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# ANALYSIS OF THE COOPERATIVE LOTKA-VOLTERRA MODEL IN THE CASE OF TWO AUTOMOTIVE INDUSTRIES IN INDONESIA

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## ABSTRACT

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*The automotive industry is one of the main sectors contributing to the national economy. Companies in the automotive industry work together to increase productivity and win the market competition. This research aims to construct the cooperative interaction between two companies into the cooperative Lotka-Volterra model. The cooperative Lotka-Volterra model was analyzed for stability at the equilibrium point and bifurcation analysis was performed. Sales data for the Calya 1.2 G product from PT Toyota Motor Manufacturing Indonesia and sales data for the New Sigra 1.2 R MT product from PT Astra Daihatsu Motor are applied to the model. The results of the study show that there is a model that explains the cooperative interaction of the two companies, namely the cooperative Lotka-Volterra model. Four equilibrium points are obtained, with three equilibrium points being unstable and the fourth being stable with conditions. The Hopf bifurcation analysis of the model shows that no parameters cause a change in stability from initially stable to unstable. The data simulation shows that the cooperation of the two companies is mutually beneficial because it increases the number of sales and creates balanced market conditions.*



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## 1. INTRODUCTION

The automotive industry sector in Indonesia is currently one of the mainstay sectors that has a sizeable contribution to the national economy [1]. The contribution that the automotive industry can make makes companies more aggressive in making improvements in various ways to be able to increase efficiency and productivity to win the market competition. One of the things that are done is to cooperate with other automotive companies to support productivity. Two companies that cooperate expect benefits for both parties.

The collaborative interaction between the two well-known automotive industries in Indonesia has been carried out for a long time. The two companies are PT Toyota Motor Manufacturing Indonesia and PT Astra Daihatsu Motor, which have the same market share, namely in the vehicle industry. The emergence of twin products produced by the two companies is a form of cooperation carried out. Some of the twin products that sell well among the public are Calya 1.2 G and New Sigra 1.2 R MT. The cooperation carried out by the two companies expects benefits for both companies. The cooperation between the two industries can be analyzed using a mathematical model, namely the cooperative Lotka-Volterra model, to determine market conditions expected to benefit the two companies [2].

The Lotka-Volterra model has been studied by Lotka[3] and Volterra[4] since 1920. The Lotka-Volterra model is in the form of a system of nonlinear ordinary differential equations ([5], [6]). The Lotka-Volterra model describes the interaction between predator and prey. An increase in the number of predators will decrease the number of prey but an increase in the number of prey will increase the number of predators ([7], [8]). Predators can function as controllers of prey populations. Based on the Lotka-Volterra model, an equilibrium number of prey and predator populations is obtained [9]. The balance of the number of prey and predators in a system can change depending on the disturbance the system receives [10]. Equilibrium changes in the Lotka-Volterra model are analyzed using the bifurcation theory [11] to minimize the occurrence of equilibrium changes in the model.

This article discusses the interaction of two companies described by the Cooperative Lotka-Volterra model. The model is then analyzed for stability at the equilibrium point and a bifurcation analysis is performed to determine an equilibrium market condition on the sales data of the two companies.

## 2. RESEARCH METHODS

This research uses a literature study method by reviewing previous research from books, journals, or articles related to the Lotka-Volterra model, stability analysis at the equilibrium point, and bifurcation analysis. The steps taken in this research are to reconstruct the cooperative Lotka-Volterra model on the interaction of two automotive industries in Indonesia. At this stage, problem identification is carried out, determining the parameters and variables used, determining assumptions and problem boundaries, and determining the relationship between parameters and variables based on the assumptions formed. The next stage is to determine the equilibrium point of the cooperative Lotka-Volterra model, determine the type of stability at each equilibrium point, perform a bifurcation analysis on the cooperative Lotka-Volterra model, and apply the two companies' product sales data to the cooperative Lotka-Volterra model.

## 3. RESULTS AND DISCUSSION

In this chapter, the cooperative Lotka-Volterra model is reconstructed based on the interaction of two automotive companies, namely PT Toyota with its product Calya 1.2 G and PT Daihatsu with its product New Sigra 1.2 R MT. In this study, data on the number of sales of the two products( [12], [13]) was used as a population which was analyzed using the Lotka-Volterra model.

### 3.1 Model Construction

This research was conducted to analyze the interaction of cooperation carried out by two companies engaged in the automotive industry in Indonesia. The interaction between the two automotive companies was described by the Lotka-Volterra model. The assumption used is a momentary change in the number of sales of Calya and Sigra products proportional to the growth rate of the number of sales of these products [14]. These assumptions can be modeled as in the following equation.

$$\begin{aligned}\frac{dx}{dt} &= \alpha x \\ \frac{dy}{dt} &= \beta y\end{aligned} \tag{1}$$

where $\boldsymbol{x(t) = x}$ is the number of sales of Calya products at time $\boldsymbol{t}$, $\boldsymbol{y(t) = y}$ is the number of sales of Sigra products at time $\boldsymbol{t}$, $\boldsymbol{\alpha}$ is the growth rate of sales of Calya products, and $\boldsymbol{\beta}$ is the growth rate of sales of Sigra products.

Another assumption is added to **Equation (1)**, namely the limited availability of material, space, and manpower resources. This development uses a logistics model [15], which assumes that the higher the product sales growth, the more limited the available resources. Therefore, product sales growth is always limited to a certain value, which is called the carrying capacity or the optimal carrying capacity of the number of sales [16]. The optimal carrying capacity for the number of sales of Calya products is symbolized by $K_1$ while the optimal carrying capacity for the number of sales of Sigra products is symbolized by $K_2$. From these assumptions, a new model is obtained as in **Equation (2)** as follows.

$$\begin{aligned}\frac{dx}{dt} &= \alpha x\left(1 - \frac{x}{K_1}\right) \\ \frac{dy}{dt} &= \beta y\left(1 - \frac{y}{K_2}\right)\end{aligned} \tag{2}$$

The interaction of the two companies affects the number of product sales from the two companies. The collaboration was carried out in the form of the emergence of twin products from the two companies. The twin products have the same component specifications so that the two companies can exist in the market share. The two companies interact mutually beneficially so that the two companies mutually support each other's sales growth. The system of differential equations is obtained as follows.

$$\begin{aligned}\frac{dx}{dt} &= \alpha x\left(1 - \frac{x}{K_1} + \frac{\mu y}{K_2}\right) \\ \frac{dy}{dt} &= \beta y\left(1 - \frac{y}{K_2} + \frac{\delta x}{K_1}\right)\end{aligned} \tag{3}$$

where $\mu$ describes the interaction contact rate of $x$ and $y$ which affects $x$ while the parameter $\delta$ describes the contact rate of interaction of $x$ and $y$ which affects $y$. The value $x(0) \geq 0$, $y(0) \geq 0$, and $\alpha, \beta, K_1, K_2, \mu, \delta > 0$. **Equation (3)** obtained is a cooperative Lotka-Volterra model [2] based on the logistic model and taking into account the dynamic interaction of two products, namely Calya and Sigra. The system of **Equation (3)** is assigned an initial value

$$\begin{aligned}x(0) &= x_0, \\ y(0) &= y_0.\end{aligned}$$

The measurement model is shown in the following equation from [17]. Since $dx \approx \Delta x$, $\Delta x = x(t+1) - x(t)$, $dt \approx \Delta t = (t+1) - t = 1$, so

$$\frac{dx}{dt} \approx \Delta x = \gamma_{11} x - \gamma_{12} x^2 + \gamma_{13} xy. \tag{4}$$

From **Equation (4)**, it is found that $\gamma_{11} = \alpha$, $\gamma_{12} = \frac{\alpha}{K_1}$, and $\gamma_{13} = \frac{\alpha\mu}{K_2}$. In the same way, the measurement model for $y$ is obtained as follows.

$$\frac{dy}{dt} \approx \Delta y = \gamma_{21} y - \gamma_{22} y^2 + \gamma_{23} xy, \tag{5}$$

with $\gamma_{21} = \beta$, $\gamma_{22} = \frac{\beta}{K_2}$, and $\gamma_{23} = \frac{\beta\delta}{K_1}$. **Equation (4)** and **Equation (5)** are used to estimate the parameters and to determine the interaction effect of the two companies' products.

### 3.2 Stability Analysis

The equilibrium point shows that the instantaneous change in the number of sales of Calya and Sigra with time $t$ no longer increases so that the number of sales no longer changes. According to Williamson **[18]**, the equilibrium point is obtained when

$$\begin{aligned} f_1 &= \alpha x\left(1 - \frac{x}{K_1} + \frac{\mu y}{K_2}\right), \\ f_2 &= \beta y\left(1 - \frac{y}{K_2} + \frac{\delta x}{K_1}\right). \end{aligned} \tag{6}$$

By solving the system of **Equation (6)**, we get four solutions which are equilibrium points, namely $P_1(0,0)$, $P_2(0,K_2)$, $P_3(K_1,0)$, and $P_4\left(\frac{K_1(1+\mu)}{1-\mu\delta}, \frac{K_2(1+\delta)}{1-\mu\delta}\right)$. Stability analysis of system's equilibrium points was carried out by linearization at each equilibrium point using the Jacobian matrix **[18]**. The Jacobian matrix at the equilibrium point $(x_0, y_0)$ obtained is as follows

$$\begin{aligned} \mathrm{J}\boldsymbol{f}(x_0,y_0) &= \begin{pmatrix} \frac{\partial f_1}{\partial x}(x_0,y_0) & \frac{\partial f_1}{\partial y}(x_0,y_0) \\ \frac{\partial f_2}{\partial x}(x_0,y_0) & \frac{\partial f_2}{\partial y}(x_0,y_0) \end{pmatrix} \\ &= \begin{pmatrix} \alpha\left(1 - \frac{2x_0}{K_1} + \frac{\mu y_0}{K_2}\right) & \frac{\alpha\mu x_0}{K_2} \\ \frac{\beta\delta y_0}{K_1} & \beta\left(1 + \frac{\delta x_0}{K_1} - \frac{2y_0}{K_2}\right) \end{pmatrix}. \end{aligned} \tag{7}$$

The next step is to evaluate the Jacobian matrix at the equilibrium point. Based on Haberman's Theorem**[19]**, the equilibrium point is said to be stable if the real part of the eigenvalues in the linearization of the Jacobian matrix at the equilibrium point are negative. The results of the linearization of the differential equation at the equilibrium point are shown in **Table 1**.

**Table 1. Jacobian Matrix Eigenvalues and Types of Stability of the Equilibrium Point**

| Equilibrium Point | Eigen Values | Stability |
|---|---|---|
| $P_1(0,0)$ | $\lambda_1 = \alpha$<br>$\lambda_2 = \beta$ | Unstable |
| $P_2(0,K_2)$ | $\lambda_1 = -\beta$<br>$\lambda_2 = \alpha(1+\mu)$ | Unstable |
| $P_3(K_1,0)$ | $\lambda_1 = -\alpha$<br>$\lambda_2 = \beta(1+\delta)$ | Unstable |
| $P_4\left(\frac{K_1(1+\mu)}{1-\mu\delta}, \frac{K_2(1+\delta)}{1-\mu\delta}\right)$ | $\lambda_{1,2} = \frac{tr(\mathrm{J}\boldsymbol{f}(P_4)) \pm \sqrt{\left(tr(\mathrm{J}\boldsymbol{f}(P_4))\right)^2 - 4\det(\mathrm{J}\boldsymbol{f}(P_4))}}{2}$ | Stable if $\mu\delta < 1$ |

trace and determinant of the Jacobian matrix at the equilibrium point $P_4$ in **Table 1** are written as

$$tr(\mathrm{J}\boldsymbol{f}(P_4)) = \frac{-\alpha(1+\mu) - \beta(1+\delta)}{1-\mu\delta} \tag{8}$$

and

$$\det(\mathrm{J}\boldsymbol{f}(P_4)) = \frac{\alpha\beta(1+\mu)(1+\delta)}{1-\mu\delta}. \tag{9}$$

Furthermore, the phase plane at the equilibrium point is presented in **Figure 1.**

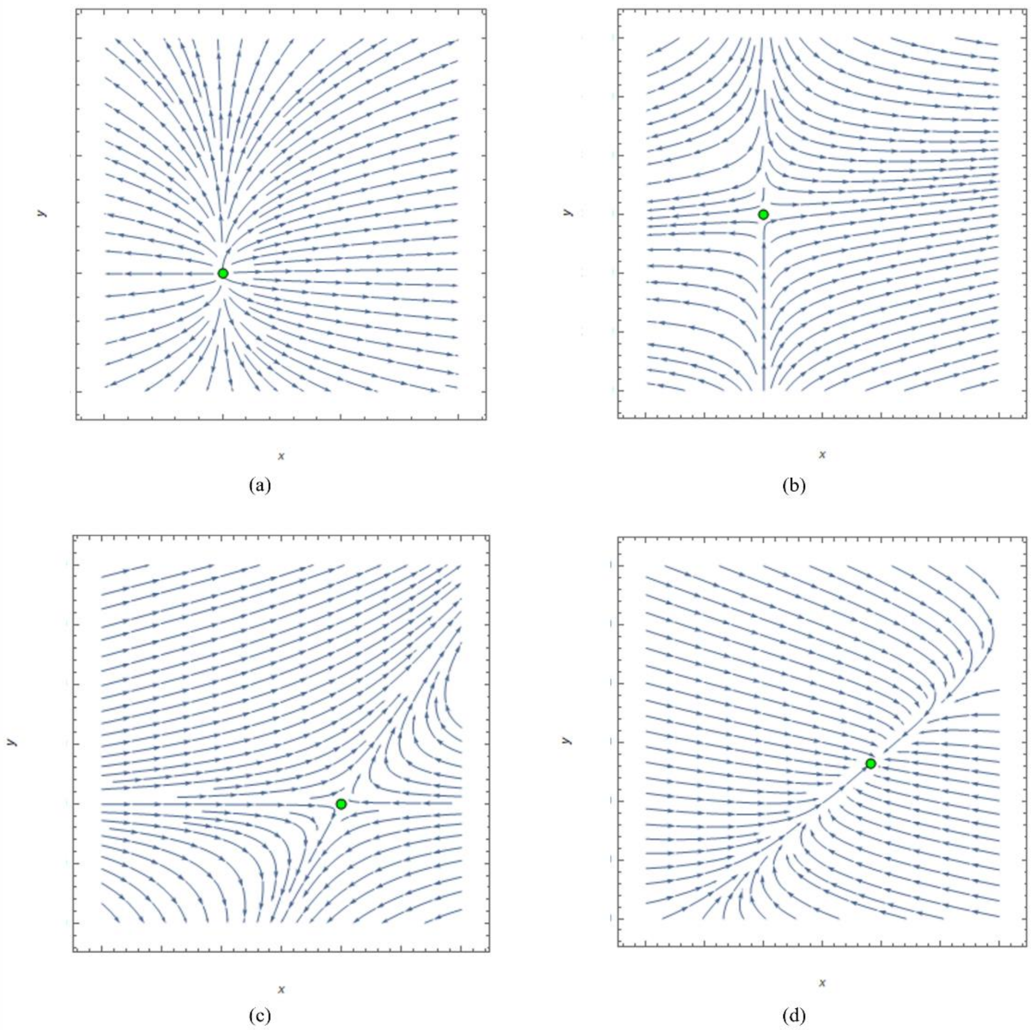


**Figure 1. Phase Plane at The Equilibrium Point (a) $P_1$, (b) $P_2$, (c) $P_3$, and (d) $P_4$**

The phase plane at the equilibrium point $P_1(0,0)$ is shown in **Figure 1**(a). The phase plane shows that the trajectory around the equilibrium point $P_1(0,0)$ is away from that point, so the stability is an unstable node. The phase planes of **Figure 1**(b) and **Figure 1**(c) show that the trajectory has two opposite directions, namely approaching and away from the fixed points $P_2(0,K_2)$ and $P_3(K_1,0)$. This trajectory causes the equilibrium point to be the unstable saddle point. **Figure 1**(d) shows that the trajectory around the equilibrium point $P_4\left(\frac{K_1(1+\mu)}{1-\mu\delta},\frac{K_2(1+\delta)}{1-\mu\delta}\right)$ is approaching that point so the stability is a stable node with the condition $\mu\delta < 1$.

### 3.3 Hopf Bifurcation Analysis

Research on bifurcation, especially Hopf bifurcation analysis, has been done a lot. Savitri **[20]** and Sundari**[21]** conducted research on the Hopf bifurcation analysis in the prey-predator model, while Wijeratne et al.**[22]** conducted a study on the Hopf bifurcation analysis in the Lotka-Volterra model in the economic market case. The stability analysis at the four equilibrium points that was carried out previously found that point $P_4$ is the only stable equilibrium point. Hopf bifurcation analysis is determined at the equilibrium point $P_4$. From the evaluation of the Jacobian matrix at the equilibrium point $P_4$ is obtained trace and determinant as written in **Equation (8)** and **Equation (9)**.

According to Kocak **[11]**, the first condition for the occurrence of a Hopf bifurcation is if the following conditions are met

$$tr\left(\mathrm{J}\boldsymbol{f}(P_4)\right) = 0. \tag{10}$$

Furthermore, the second condition is fulfilled under the following conditions

$$\det\left(\mathrm{J}\boldsymbol{f}(P_4)\right) > 0. \tag{11}$$

From the first condition, four different parameters are obtained, which might cause the Hopf bifurcation to occur. Parameter results are substituted in $\det\left(\mathrm{J}\boldsymbol{f}(P_4)\right)$ to obtain results as shown in **Table 2**.

**Table 2. Hopf Bifurcation Analysis at the $P_4$ Equilibrium Point**

| Equilibrium Point | Parameters | $\det\left(\mathrm{J}\boldsymbol{f}(P_4)\right)$ |
|---|---|---|
| $P_4\left(\frac{K_1(1+\mu)}{1-\mu\delta}, \frac{K_2(1+\delta)}{1-\mu\delta}\right)$ | $\alpha_0 = -\frac{\beta(1+\delta)}{1+\mu}$ | $-\frac{\beta^2(1+\delta)^2}{1-\mu\delta}$ |
| | $\beta_0 = -\frac{\alpha(1+\mu)}{1+\delta}$ | $-\frac{\alpha^2(1+\mu)^2}{1-\mu\delta}$ |
| | $\mu_0 = -\frac{\beta(1+\delta)}{\alpha} - 1$ | $-\frac{\alpha\beta^2(1+\delta)}{\alpha+\beta\delta}$ |
| | $\delta_0 = -\frac{\alpha(1+\mu)}{\beta} - 1$ | $-\frac{\alpha^2\beta(1+\mu)}{\beta+\alpha\mu}$ |

Based on the $\det\left(\mathrm{J}\boldsymbol{f}(P_4)\right)$ value in **Table 2**, it can be seen that the determinant is negative. This means that the value of the determinant does not comply with the conditions for the occurrence of a Hopf bifurcation as in **Equation (11)**. Therefore, the cooperative Lotka-Volterra model in **Equation (3)** does not occur a Hopf bifurcation at the equilibrium point $P_4$ because there are no parameter values that meet the requirements for a Hopf bifurcation.

At the equilibrium points $P_1(0,0)$, $P_2(0,K_2)$, and $P_3(K_1,0)$, the same analysis was carried out and it gave the result that the determinant is negative. It can be concluded that there is no Hopf bifurcation at all equilibrium points of the cooperative Lotka-Volterra model.

### 3.4 Numerical Experiment

In this section, the cooperative Lotka-Volterra model is applied to analyze the interaction of the two companies, namely the Calya 1.2 G product from PT Toyota Motor Manufacturing Indonesia and the New Sigra 1.2 R MT product from PT Astra Daihatsu Motor. The data used in this section is data on the number of sales of Calya and Sigra products for 19 months with an interval of one month. Data on the total sales of the two products is shown in **Table 3**.

**Table 3. Sales amount of Calya and New Sigra**

| Period | Sales Amount | | Period | Sales Amount | |
|---|---|---|---|---|---|
| | Calya | New Sigra | | Cancer | Stroke |
| Jun 2021 | 1318 | 823 | Apr 2022 | 2504 | 1164 |
| Jul 2021 | 1523 | 628 | May 2022 | 1023 | 893 |
| Aug 2021 | 2809 | 1360 | Jun 2022 | 2899 | 2282 |
| Sep 2021 | 3266 | 1722 | Jul 2022 | 2958 | 2175 |
| Oct 2021 | 1041 | 1563 | Aug 2022 | 1409 | 1630 |
| Nov 2021 | 2094 | 1453 | Sep 2022 | 3280 | 2539 |
| Dec 2021 | 2306 | 1722 | Oct 2022 | 3505 | 2171 |
| Jan 2022 | 624 | 2405 | Nov 2022 | 2119 | 2692 |
| Feb 2022 | 1839 | 1212 | Dec 2022 | 2602 | 2698 |
| Mar 2022 | 2635 | 1120 | | | |

***Data source***: *https://www.gaikindo.or.id/indonesian-automobile-industry-data/*

Parameter estimation with nonlinear regression using Minitab software. Nonlinear regression produces a value of $\gamma$ as in equation $\gamma$.

The value of $\gamma$ is substituted in the measurement model of **Equation (4)** and **Equation (5)** so that the estimated values of the parameters $\alpha$, $\beta$, $\mu$ and $\delta$ and the values of $K_1$ and $K_2$ are obtained as shown in **Table 4**.

**Table 4. Estimation of parameter values $\alpha$, $\beta$, $\mu$, $\delta$, $K_1$, and $K_2$**

| Parameter | Value Estimation |
|---|---|
| $\alpha$ | 0.505594 |
| $\beta$ | 0.209412 |
| $\mu$ | 0.319288 |
| $\delta$ | 0.411330 |
| $K_1$ | 3600 |
| $K_2$ | 2800 |

The calculation results show that the growth in the number of product sales for each Calya and Sigra product is positive. The interaction that occurred between Calya and Sigra caused the number of sales of Calya and Sigra products to increase. This means that the collaboration between the two companies, Toyota and Daihatsu, has increased the number of sales of these two products.

The stability at the equilibrium point $P_4(5467{,}4549)$ is fulfilled with the condition that is $\mu\delta < 1$, so that $\mu\delta = 0.373688 < 1$ is obtained. Therefore, the stability requirements at the equilibrium point $P_4(5467{,}4549)$ are fulfilled. This proves that the system in **Equation (3)** is stable at the equilibrium point $P_4(5467{,}4549)$.

The Lotka-Volterra model was solved using the Runge-Kutta numerical method of order 4. The solution was done by substituting the estimated parameter values and initial values into the Lotka-Volterra equation system. The initial values used are $x(0) = 1318$ and $y(0) = 823$ and the parameter values are in **Table 4**. The simulated graph of the number of sales of the two products at time $t$ in a month is shown in **Figure 2**.

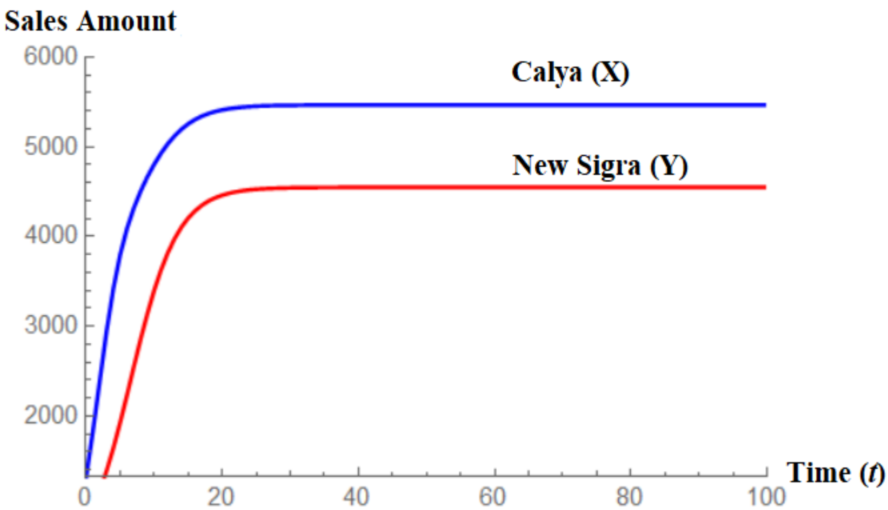


**Figure 2. Total sales of calya and New Sigra at Time $t$**

The simulation graph of Calya and Sigra's sales over time increases and reaches stability at $t = 38$ with Calya's sales of 5467 units, while Sigra's sales of 4549 units. This shows that the interaction or cooperation of the two companies has caused the number of sales of the two companies to increase until they reach stability. The stability of the number of sales shows the number of products that can be sold to create balanced market conditions.

## 4. CONCLUSIONS

Based on the results and discussion, it is concluded that the cooperative Lotka-Volterra model was constructed based on sales data for Calya and Sigra products in Indonesia. The model formed is a system of nonlinear ordinary differential equations. The Lotka-Volterra model has 4 equilibrium points, namely $P_1(0,0)$, $P_2(0,K_2)$, $P_3(K_1,0)$, and $P_4\left(\frac{K_1(1+\mu)}{1-\mu\delta},\frac{K_2(1+\delta)}{1-\mu\delta}\right)$. The equilibrium point $P_1$ is an unstable node, equilibrium points $P_2$ and $P_3$ are unstable saddle points, while point $P_4$ is a stable node with the condition $\mu\delta < 1$. The Hopf bifurcation did not occur in the cooperative Lotka-Volterra model used in this study, so the Hopf bifurcation analysis could not be performed in this study. In applying the data, it was found that the number of sales of Calya and Sigra increased over time and reached stability at $t = 38$ with the number of sales of Calya 5467 units, while the number of sales of Sigra was 4549 units. This shows that the interaction or cooperation of the two companies causes the number of sales to increase until it reaches stability and creates a balanced market condition.

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