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THE LEVI DECOMPOSITION OF THE LIE ALGEBRA $\mathbf{M_2}(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$

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ABSTRACT

The idea of the Lie algebra $\mathfrak{g}_2 \coloneqq M_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ is studied in this research. The decomposition between Levi sub-algebra and the radical can be used to define the finite dimensional Lie algebra. The Levi decomposition is the name for this type of decomposition. The goal of this study is to obtain a Levi decomposition of the Lie algebra $\mathfrak{g}_2$ of dimension 8. We compute its Levi sub-algebra and the radical of Lie algebra $\mathfrak{g}_2$ with respect to its basis to achieve this goal. We use literature studies on the Levi decomposition and Lie algebra in Dagli result to produce the radical and Levi sub-algebra. It has been shown that $\mathfrak{g}_2$ can be decomposed in the terms of the Levi sub-algebra and its radical. In this research, the result is obtained by direct computations and we obtained that the explicit formula of Levi decomposition of the affine Lie algebra $\mathfrak{g}_2$ whose basis is $G = \{x_i\}_{i=1}^{8}$ written by $\mathfrak{g}_2 = span\{x_5, x_6, x_7, x_8\} \rtimes span\{x_1, x_2, x_3, x_4\} = \mathrm{Rad}(\mathfrak{g}_2) \rtimes \mathfrak{s}$ with $\mathfrak{s}$ is the Levi sub-algebra of $\mathfrak{g}_2$.

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1. INTRODUCTION

A Lie algebra is a vector space over a field equipped by a bilinear map which is skew-symmetric and satisfying Jacobi identity. This bilinear map is called a Lie bracket [1]. The concept of Lie algebra has received a lot of attention and it can be applied in many areas of mathematics. One of them is Levi's decomposition research of a Lie algebra with 6 dimensions which had been done [2]. A Levi decomposition may be written as a semi-direct sum of Levi sub-algebra or a Lie sub-algebra and the radical as an solvable maximum ideal for any finite dimensional Lie algebra [3]. Let $\mathfrak{g}$ be a finite dimenisonal Lie algebra and let the Lie algebra $\mathfrak{g}$ be expressed in Levi decomposition form $\mathfrak{g} = \mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{g})$ where $\mathrm{Rad}(\mathfrak{g})$ is the solvable maximum ideal or a radical of a Lie algebra $\mathfrak{g}$ and $\mathfrak{s}$ is its Levi sub-algebra of $\mathfrak{g}$. The Lie algebra notion $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ where $\mathfrak{gl}_n(\mathbb{R})$ is the Lie algebra of a vector space of matrices of size $n \times n$ equipped with Lie brackets and $M_{n,p}(\mathbb{R})$ is a vector space of matrices of size $n \times p$ had been introduced in [4]. Furthermore, we can study more about Lie algebra in the following articles: ([5], [6], [7], [8]).

The Lie algebra $\mathfrak{g}$ whose index equals 0 is called Frobenius. An example of Frobenius Lie algebra is the Lie algebra $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ for cases $n = p = 3$ [9]. Frobenius Lie algebra have been the object of several investigations throughout the years. Frobenius Lie algebra classification has been carried out for Lie algebra of dimension 6 into 16 isomorphism classes [10]. The characteristics of principal elements on Frobenius Lie algebra one of them is cannot be nilpotent [11]. Frobenius Lie algebra with dimension ≤ 6 can constructed from non-commutative nilpotent Lie algebra with dimensions ≤ 4 [12]. Furthermore, the Lie algebra $M_{n,1}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ which is known as the affine Lie algebras is Frobenius Lie algebra of dimension $n(n+1)$[13]. Readers can see the notions of Frobenius Lie algebra in [14] and [15].

In this research, we study the real Lie algebra $M_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ which is denoted by by $\mathfrak{g}_2$. The elements of the Lie algebra $\mathfrak{g}_2 := \mathrm{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R})$ can be expressed in the form of matrices $A = \begin{pmatrix} K & L \\ 0 & 0 \end{pmatrix}$ where a matrix K is contained in $\mathfrak{gl}_2(\mathbb{R})$ and a matrix L is contained in $\mathrm{M}_2(\mathbb{R})$. Namely, we can take $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $L = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ such that the element $A \in \mathfrak{g}_2$ can be written of the form $A = \begin{pmatrix} a & b & w & x \\ c & d & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Indeed, we can see that the Lie algebra $\mathfrak{g}_2$ is contained in the Lie algebra $\mathfrak{gl}_4(\mathbb{R})$ of all 4×4 real matrices. In the other words, $\mathfrak{g}_2$ is a Lie sub-algebra of $\mathfrak{gl}_4(\mathbb{R})$. The research aims to decompose this Lie algebra $\mathfrak{g}_2$ using the Levi decomposition by construction its Levi sub-algebra and its radical.

The significance of this research is to give the role model in understanding the Levi decomposition in the term of matrix notations. This is why in this research it is taken for the case $n = 2$. Different from previous researches, in this paper we give more concrete computations and more direct proof. We believe that our obtained result can be applied for higher n cases in determining a Levi decomposition structure of a Lie algebra $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$.

The following paper is organized as follows: We discuss the research background, research motivation, and research purpose in the introduction section. For this work, we employed the literature review technique, as describe in the research method section. We give more concrete and comprehensive computations and more direct proof for getting a Levi sub-algebra and radical of $\mathfrak{g}_2$. The main result of this paper stated in Proposition 2 of the results and discussion section.

2. RESEARCH METHODS

Literature review was employed as a research method, with a focus on Lie algebra $\mathfrak{g}_2$ and Levi decomposition of Lie algebra [2]. To begin, given a Lie algebra $\mathfrak{g}_2$, it is shown that $\mathfrak{g}_2$ can be decomposed into its sub-algebra and radical. To determine the basis radical and Levi sub-algebra of Lie algebra in further detail,

1. Let $\mathfrak{g}$ be a Lie algebra with basis $G = \{x_1, x_2, \dots, x_n\}$. First, we find the product space $[\mathfrak{g}, \mathfrak{g}]$ and radical of $\mathfrak{g}$, denoted by $\mathrm{Rad}(\mathfrak{g})$, of Lie algebra $\mathfrak{g}$.

Step 1. Compute the set E of elements $[x_i, x_i]$ for all $1 \leq i \leq n$.
Step 2. Find the set $E = \{y_1, y_2, y_3, \ldots, y_m\}$ of a maximal linearly independent of G for obtain basis of the product space $[\mathfrak{g}, \mathfrak{g}]$
Step 3. Compute adjoint representation of x_i and y_j for $1 \leq i \leq n, 1 \leq j \leq m$.
Step 4. Compute basis of $\mathrm{Rad}(\mathfrak{g})$ by showing that $x_i \in \mathfrak{g}$ satisfies

$$Tr\left(\mathrm{ad}_{\mathfrak{g}} x_i \cdot \mathrm{ad}_{\mathfrak{g}} y_j\right) = 0 \tag{1}$$

for all $y_i \in E$.
Step 5. We obtain a basis of radical $\mathrm{Rad}(\mathfrak{g})$ i.e $\{r_1, r_2, \ldots, r_k\}$.

2. If we get the result of the radical is abelian i.e. $\left[r_i, r_j\right] = 0$ for all $1 \leq i, j \leq k$, then we can obtain the Levi sub-algebra of $\mathfrak{g}$.

Step 1. Find a complement basis $\{x_1, x_2, \ldots, x_n\}$ from $\mathrm{Rad}(\mathfrak{g})$ and we denote by $\mathfrak{h}$
Step 2. To obtain the basis of Levi sub-algebra by showing $\mathbf{Rad}(\boldsymbol{\mathfrak{h}}) = \{\mathbf{0}\}$ which $\boldsymbol{\mathfrak{h}}$ is a semisimple Lie algebra.

3. RESULTS AND DISCUSSION

Before we get into the discussion, we shall recall some basic concepts which is important in our result and discussion.

Definition 1 [3]. Let $\mathfrak{g}$ be a real vector space and $[.,.]: \mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto [x, y] \in \mathfrak{g}$ be a bilinear form. The bilinear form $[.,.]$ is called a Lie bracket for $\mathfrak{g}$ if the following conditions are fulfilled:

1. $[x, y] = -[y, x]; \forall\, x, y \in \mathfrak{g}$
2. $\left[x, [y, z]\right] + \left[y, [z, x]\right] + \left[z, [x, y]\right] = 0; \forall\, x, y, z \in \mathfrak{g}$.

A Lie algebra is the vector space $\mathfrak{g}$ that is equipped with Lie bracket.

Definition 2 [3]. Let $\mathfrak{h}$ be a linear subspace of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ then $\mathfrak{h}$ is said to be a Lie sub-algebra and we denote it by $\mathfrak{h} < \mathfrak{g}$. We call $\mathfrak{h}$ an ideal of $\mathfrak{g}$ if we have $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, and then denote by $\mathfrak{h} \trianglelefteq \mathfrak{g}$.

Definition 3 [3]. Let $\mathfrak{g}$ be a Lie algebra. The derived series of $\mathfrak{g}$ is defined by

$$D^0(\mathfrak{g}) = \mathfrak{g} \text{ and } D^n(\mathfrak{g}) = [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]\ \forall\, n \in \mathbb{N}. \tag{2}$$

The Lie algebra $\mathfrak{g}$ is called solvable if there exist an $n \in \mathbb{N}$ with $D^n(\mathfrak{g}) = \{0\}$.

Example 1. The familiar examples of Lie algebras are matrix Lie algebras: $\mathfrak{gl}(n, \mathbb{R}), \mathfrak{D}(n), \mathfrak{so}(n)$, Heisenberg Lie algebra $\mathfrak{h}_n$ of dimension $2n + 1$.

Remark 1. In every finite-dimensional Lie algebra $\mathfrak{g}$, there is a maximal solvable ideal. This ideal is called the radical of $\mathfrak{g}$ and we denoted by $Rad(\mathfrak{g})$.

Theorem 1 [2]. *Let V be vector space over a field $\mathbb{F}$ and let $\mathfrak{g}$ be a sub-algebra of $\mathfrak{gl}(V)$, the Lie algebra $\mathfrak{g}$ is called solvable if $Tr(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proposition 1 [2]. *Let $\mathfrak{g}$ be Lie algebra over a field $\mathbb{F}$, then*

$$Rad(\mathfrak{g}) = \{x \in \mathfrak{g} \mid Tr(\mathrm{ad}\, x \cdot \mathrm{ad}\, y) = 0\} \tag{3}$$

for all $y \in [\mathfrak{g}, \mathfrak{g}]$.

Definition 4 [3]. Let $\mathfrak{g}$ be a Lie algebra. Then the Lie algebra $\mathfrak{g}$ is called semisimple, if its radical is trivial, namely $\mathrm{Rad}(\mathfrak{g}) = \{0\}$. The Lie algebra $\mathfrak{g}$ is called simple if it is not abelian and contains no ideal other than $\mathfrak{g}$ and $\{0\}$.

Definition 5 [1]. Let V be a space vector. A linear map $\rho: V \to V$ is called an endomorphism on V, if the following condition satisfied:

1. $\rho(x + y) = \rho(x) + \rho(y)$

2. $\rho(xy) = (\rho(x))y = x(\rho(y))$

for all $x, y \in V$. $End(V)$ denotes the set of all endomorphism on V.

Furthermore, the endomorphism $End(V)$ equipped by Lie bracket $[x, y] = xy - yx$, for all $x, y \in End(V)$ is Lie algebra and it is said to be a general linear algebra, we denote it by $\mathfrak{gl}(V)$.

Definition 6 [3]. Let $\mathfrak{g}$ be a Lie algebra and x be an arbitrary element of $\mathfrak{g}$. The linear map ad: $\mathfrak{g} \to \mathfrak{g}$ defined by

$$\text{ad}\, x : \mathfrak{g} \ni y \mapsto \text{ad}\, x(y) = [x, y] \in \mathfrak{g} \tag{4}$$

is a derivation of $\mathfrak{g}$. The map ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is called the adjoint representation.

We also go over some of the basic notations used in Levi decomposition. A finite dimensional Lie algebra can be written as the semi direct sum between the Levi sub-algebra and the radical, according to Levi's theorem.

Theorem 2 [2]. *Let $\mathfrak{g}$ be a Lie algebra. If Lie algebra $\mathfrak{g}$ be non solvable Lie algebra, then $\mathfrak{g}/Rad(\mathfrak{g})$ is a semisimple Lie sub-algebra.*

Theorem 3 [2]. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. If $\mathfrak{g}$ is not solvable, there exist a semisimple sub-algebra $\mathfrak{s}$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{s} \oplus Rad(\mathfrak{g}) \tag{5}$$

In this decomposition, then $\mathfrak{s} \cong \mathfrak{g}/\text{Rad}(\mathfrak{g})$ and we have the following commutation relations:

$$[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}, \quad [\mathfrak{s}, Rad(\mathfrak{g})] \subseteq Rad(\mathfrak{g}), \quad [Rad(\mathfrak{g}), Rad(\mathfrak{g})] \subseteq Rad(\mathfrak{g}) \tag{6}$$

Example 2. Another example of Lie algebra is $\mathfrak{g}_2$, whose standard basis is represented in the following formulas

$$G = \left\{ x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \tag{7}$$

Moreover, to obtain all Lie brackets of $\mathfrak{g}_2$, we compute them by $[S, T] = ST - TS$ for all matrices $S, T \in \mathfrak{g}_2$. Then all non-zero Lie brackets of the Lie algebra $\mathfrak{g}_2$ can be written in the following forms:

$$\begin{array}{llll} [x_1, x_2] = x_2 & [x_1, x_3] = -x_3 & [x_1, x_5] = x_5 & [x_1, x_6] = x_6 \\ [x_2, x_3] = x_1 - x_4 & [x_2, x_4] = x_2 & [x_2, x_7] = x_5 & [x_2, x_8] = x_6 \\ [x_3, x_4] = -x_3 & [x_3, x_5] = x_7 & [x_3, x_6] = x_8 & [x_4, x_7] = x_7 \\ [x_4, x_8] = x_8 & & & \end{array} \tag{8}$$

The following proposition is our main result, which we shall establish in this section of the discussion

Proposition 2. *Let $\mathfrak{g}_2$ be the affine Lie algebra and $G = \{x_i\}_{i=1}^{8}$ be a basis for $\mathfrak{g}_2$. The non-zero Lie brackets for $\mathfrak{g}_2$ are given in the* **Equation (8)**. *Then the Lie algebra $\mathfrak{g}_2$ can be expressed in the Levi decomposition of the form $\mathfrak{g}_2 = Rad(\mathfrak{g}_2) \rtimes \mathfrak{s}$ or*

$$\mathfrak{g}_2 = span\{x_5, x_6, x_7, x_8\} \rtimes span\{x_1, x_2, x_3, x_4\}. \tag{9}$$

Proof. Firstly, we find the maximal linearly independent set in the structure matrix such that basis $E = \{y_1 = x_1 - x_4, y_2 = x_2, y_3 = x_3, y_4 = x_5, y_5 = x_6, y_6 = x_7, y_7 = x_8\}$ of the product space $[\mathfrak{g}_2, \mathfrak{g}_2]$. Next, we obtain ad x_i and ad y_j for $1 \le i \le 8, 1 \le j \le 7$,

$$\text{ad } x_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \text{ad } x_2 = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ -1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$\text{ad } x_3 = \begin{pmatrix} 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&-1&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \text{ad } x_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$\text{ad } x_5 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \text{ad } x_6 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix},$$

$$\text{ad } x_7 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \text{ad } x_8 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{pmatrix},$$

$$\text{ad } y_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 0&0&-2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1 \end{pmatrix}. \qquad (10)$$

Furthermore, to obtain the radical of $\mathfrak{g}_2$ find $x_i \in \mathfrak{g}_2$ which $Tr\left(ad\ x_i \cdot ad\ y_j\right) = 0$ for all $y_j \in E$.
Compute $Tr\left(\text{ad } x_i \cdot \text{ad } y_j\right)$ for $i = 1$ and $1 \leq j \leq 7$.

$$\begin{aligned} Tr\ (\text{ad } x_1 \cdot \text{ad } y_1) &= 6\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_2) &= 0\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_3) &= 0\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_4) &= 0\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_5) &= 0\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_6) &= 0\\ Tr\ (\text{ad } x_1 \cdot \text{ad } y_7) &= 0 \end{aligned} \qquad (11)$$

We have $Tr\ (\text{ad } x_1 \cdot \text{ad } y_1) = 6 \neq 0$ then $x_1 \notin \text{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\text{ad } x_i \cdot \text{ad } y_j\right)$ for $i = 2$ and $1 \leq j \leq 7$.

$$\begin{aligned} Tr\ (\text{ad } x_2 \cdot \text{ad } y_1) &= 2\\ Tr\ (\text{ad } x_2 \cdot \text{ad } y_2) &= 0\\ Tr\ (\text{ad } x_2 \cdot \text{ad } y_3) &= 6 \end{aligned}$$

$$Tr\ (\mathrm{ad}\ x_2 \cdot \mathrm{ad}\ y_4) = 0 \quad (12)$$
$$Tr\ (\mathrm{ad}\ x_2 \cdot \mathrm{ad}\ y_5) = 0$$
$$Tr\ (\mathrm{ad}\ x_2 \cdot \mathrm{ad}\ y_6) = 0$$
$$Tr\ (\mathrm{ad}\ x_2 \cdot \mathrm{ad}\ y_7) = 0$$

We have $Tr\ (\mathrm{ad}\ x_2 \cdot \mathrm{ad}\ y_i) \neq 0$ for $i = 1,3$ then $x_2 \notin \mathrm{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\mathrm{ad}\ x_i \cdot \mathrm{ad}\ y_j\right)$ for $i = 3$ and $1 \leq j \leq 7$.

$$\begin{gathered} Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_1) = 0 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_2) = 6 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_3) = 0 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_4) = 0 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_5) = 0 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_6) = 0 \\ Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_7) = 0 \end{gathered} \quad (13)$$

We have $Tr\ (\mathrm{ad}\ x_3 \cdot \mathrm{ad}\ y_2) = 6 \neq 0$ then $x_3 \notin \mathrm{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\mathrm{ad}\ x_i \cdot \mathrm{ad}\ y_j\right)$ for $i = 4$ and $1 \leq j \leq 7$.

$$\begin{gathered} Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_1) = -2 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_2) = 0 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_3) = 0 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_4) = 0 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_5) = 0 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_6) = 0 \\ Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_7) = 0 \end{gathered} \quad (14)$$

We have $Tr\ (\mathrm{ad}\ x_4 \cdot \mathrm{ad}\ y_1) = -2 \neq 0$ then $x_4 \notin \mathrm{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\mathrm{ad}\ x_i \cdot \mathrm{ad}\ y_j\right)$ for $i = 5$ and $1 \leq j \leq 7$.

$$\begin{gathered} Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_1) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_2) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_3) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_4) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_5) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_6) = 0 \\ Tr\ (\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_7) = 0 \end{gathered} \quad (15)$$

We have $Tr\left(\mathrm{ad}\ x_5 \cdot \mathrm{ad}\ y_j\right) = 0$ for $1 \leq j \leq 7$ then $x_5 \in \mathrm{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\mathrm{ad}\ x_i \cdot \mathrm{ad}\ y_j\right)$ for $i = 6$ and $1 \leq j \leq 7$.

$$\begin{gathered} Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_1) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_2) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_3) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_4) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_5) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_6) = 0 \\ Tr\ (\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_7) = 0 \end{gathered} \quad (16)$$

We have $Tr\left(\mathrm{ad}\ x_6 \cdot \mathrm{ad}\ y_j\right) = 0$ for $1 \leq j \leq 7$ then $x_6 \in \mathrm{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\mathrm{ad}\ x_i \cdot \mathrm{ad}\ y_j\right)$ for $i = 7$ and $1 \leq j \leq 7$.

$$\begin{gathered} Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_1) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_2) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_3) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_4) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_5) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_6) = 0 \\ Tr\ (\mathrm{ad}\ x_7 \cdot \mathrm{ad}\ y_7) = 0 \end{gathered} \quad (17)$$

We have $Tr\left(\text{ad } x_7 \cdot \text{ad } y_j\right) = 0$ for $1 \le j \le 7$ then $x_7 \in \text{Rad}(\mathfrak{g}_2)$.
Compute $Tr\left(\text{ad } x_i \cdot \text{ad } y_j\right)$ for $i = 8$ and $1 \le j \le 7$.

$$\begin{aligned} Tr\,(\text{ad } x_8 \cdot \text{ad } y_1) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_2) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_3) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_4) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_5) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_6) &= 0 \\ Tr\,(\text{ad } x_8 \cdot \text{ad } y_7) &= 0 \end{aligned} \tag{18}$$

We have $Tr\left(\text{ad } x_8 \cdot \text{ad } y_j\right) = 0$ for $1 \le j \le 7$ then $x_8 \in \text{Rad}(\mathfrak{g}_2)$.

Based on the calculation results above, we find the radical of $\mathfrak{g}_2$ is written follows

$$Rad(\mathfrak{g}_2) = span\{x_5, x_6, x_7, x_8\} = span\{r_1, r_2, r_3, r_4\}. \tag{19}$$

Next, we can calculate basis Levi sub-algebra of $\mathfrak{g}_2$. After we have radical of $\mathfrak{g}_2$, we know that $Rad(\mathfrak{g}_2)$ is abelian because $\left[r_i, r_j\right] = 0$ for all $1 \le i, j \le 4$. The complement on $\mathfrak{g}_2$ respect to $Rad(\mathfrak{g}_2)$ spanned by $\{x_1, x_2, x_3, x_4\}$. We set $\mathfrak{h}$ Levi sub-algebra spanned by $\{x_1, x_2, x_3, x_4\}$ and we have its brackets as follows

$$[x_1, x_2] = x_2,\ [x_1, x_3] = -x_3,\ [x_2, x_3] = x_1 - x_4,\ [x_2, x_4] = x_2, [x_3, x_4] = -x_3 \tag{20}$$

Next, we prove that $\mathfrak{h}$ semisimple Lie algebra by showing that there not exist $x_i \in \mathfrak{h}$ for $i = 1,2,3,4$ which $Tr\left(ad\ x_i \cdot ad\ y_j\right) = 0$ for all $1 \le j \le 4$ such that $Rad(\mathfrak{h}) = \{0\}$. From **Equation (11)** to **Equation (14)** we obtain that $\{x_1, x_2, x_3, x_4\} \notin Rad(\mathfrak{h})$ then $Rad(\mathfrak{h}) = \{0\}$. Final, we find the Levi sub-algebra of $\mathfrak{g}_2$ is

$$\mathfrak{h} = \text{span}\{x_1, x_2, x_3, x_4\}.$$

Furthermore, it is obtained that following conditions are satisfied

$$[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h},$$

$$[\mathfrak{h}, \text{Rad}(\mathfrak{g}_2)] \subseteq \text{Rad}(\mathfrak{g}_2),$$

$$[\text{Rad}(\mathfrak{g}_2), \text{Rad}(\mathfrak{g}_2)] \subseteq \text{Rad}(\mathfrak{g}_2).$$

■

Example 3. Let $\mathfrak{g}_1$ be a first type of four-dimensional Lie algebra in **[10]**. The Levi decomposition of $\mathfrak{g}_1$ can be written as $\mathfrak{g}_1 = \langle e_1, e_2, e_3\rangle \rtimes \langle e_4\rangle$. Furthermore, we see that the Heisenberg Lie algebra $\mathfrak{h}_3 = \langle x_1\ , x_2, x_3\rangle$ of dimension 3 with $[x_1, x_2] = x_3$ has the Levi decomposition $\mathfrak{h}_3 = \langle x_2, x_3\rangle \rtimes \langle x_1\rangle$.

3. CONCLUSIONS

We concluded that the Levi decomposition of the Lie algebra $\mathfrak{g}_2 = \text{M}_2(\mathbb{R}) \rtimes \mathfrak{gl}_2(\mathbb{R}) = \text{span}\{x_i\}_{i=1}^{8}$ is written as $\text{Rad}(\mathfrak{g}_2) \rtimes \mathfrak{s}$ where $\text{Rad}(\mathfrak{g}_2) = \text{span}\{x_5, x_6, x_7, x_8\}$ and $\mathfrak{s} = \text{span}\{x_1, x_2, x_3, x_4\}$. This result is written in **Proposition 2** and it is realized in **Equation (9)** explicitly. For further research, Levi decomposition of the general formula of the Lie algebra $M_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ is interesting to be investigated.

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