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ZILLMER RESERVE ON ENDOWMENT LAST SURVIVOR LIFE INSURANCE USING LOMAX DISTRIBUTION

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ABSTRACT

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This article discusses Zillmer's reserves for endowment last survivor of life insurance. Zillmer reserves are a type of modification of premium reserves which are calculated using prospective reserves and the Zillmer rate. In Zillmer reserves, loading which is the difference between gross premium and net premium in the first policy year is greater than standard loading. In this article, the life insurance used is endowment last survivor of life insurance, where the reserve calculation for last survivor status is calculated for 3 cases, namely, both participants survive until the end of the policy, participant x survive but participant y died, and participant y survive but participant x died. So, the purpose of this research is to find a way to make the loading value in 3 cases on the dwiguna last survivor of life insurance Zillmer reserves smaller. To achieve this goal, this article uses the Lomax distribution with the parameters estimated using maximum likelihood estimation and then determined by a Newton-Raphson iteration method. Based on the illustration, even though in the first policy year in cases where both participants survive until the end of the policy there was still a negative loading, overall Zillmer's reserves in each case continues to increase over time

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1. INTRODUCTION

Basically humans will face various kinds of unpredictable risks, one way to overcome these risks is to minimize the risk, namely by participating in a life insurance program [1]. Endowment life insurance with *last survivor* status is life insurance where premium payments are made until the death of the last insurance participant has not occured until the sum insured is given by the insurance company after the death of the last insured person [2]. In an insurance company, when someone becomes an insurance participant, he or she must pay a premium to the insurance company [3]. From this premium, some interest income will be obtained during the premium payment period.

The income from these premiums will later be used to pay for a number of needs from the insurance company. Reserves are the amount of money the insurance company has within the coverage period [4]. The reserves in the insurance company will be used to pay the insurance money to insurance participants. In calculating the reserve value, it is necessary to first know the value of the premium and life annuity which is influenced by the chance of life and the chance of death of the insurance participant [5]. In determining the chance of life and the chance of death in *last survivor* life insurance, the Lomax distribution is used, also known as the second type of Pareto distribution introduced by K.S. Lomax in 1954 [6], [7], with its parameters need to be estimated [8], [9].

At the beginning of the policy year, large costs are usually required for various purposes, for example paying provision fees as compensation to premium collection officers and others [3]. Therefore, insurance companies at the beginning of their existence needed to have a way to prevent losses. So this article aims to obtain a calculation of the reserve value that does not have a negative value in the first policy year, or if there is still a negative value then it does not indicate a number that is too large. Negative premium reserves will provide a greater chance of insurance company bankruptcy [10].

Insurance companies need to consider including operational costs in reserve management to avoid losses, one way is by modifying the premium reserve calculation, such as using the Zillmer reserve calculation. Zillmer reserves are a type of modified premium reserves whose calculation uses prospective reserves and a Zillmer rate of α [11]. Research related to the formation of life insurance premium for *last survivor* status, premium reserves and Zillmer reserves has been carried out by many researchers. Hasriati *et al*. [3] explained in his research that the calculation of single premiums and annual premiums for combined life insurance with *last survivor* status is enough to be combined in one insurance policy so that insurance participants pay less premiums. To determine the premium, you need to know the cash value of the initial annuity which is influenced by the chances of living and the chances of dying based on the Pareto distribution, whereas in Hasriati's *et al.* [12] research calculation of single premium and annual premium for combined status life insurance last *survivor* based on the formula of Makeham's Law which is a development of Gomperzt's law.

Research Dwipayana *et al.* [13] determined the *last survivor* life insurance premium reserve formula using New Jersey method where the calculation starts in the second year, for the t years, with $t = 2,3,4, \dots, n$, where n represents the term of the insurance participant's contract. In the research Hasriati's et *al*. [14] discussed determining prospective reserves in endowment life insurance combined *last survivor* and *joint life* for life insurance participants by formulating the chances of life and death of insurance participants based on the Gompertz distribution.

Research by Iriana *et al*. [1] calculated the Zilmer reserve of whole life insurance for insurance participants aged $23 - 25$ years using the Indonesian Mortality Table 1999 and 2011 as well as gender differences, which shows that the Zillmer reserve result when calculated with the 1999 TMI is greater than the 2011 TMI, if in terms of gender, the Zillmer reserve results for male insurance participants are greater than for female insurance participants. On Hasriati *et al*. [5] using the Zillmer method for calculating prospective reserves in modified endowment life insurance using the CIR interest rate expressed in the form of a discount factor with two parameter estimates using Indonesian interest rate data from 2010 to 2019, producing the Zillmer Reserve formula using the CIR interest rate, for insured participants who are x years, with a coverage period of m years, and payment n years with a certain Zillmer rate which is useful for insurance companies to predict the reserves they have so that they are sufficient in the event of an insurance claim. Meanwhile, in research Hasriati's et *al*. [10] discussed determining the loading value of the Zillmer reserve which is the difference between the gross premium and the net premium in the first policy year which is smaller than the standard loading using the *survival* function based on the Pareto distribution and interest rates using the CIR interest rate model.

Based on the background that has been developed and the references cited, the aim of this research is to determine how to achieve premium reserves using the Zillmer method *last survivor* status there were 3 cases in the first policy year each case does not have negative value, by formulating the chance of life and the chance of death of life insurance participants based on the Lomax distribution by estimating the parameters, this research is useful for insurance companies in calculating *last survivor* status life insurance premiums so that the premium reserve in the first year of the policy does not cause problems for calculations at a later time.

2. RESEARCH METHODS

The method used in this research is a qualitative method based on relevant literature studies and journals. This research does not include data so no case studies are discussed. However, in this research an example is given as an illustration of the use of last survivor status and Lomax distribution in calculating the Zillmer reserves.

This section explains the statistical and actuarial theories used to analyze the problems discussed. These theories include the survival function of last survivor function, the Lomax distribution and parameter estimation on Lomax distribution.

2.1 Survival Function of Last Survivor Function

Endowment *last survivor* of life insurance is a combined life insurance where premium payments are made until the last death of the insurance participant. Combined life insurance in determining the amount of premium required *survival* function for combined status [15], which is obtained from the relationship between *survival* function on individual status. The random variable X is said to be a continuous random variable if there is a function $f(x)$ so that the cumulative distribution function can be expressed [16]

$$F(x) = P(\mathrm{X} \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$

The *survival* function is denoted by $S(x)$,

$$S(x) = P(\mathrm{X} > x).$$

The relationship between the *survival* function and the cumulative distribution function can be obtained as follows:

$$S(x) = 1 - P(X \leq x),$$

$$S(x) = 1 - F(x). \quad (1)$$

The function $F_{T(x)}(t)$ is the probability that a person aged x dies within a period of t years, with $F_{T(x)}(t)$ denoted by ${}_tq_x$. Based on **Equation (1)** the *survival* function can be stated

$$S(x) = 1 - {}_tq_x\ . \quad (2)$$

The relationship between the probability of living ${}_tp_x$ nd the probability of dying ${}_tq_x$ for life insurance participants who are x years old and can survive up to t years, namely

$${}_tp_x = 1 - {}_tq_x\ .$$

Suppose $T(x)$ represents a continuous random variable for life insurance participants aged x years and T(y) represents a continuous random variable for life insurance participants aged y tahun. The *last survivor* life insurance continuous random variable becomes $T(xy) = \max[T(x), T(y)]$ with its cumulative distribution function, namely $F_{T(\overline{xy})}(t) = P(\max[T(x), T(y)] \leq t)$ [15],

$$F_{T(\overline{xy})}(t) = P(\mathrm{T(x)} \leq t\ dan\ \mathrm{T(y)} \leq t).$$

Obtained

$$F_{T(\overline{xy})}(t) = {}_tq_x \cdot {}_tq_y\ ,$$

$${}_tq_{\overline{xy}} = {}_tq_x \cdot {}_tq_y\ .$$

So,

$$1-{}_tp_{\overline{xy}}=(1-{}_tp_x)(1-{}_tp_y),$$

$$1-{}_tp_{\overline{xy}}=1-{}_tp_x-{}_tp_y+{}_tp_x\,{}_tp_y,$$

$${}_tp_{\overline{xy}}={}_tp_x+{}_tp_y-{}_tp_x\,{}_tp_y.$$

Thus, endowment *last survivor* insurance can be obtained as follows:

$${}_tp_{\overline{xy}}={}_tp_x+{}_tp_y-{}_tp_{xy}. \tag{3}$$

2.2 Lomax Distribution

The Lomax or Pareto II distribution introduced by K.S. Lomax in 1954, is a special case of the Pareto distribution whose interval starts from zero and has been widely used in *survival* analysis [17]. The probability density function of the Lomax distribution, namely [6]

$$f(x,\theta,\lambda)=\theta\lambda^{\theta}(x+\lambda)^{-(\theta+1)},\ \ x,\theta,\lambda>0,$$

The parameter θ is a shape parameter and λ is a scalar parameter.

The cumulative distribution function $f(x)$ in the Lomax distribution can be obtained as follows:

$$\begin{aligned}F(x)&=\int_0^x f(t)\,dt\\&=\int_0^x \theta\lambda^{\theta}(t+\lambda)^{-(\theta+1)}\,dt\\&=\theta\lambda^{\theta}\int_0^x\frac{1}{(t+\lambda)^{1+\theta}}dt\\&=\theta\lambda^{\theta}\left[-\frac{1}{\theta(t+\lambda)^{\theta}}\right]_0^x\\&=-\frac{\theta\lambda^{\theta}}{\theta(x+\lambda)^{\theta}}+\frac{\theta\lambda^{\theta}}{\theta\lambda^{\theta}},\end{aligned}$$

$$F(x)=1-\left(\frac{\lambda}{\lambda+x}\right)^{\theta}. \tag{4}$$

Based on **Equation (1)** and **Equation (4)**, the *survival* function of the Lomax distribution is obtained

$$S(x)=1-\left(1-\left(\frac{\lambda}{\lambda+x}\right)^{\theta}\right),$$

$$S(x)=\left(\frac{\lambda}{\lambda+x}\right)^{\theta}.$$

The cumulative distribution function of the continuous random variable in the Lomax distribution can also be expressed as follows [18]:

$$F_{T(x)}(t)=\frac{F(x+t)-F(x)}{S(x)}$$

$$=\frac{\left(1-\left(\frac{\lambda}{\lambda+x+t}\right)^{\theta}\right)-\left(1-\left(\frac{\lambda}{\lambda+x}\right)^{\theta}\right)}{\left(\frac{\lambda}{\lambda+x}\right)^{\theta}}$$

$$= 1 - \frac{\left(\frac{\lambda}{\lambda + x + t}\right)^{\theta}}{\left(\frac{\lambda}{\lambda + x}\right)^{\theta}},$$

$$F_{T(x)}(t) = 1 - \left(\frac{\lambda + x}{\lambda + x + t}\right)^{\theta}.$$

So obtained

$${}_{t}q_{x} = 1 - \left(\frac{\lambda + x}{\lambda + x + t}\right)^{\theta}.$$

The probability of life for life insurance participants aged x years, denoted by ${}_{t}p_{x}$ can be expressed

$${}_{t}p_{x} = \left(\frac{\lambda + x}{\lambda + x + t}\right)^{\theta}. \quad (5)$$

Based on **Equation (3)** endowment *last survivor* life insurance can be stated as follows:

$${}_{t}p_{\overline{xy}} = \left(\frac{\lambda + x}{\lambda + x + t}\right)^{\theta} + \left(\frac{\lambda + y}{\lambda + y + t}\right)^{\theta} - \left(\left(\frac{\lambda + x}{\lambda + x + t}\right)^{\theta}\left(\frac{\lambda + y}{\lambda + y + t}\right)^{\theta}\right). \quad (6)$$

2.3 Parameter Estimation on Lomax Distribution

In the Lomax distribution, there are several parameters whose values need to be known. Therefore, parameter estimation is carried out using the *Maximum Likelihood Estimation (MLE)* method, the way this method works is to maximize the function *likelihood* function **[19]**.

The procedure for finding the maximum *likelihood* of a parameter is to determine the *likelihood* function of the probability density function in the Lomax distribution as follows:

$$L(\theta, \lambda | x_1, x_2, \dots, x_n) = \left(\frac{\theta}{\lambda}\right)^{n} \frac{1}{\prod_{i=1}^{n}\left(1 + \frac{x_i}{\lambda}\right)^{1+\theta}}. \quad (7)$$

Equation (7) is expressed in terms of the natural logarithm (ln), the *log-likelihood* function

$$\ln L(\theta, \lambda | x_1, x_2, \dots, x_n) = n \ln(\theta) - n \ln(\lambda) - (1 + \theta) \sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right). \quad (8)$$

The next step after obtaining the *log-likelihood* function that is to determine the first derivative of the parameters θ and λ as follows:

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right). \quad (9)$$

$$\frac{\partial \ln L}{\partial \lambda} = -\frac{n}{\lambda} + \left(\frac{1 + \theta}{\lambda}\right) \sum_{i=1}^{n} \left(\frac{x_i}{\lambda + x_i}\right). \quad (10)$$

Furthermore, the first derivative equation is equal to zero so that the a c*losed form* equation is formed to obtain parameter estimates on the Lomax distribution, denoted by $\hat{\theta}$ and $\hat{\lambda}$ as follows:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}. \quad (11)$$

$$\hat{\lambda} = \frac{n}{\left(\frac{1 + \theta}{\lambda}\right) \sum_{i=1}^{n} \left(\frac{x_i}{\lambda + x_i}\right)}. \quad (12)$$

Equation (12) is an equation that is not *closed form* because in the final equation there is still a parameter λ. One method of solving using the Newton-Raphson method [9].

The Newton-Raphson method uses the iteration approach to produce convergent values. The general equation of the Newton-Raphson method is as follows [20]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{13}$$

Based on Equation (11), Equation (10) which is equated to zero can be stated

$$0 = -\frac{n}{\lambda} + \frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda + x_i}\right)}{\lambda}. \tag{14}$$

The iteration formula for λ based on Equation (14) is obtained

$$\lambda = \frac{n}{\frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda + x_i}\right)}{\lambda}}. \tag{15}$$

The function f with respect to λ is based on Equation (15)

$$f(\lambda) = -\frac{n}{\lambda} + \frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda + x_i}\right)}{\lambda}. \tag{16}$$

The first derivative of the function f with respect to λ based on Equation (16) is obtained

$$f'(\lambda) = \frac{n}{\lambda^2} - \frac{n\left(\sum_{i=1}^{n}\left(\frac{-x_i}{\lambda^2\left(1 + \frac{x_i}{\lambda}\right)}\right)\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda + x_i}\right)}{\lambda\left(\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)\right)^2} - \frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda + x_i}\right)}{\lambda_n^2} + \frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda}\right)}\right)\sum_{i=1}^{n}\left(\frac{-x_i}{(\lambda + x_i)^2}\right)}{\lambda}. \tag{17}$$

The estimation of parameter λ in Equation (13) can be expressed in the Newton-Raphson method based on Equation (15), Equation (16), and Equation (17) for the following:

$$\hat{\lambda}_{n+1} = \frac{n}{\frac{\left(1 + \frac{n}{\sum_{i=1}^{n} \ln\left(1 + \frac{x_i}{\lambda_n}\right)}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda_n + x_i}\right)}{\lambda_n}}$$

$$-\frac{\dfrac{-n}{\lambda_n}+\dfrac{\left(1+\dfrac{n}{\sum_{i=1}^{n}\ln\left(1+\dfrac{x_i}{\lambda_n}\right)}\right)\sum_{i=1}^{n}\left(\dfrac{x_i}{\lambda_n+x_i}\right)}{\lambda_n}}{\left(\begin{array}{l}\dfrac{n}{\lambda_n^2}-\dfrac{n\left(\sum_{i=1}^{n}\left(\dfrac{-x_i}{\lambda_n^2\left(1+\dfrac{x_i}{\lambda_n}\right)}\right)\right)\sum_{i=1}^{n}\left(\dfrac{x_i}{\lambda_n+x_i}\right)}{\lambda_n\left(\sum_{i=1}^{n}\ln\left(1+\dfrac{x_i}{\lambda_n}\right)\right)^2}-\dfrac{\left(1+\dfrac{n}{\sum_{i=1}^{n}\ln\left(1+\dfrac{x_i}{\lambda_n}\right)}\right)\sum_{i=1}^{n}\left(\dfrac{x_i}{\lambda_n+x_i}\right)}{\lambda_n^2}\\[2ex] +\dfrac{\left(1+\dfrac{n}{\sum_{i=1}^{n}\ln\left(1+\dfrac{x_i}{\lambda_n}\right)}\right)\sum_{i=1}^{n}\left(\dfrac{-x_i}{(\lambda_n+x_i)^2}\right)}{\lambda_n}\end{array}\right)}. \quad (18)$$

The value of iteration λ in **Equation (18)** will be used to determine the estimate of parameter θ in **Equation (11)**. Parameter estimation that depends on life insurance participants for age x years is denoted by $\hat{\theta}_x$ and $\hat{\lambda}_x$ while for age y years is denoted by $\hat{\theta}_y$ and $\hat{\lambda}_y$.

3. RESULTS AND DISCUSSION

3.1 Futures Life Annuity of Last Survivor Endowment Life Insurance

Annuity is a series of payments to an insurance company in a certain amount at certain time intervals as long as the insurance participant is still alive [21]. In an initial life annuity, payments are made at the beginning of the period up to the $n-1$ year period. In the calculation of the annuity life is strongly influenced by the interest rate and there is also a discount factor function, namely

$$v=\frac{1}{1+i}.$$

The cash value of a term initial life annuity for insurance participants aged x years with a coverage period of n years can be stated [18]

$$\ddot{a}_{x:\overline{n}|}=1+vp_x+v^2\ {}_2p_x+\cdots+v^{n-1}\ {}_{n-1}p_x\ ,$$

$$\ddot{a}_{x:\overline{n}|}=\sum_{t=0}^{n-1}v^t\ {}_tp_x\ . \quad (19)$$

Based on **Equation (5)** the term initial life annuity for insurance participants aged x years with a coverage period of n years uses the Lomax distribution with parameters that depend on age x years denoted θ_x and λ_x, namely

$$\ddot{a}_{x:\overline{n}|}=\sum_{t=0}^{n-1}v^t\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}. \quad (20)$$

Initial term life annuity for insurance participants aged y years with a coverage period of n years using the Lomax distribution with age-dependent parameters y years denoted θ_y and λ_y based on **Equation (20)** can be stated, namely

$$\ddot{a}_{y:\overline{n}|}=\sum_{t=0}^{n-1}v^t\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}. \quad (21)$$

Term initial life annuity for insurance participants aged x years with a coverage period of h years using the Lomax distribution with parameters that depend on age x years based on **Equation (19)** is stated

$$\ddot{a}_{x:\overline{h}|} = \sum_{t=0}^{h-1} v^t \left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x}. \tag{22}$$

Based on **Equation (22)** the term initial life annuity for insurance participants aged y years with a coverage period of h years using the Lomax distribution with age-dependent parameters y years can be expressed

$$\ddot{a}_{y:\overline{h}|} = \sum_{t=0}^{h-1} v^t \left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y}. \tag{23}$$

Initial life annuity cash value *last survivor* futures are affected by the discount factor v and the combined survival probability of ${}_tp_{\overline{xy}}$ denoted by $\ddot{a}_{\overline{xy}:\overline{n}|}$, ie **[11]**

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{t=0}^{n-1} v^t \; {}_tp_{\overline{xy}}. \tag{24}$$

Based on **Equation (6)** the initial life annuity *last survivor* term in **Equation (24)** with a coverage period of n years using the Lomax distribution with parameters that depend on age x years and y years, namely

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \sum_{t=0}^{n-1} v^t \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x} + \left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x}\left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y}\right)\right). \tag{25}$$

Based on **Equation (25)**, the initial life annuity of *last survivor* a term with a coverage period of h years using the Lomax distribution with age-dependent parameters x years and y years can be expressed

$$\ddot{a}_{\overline{xy}:\overline{h}|} = \sum_{t=0}^{h-1} v^t \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x} + \left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x}\left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y}\right)\right). \tag{26}$$

3.2 Premium and Prospective Reserve of *Last Survivor* Endowment Life Insurance

In endowment life insurance, the sum insured is based on the amount of a premium. Premiums are a series of payments made by insurance participants to insurance companies over a certain period of time with a predetermined amount. The single premium for *last survivor* endowment life insurance is the sum of the single premium for pure *last survivor* ife insurance and term *last survivor* life insurance, is **[14]**

$$A_{\overline{xy}:\overline{n}|} = A_{\overline{xy}:\overset{1}{\overline{n}|}} + A^{1}_{\overline{xy}:\overline{n}|}.$$

So that

$$\begin{aligned} A_{\overline{xy}:\overline{n}|} &= v^n \; {}_np_{\overline{xy}} + \sum_{t=0}^{n-1} v^{t+1} \; {}_{t|}q_{\overline{xy}} \\ &= v^n \; {}_np_{\overline{xy}} + \sum_{t=0}^{n-1} v^{t+1} (\; {}_tp_{\overline{xy}} - {}_{t+1}p_{\overline{xy}}), \\ A_{\overline{xy}:\overline{n}|} &= v^n \; {}_np_{\overline{xy}} + v\ddot{a}_{\overline{xy}:\overline{n}|} - \left(\ddot{a}_{\overline{xy}:\overline{n}|} - (1 - v^n \; {}_np_{\overline{xy}})\right). \end{aligned}$$

So

$$\begin{aligned} A_{\overline{xy}:\overline{n}|} &= 1 + v\ddot{a}_{\overline{xy}:\overline{n}|} - \ddot{a}_{\overline{xy}:\overline{n}|}, \\ A_{\overline{xy}:\overline{n}|} &= 1 - d\ddot{a}_{\overline{xy}:\overline{n}|}. \end{aligned} \tag{27}$$

for d is a function of the discount rate expressed by $d = 1 - v$. The single premium for the *last survivor* endowment life insurance with a coverage period of n years uses the Lomax distribution as follows:

$$A_{\overline{xy}:\overline{n}|} = 1 - d\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x} + \left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x}\left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y}\right)\right), \quad (28)$$

with a sum insured of R paid at the end of the single premium life insurance *last survivor* endowment policy year to $RA_{\overline{xy}:\overline{n}|}$.

Annual premium is a premium paid at the beginning of each year whose amount can change or be the same every year **[4]**. The Annual premium of the endowment *last survivor* life insurance with a coverage period of n years is denoted by $P_{\overline{xy}:\overline{n}|}$ and can be expressed as

$$P_{\overline{xy}:\overline{n}|} = \frac{A_{\overline{xy}:\overline{n}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}},$$

$$P_{\overline{xy}:\overline{n}|} = \frac{1}{\ddot{a}_{\overline{xy}:\overline{n}|}} - d. \quad (29)$$

The annual premium for *last survivor* endowments life insurance with a coverage period of n years using the Lomax distribution as follows:

$$P_{\overline{xy}:\overline{n}|} = \frac{1}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x} + \left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x + x}{\lambda_x + x + t}\right)^{\theta_x}\left(\frac{\lambda_y + y}{\lambda_y + y + t}\right)^{\theta_y}\right)\right)} - d. \quad (30)$$

The sum insured if R is paid at the end of the policy year, annual premium for the endowment *last survivor* life insurance will be $RP_{\overline{xy}:\overline{n}|}$.

Prospective reserves of endowment *last survivor* life insurance with the sum insured paid at the end of the policy year, the single premium is $A = 1 - d\ddot{a}$, and the annual premium is $P_{\overline{xy}:\overline{n}|} = \frac{1}{\ddot{a}} - d$, can be expressed in 3 cases as follows **[14]**:

1. Both insurance participants live until the end of the policy year

$$_tV_{\overline{xy}:\overline{n}|} = A_{\overline{x+t,y+t}:\overline{n-t}|} - P_{\overline{xy}:\overline{n}|}\ddot{a}_{\overline{x+t,y+t}:\overline{n-t}|}. \quad (31)$$

Based on **Equation (24)** and **Equation (27)** for insurance participants aged $(x + t)$ and $(y + t)$ years with coverage period $(n - t)$ years, the prospective reserves of endowment *last survivor* life insurance in **Equation (31)** can be stated

$$_tV_{\overline{xy}:\overline{n}|} = \left(1 - d\ddot{a}_{\overline{x+t,y+t}:\overline{n-t}|}\right) - \left(\frac{1}{\ddot{a}_{\overline{xy}:\overline{n}|}} - d\right)\ddot{a}_{\overline{x+t,y+t}:\overline{n-t}|}$$

$$_tV_{\overline{xy}:\overline{n}|} = 1 - \frac{\ddot{a}_{\overline{x+t,y+t}:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}}. \quad (32)$$

Using the Lomax distribution of prospective reserves of endowment *last survivor* life insurance in **Equation (32)** can be obtained

$$_tV_{\overline{xy}:\overline{n}|} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k\left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x} + \left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}\left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}\right)\right)}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x} + \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}.$$

2. When x is alive and y is dead

$$_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{\overline{xy}:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|}. \quad (33)$$

Based on **Equation (19)** and **Equation (27)** for insurance participants aged $(x + t)$ years with a coverage period $(n - t)$ years, prospective reserves of endowment *last survivor* life insurance in **Equation (33)** can be stated

$$ {}_tV_{x:\overline{n}|} = \left(1 - d\ddot{a}_{x+t:\overline{n-t}|}\right) - \left(\frac{1}{\ddot{a}_{\overline{xy}:\overline{n}|}} - d\right)\ddot{a}_{x+t:\overline{n-t}|} $$

$$ {}_tV_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}}. \tag{34} $$

Using the Lomax distribution of prospective reserves of endowment *last survivor* life insurance in **Equation (34)** can be obtained

$$ {}_tV_{x:\overline{n}|} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k\left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}\right)}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}+\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}-\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}. $$

3. When y is alive and x dies

$$ {}_tV_{y:\overline{n}|} = A_{y+t:\overline{n-t}|} - P_{\overline{xy}:\overline{n}|}\ddot{a}_{y+t:\overline{n-t}|}. \tag{35} $$

Based on **Equation (19)** and **Equation (27)** for insurance participants aged $(y+t)$ years with coverage period $(n-t)$ years, the prospective reserve of endowment *last survivor* life insurance in **Equation (35)** can be stated

$$ {}_tV_{y:\overline{n}|} = \left(1 - d\ddot{a}_{y+t:\overline{n-t}|}\right) - \left(\frac{1}{\ddot{a}_{\overline{xy}:\overline{n}|}} - d\right)\ddot{a}_{y+t:\overline{n-t}|} $$

$$ {}_tV_{y:\overline{n}|} = 1 - \frac{\ddot{a}_{y+t:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}}. \tag{36} $$

Using the Lomax distribution of prospective reserves of endowment *last survivor* life insurance in **Equation (36)** can be obtained

$$ {}_tV_{y:\overline{n}|} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k\left(\left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}\right)}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}+\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}-\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}. $$

3.3 Zillmer Reserve Last Survivor Endowment Life Insurance

Zillmer reserves are one type of modified premium reserves which are calculated using prospective reserves and a Zillmer level of α, with an α value of $0{,}025$. In Zillmer reserves, there are modified premiums, namely P_1 and P_2, each of which is the net premium in the first year of the policy and the net premium in the second year of the policy until the h-year of the policy where h is Zillmer's time with $P_1 < P_{\overline{xy}:\overline{n}|} < P_2$. The relationship of the Zillmer rate and net premium, ie [5]

$$ P_2 - P_1 = \alpha $$

$$ P_2 = \alpha + P_1. \tag{37} $$

Premiums paid by *last survivor* life insurance participants with a coverage period of n years are illustrated by the following timeline:

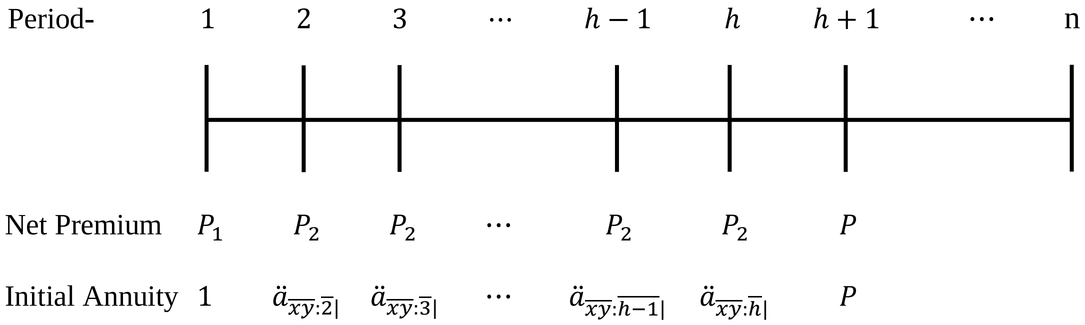

Figure 1. Modified *Last Survivor* Premium Timeline

Based on the timeline in **Figure 1** and that the P value is equivalent to the annual endowment life insurance premium **[22]**. Then to obtain the cash value of the modified premiums P_1 and P_2 can be determined as follows:

$$P_{\overline{xy}:\overline{n}|}\ddot{a}_{\overline{xy}:\overline{h}|} = P_1 + P_2\left(\ddot{a}_{\overline{xy}:\overline{h}|} - 1\right)$$
$$= P_1 - P_2 + P_2\ddot{a}_{\overline{xy}:\overline{h}|},$$
$$P_2 = \frac{\alpha}{\ddot{a}_{\overline{xy}:\overline{h}|}} + P_{\overline{xy}:\overline{n}|}.$$

Based on **Equation (37)** is obtained

$$P_1 = \frac{\alpha}{\ddot{a}_{\overline{xy}:\overline{h}|}} + P_{\overline{xy}:\overline{n}|} - \alpha,$$
$$P_1 = P_{\overline{xy}:\overline{n}|} - \alpha\left(1 - \frac{1}{\ddot{a}_{\overline{xy}:\overline{h}|}}\right).$$

Calculation of Zillmer reserves *last survivor* endowment life insurance with t years of reserve calculation time, h years of Zillmer time $1 \le t \le h$, and coverage period of n years, denoted ${}_tV^{(hz)}_{\overline{xy}:\overline{n}|}$ can be expressed in 3 cases as follows:

1. Both insurance participants live until the end of the policy year

$${}_tV^{(hz)}_{\overline{xy}:\overline{n}|} = {}_tV_{\overline{xy}:\overline{n}|} - \frac{\alpha}{\ddot{a}_{\overline{xy}:\overline{h}|}}\ddot{a}_{\overline{x+t,y+t}:\overline{h-t}|}. \quad (38)$$

Based on **Equation (32)**, Zillmer reserves *last survivor* endowment life insurance in **Equation (38)**, ie

$${}_tV^{(hz)}_{\overline{xy}:\overline{n}|} = 1 - \frac{\ddot{a}_{\overline{x+t,y+t}:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}} - \frac{\alpha}{\ddot{a}_{\overline{xy}:\overline{h}|}}\ddot{a}_{\overline{x+t,y+t}:\overline{h-t}|}. \quad (39)$$

Using the Lomax distribution of Zillmer reserve of *last survivor* endowment life insurance in **Equation (39)** can be stated

$${}_tV^{(hz)}_{\overline{xy}:\overline{n}|} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k\left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x} + \left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}\left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}\right)\right)}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x} + \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}$$
$$- \frac{\alpha\sum_{k=0}^{(h-t)-1} v^k\left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x} + \left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}\left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}\right)\right)}{\sum_{t=0}^{h-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x} + \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}.$$

2. When x is alive and y is dead

$${}_tV^{(hz)}_{x:\overline{n}|} = {}_tV_{x:\overline{n}|} - \frac{\alpha}{\ddot{a}_{x:\overline{h}|}}\ddot{a}_{x+t:\overline{h-t}|}. \quad (40)$$

Based on **Equation (34)**, Zillmer reserves of *last survivor* endowment life insurance in **Equation (40)**, ie

$${}_tV^{(hz)}_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}} - \frac{\alpha}{\ddot{a}_{x:\overline{h}|}}\ddot{a}_{x+t:\overline{h-t}|}. \quad (41)$$

Using the Lomax distribution of Zillmer reserve of *last survivor* endowment life insurance in **Equation (41)** can be stated

$${}_tV^{(hz)}_{x:\overline{n}|} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k\left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}}{\sum_{t=0}^{n-1} v^t\left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x} + \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}$$

$$-\frac{\alpha}{\sum_{t=0}^{h-1} v^t \left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}} \sum_{k=0}^{(h-t)-1} v^k \left(\frac{\lambda_x+x+t}{\lambda_x+x+t+k}\right)^{\theta_x}.$$

3. When y is alive and x dies

$$ {}_tV_{y:\overline{n}|}^{(hz)} = {}_tV_{y:\overline{n}|} - \frac{\alpha}{\ddot{a}_{y:\overline{h}|}} \ddot{a}_{y+t:\overline{h-t}|}. \tag{42}$$

Based on **Equation (36)**, Zillmer reserves of *last survivor* endowment life insurance in **Equation (42)**, ie

$$ {}_tV_{y:\overline{n}|}^{(hz)} = 1 - \frac{\ddot{a}_{y+t:\overline{n-t}|}}{\ddot{a}_{\overline{xy}:\overline{n}|}} - \frac{\alpha}{\ddot{a}_{y:\overline{h}|}} \ddot{a}_{y+t:\overline{h-t}|}. \tag{43}$$

Using the Lomax distribution of Zillmer reserve of *last survivor* endowment life insurance in **Equation (43)** can be stated

$$ {}_tV_{y:\overline{n}|}^{(hz)} = 1 - \frac{\sum_{k=0}^{(n-t)-1} v^k \left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}}{\sum_{t=0}^{n-1} v^t \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x} + \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y} - \left(\left(\frac{\lambda_x+x}{\lambda_x+x+t}\right)^{\theta_x}\left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}\right)\right)}$$

$$-\frac{\alpha}{\sum_{t=0}^{h-1} v^t \left(\frac{\lambda_y+y}{\lambda_y+y+t}\right)^{\theta_y}} \sum_{k=0}^{(h-t)-1} v^k \left(\frac{\lambda_y+y+t}{\lambda_y+y+t+k}\right)^{\theta_y}.$$

Example: A husband and wife whose respective ages are x, which is 38 years and y, which is 35 years, following the *last survivor* endowment life insurance program with a term of 20 years. If the sum insured received by the heirs is Rp100.000.000, the current interest rate is 0,02, the Zillmer rate is 0,025 and the Zillmer time is 16 years, then determine the calculation for the following cases:

(i) If both participants live until the end of the policy year

(ii) At the time the husband was still alive and the wife passed away

(iii) At the time the husband passed away and the wife was still alive

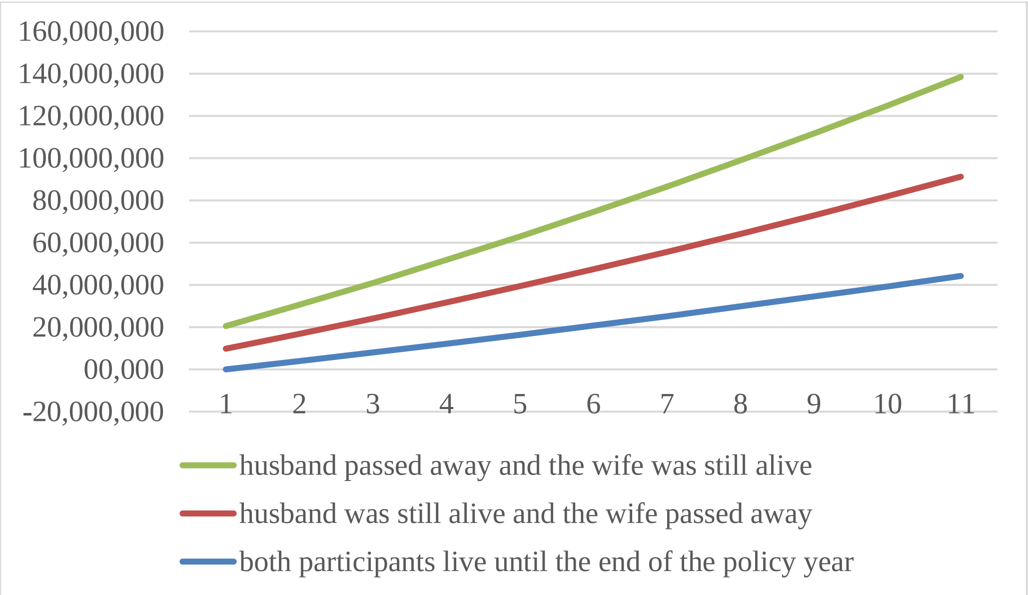

Figure 2. Zillmer Reserve of *Last Survivor* Endowment Life Insurance

4. CONCLUSIONS

Zillmer reserves are an alternative method that can be used to determine of reserves obtained from net premiums with a value of α which is the Zillmer level. In this case, calculating on Zillmer reserves using the Lomax distribution as a modification of prospective reserves which is suitable for modeling the survival function of both men and women, and endowment *last survivor* status, so by using a Zillmer rate α of 0.025, we obtain increasingly increasing Zillmer reserves. This research is very useful for insurance companies in predicting the reserves that must be owned by the insurance company when a claim occurs

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REFERENCES

[1] N. Iriana, I. Purnamasari and Y. N. Nasution, "Penentuan Cadangan Premi Asuransi Jiwa Seumur Hidup Menggunakan Metode Zillmer," *Jurnal Matematika, Statistika, dan Komputasi,* vol. 16, no. 2, pp. 219-225, January 2020.

[2] Ruhiyat, K. W. Erlina, Lamberto and E. Ardelia, "Last-Survivor Insurance Premium and Benefit Reserve Calculation using Gamma-Gompertz Mortality Law," *J. Mat. Integr,* vol. 18, no. 1, pp. 9-18, May 2022.

[3] Hasriati and T. P. Nababan, "Private Premium of Endowment Last Survivor and Joint Life Insurance with Pareto Distribution," *International Journal of Statistical Distributions and Applications,* vol. 5, no. 4, pp. 76-81, December 2019.

[4] T. Futami, Matematika Asuransi Jiwa Bagian I, terjemahan dari seimei hoken sugaku, jokan (92 Revision), Japan: Incorporated Foundation Oriental Life Insurance Cultural Development Center, 1993.

[5] Hasriati, Sukono, I. Hasbiyati, T. P. Nababan and A. Kirana, "Zillmer Reserves in Dwiguna Life Insurance with the Cox-Ingersoll-Ross (CIR) Interest Rate," *Komputer: J. Ilmiah Ilmu Komputer dan Matematika,* vol. 20, no. 1, pp. 01-09, January 2023.

[6] A. S. Hassan and A. N. Zaky, "Entropy Bayesian Estimation for Lomax Distribution Based on Record," *Thail. Stat,* vol. 19, no. 1, pp. 95-114, January 2020.

[7] V. B. V. Nagarjuna, R. V. Vardhan and C. Chesneau, "Nadarajah–Haghighi Lomax Distribution and Its Applications," *Math. Comput. Appl,* vol. 27, no. 2, pp. 01-13, April 2022.

[8] C. V. Aruna and G. V. S. R. Anjaneyulu, Lomax Distribution Parameters Estimation Using Different Methods, Inggris: LAMBERT Academic Publising, 2022.

[9] H. Sirait, Hasriati and S. Putra, "Estimating Expected Time For Recruitment of Human Recource Companies Health Insurance based on Lomax Distribution," in *Proc. 2nd African Int. Conf. on Industrial Engineering and Operations Management Detroit*, pp. 2498-2501, Aug. 10-14, 2020.

[10] Hasriati, I. Hasbiyati, A. Kirana and A. Prabowo, "Cadangan Zillmer dengan Disribusi Pareto dan Tingkat Bunga Cox-Ingersoll-Ross," *Jurnal Matematika, Sains, dan Teknologi*, vol. 23, no. 2, pp. 24-36, September 2022.

[11] T. Futami, Matematika Asuransi Jiwa Bagian II, terjemahan. dari seimei hoken sugaku, jokan (92 Revision), Japan: Incorporated Foundation Oriental Life Insurance Cultural Development Center, 1994.

[12] Hasriati, H. Sirait, A. Prabowo, Sukono and A. T. Bon, "Life Insurance Premiums Dwiguna Joint Life and Last Survivor with Makeham Law," in *Proc. 11th Annual Int. Conf. on Industrial Engineering and Operations Management Singapura*, pp. 3270-32279, March 7-11, 2021.

[13] I. G. A. G. Dwipayana, I. N. Widana and K. Sari, "Menentukan Formula Cadangan Premi Asuransi Jiwa Last Survivor Menggunakan Metode New Jersey," *E-Jurnal Matematika,* vol. 8, no. 4, pp. 264-268, November 2019.

[14] Hasriati, T. P. Nababan and I. Hasbiyati, "Reserve of Life Insurance Prospective Dwiguna Joint Life and Last Survivor with Gompertz Law," in *Proc. 2nd African Int. Conf. on Industrial Engineering and Operations Management Harare*, pp. 1815-1824, Dec. 7-10, 2020.

[15] N. L. Bowers, H. U. Geerber, J. C. Hickman, D. A. Jones and C. J. Nesbitt, Actuarial Mathematics, Schaumburg: The Society of Actuaries, 1997.

[16] L. J. Bain and M. Engelhardt, Introduction to Probability and Mathematical Statistics, Second Edition, california: Duxbury, 1992.

[17] P. Kumar, K. Kour and J. Kour, "Estimation of the probability density function of Lomax distribution," *Int. j. stat. econ,* vol. 19, no. 2, pp. 78-88, April 2018.

[18] D. C. M. Dickson, M. R. Hardy and H. R. Waters, Actuarial Mathematics for Life Contingent Risk, New York: Cambridge University Press, 2009.

[19] K. M. Ramacandran and C. P. Tsokos, Mathematical Statistics with Applications, California: Elsevier Academic Press, 2009.

[20] K. Atkinson and W. Han, Elementary Numerical Analysis, Third Edition, Hoboken: ohn Wiley & Sons, 2004.

[21] S. G. Kellison, The theory of interest, Homewood: Richard D. Irwin Inc, 1991.

[22] W. O. Menge and C. H. Fischer, The Mathematics of Life Insurance, Second Edition, New York: The Macmillan Company, 1965.