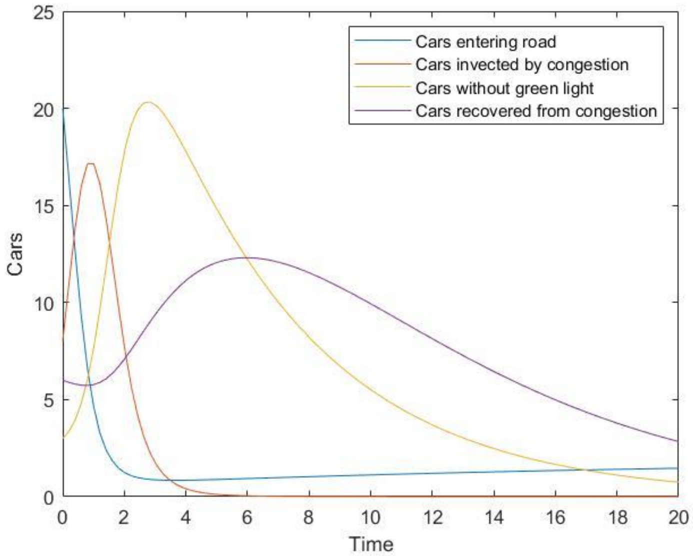

Figure 2. Numerical Simulation of System 1.

In the **Figure 2** shows that the dynamic of **Equation (1)** for the given initial point the number of entering cars on the road with the rate of $\alpha = 0.1$ contributed to the increase in the number of vehicles on congestion and increasing the number of vehicles queuing at red light conditions. It is confirmed that the dynamic achieves a free equilibrium point while $t \to \infty$ as the suspected cars in the congestion converge at $\frac{\alpha}{\gamma} = 2$. It is also confirmed that the system is locally stable for the disease-free equilibrium point E_0.

4. CONCLUSIONS

In this paper we presented a dynamic model for congestion in urban city with some assumption. The model constructed adopts the SIR disease spreading model regarding to the urban road characteristic. We considered that there is no treatment for congested condition, such that the model constructed in this work generate a simple SIR model. We obtain the disease-free equilibrium point and the general equilibrium point. The existence of the equilibrium point and the stability analysis were obtained. To clarify the stability analysis, the numerical simulation was given. The result shows that the system is locally stable achieved for an infinite time while the suspectable cars achieved to two for both ordinary and rush hours.

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