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THE BENEFITS OF FAMILY ANNUITY CALCULATION WITH VINE'S COPULA AND FUZZY INTEREST RATE

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ABSTRACT

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One example of a multiple life annuity product (covering more than one person) is a reversionary annuity, which is a life annuity product for two or more annuitants whose annuity payments will begin after one of the annuitants specified in the contract dies first until the other annuitant also dies. This type of annuity is modified into a family annuity consisting of husband, wife, and child. The marginal distribution is constructed from a combined model of several mortality models such as Heligman-Pollard, Costakis, and Kannisto-Makeham models to capture mortality at young and old ages.This study takes this dependency into account when modeling the joint distribution of remaining life expectancy between the parties. The joint distribution of remaining lifetime between annuitants is modeled with a Vine's copula constructed from the marginal distribution of each annuitant. This research also takes account the actuarial margin rate using BI-7-day (reverse) repo rate data estimated with fuzzy sets. The annuity benefits calculation is assumed with some Kendall's tau (τ) values. The result shows the value of annuity benefits increases as the value of τ increases.

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1. INTRODUCTION

One example of a multiple life annuity product is a reversionary annuity, which is a life annuity product for two/more annuitants whose annuity payments will begin after one of the annuitants specified in the contract dies first until the other annuitant also dies [1]. Reversionary annuity products can be developed into family annuity products consisting of husband, wife, and child. The determination of the annuity greatly affects the amount of the pure premium that must be paid by the insured. The calculation of pure premiums for multiple lives requires a marginal and joint distribution of their lifetimes [2].

In more recent studies, it has alternatively captured old-age mortality rates through the proposition of multi-factor exponential models based on estimating mortality measures with Laguerre functions [3]. These models cannot capture the unobserved heterogeneity of individuals. There is a Gamma-Gompertz model with a death rate function that can overcome the delay and capture the unobserved heterogeneity in individuals formed by the frailty proportional hazard model [4]. However, these models cannot capture mortality in infancy, childhood, and young adolescence well, so a combined model of several mortality models such as Heligman-Pollard, Kostaki, and Kannisto-Makeham models to capture mortality at old ages.

In premium calculations assuming dependence between the insured parties, the copula model is the most widely used. The use of the Archimedean Copula family is more widely used than the Eliptical Copula family which is less able to capture asymmetric data shapes in joint life span modeling [5]. Furthermore, studies on joint life annuities from survey data in Ghana [6]. The dependency pattern of more than 2 random variables using one Copula cannot capture the dependency between random variables specifically. Therefore, Vine's Copula can be used as an alternative to modeling the dependency [7].

One of the factors that determine the contribution price is the actuarial margin level which is taken from the interest rate or yield. Determination of the APV can also use Fuzzy interest rates. Life insurance issues such as calculating the price of life insurance policies, life insurance portfolios, life contingencies, life actuarial obligations, and life annuities can use Fuzzy sets [8]. The use of Fuzzy interest rates causes investment gains and surplus processes in the form of intervals [9]. Based on this background, this study will apply the vine's copula to model the joint distribution of the insured parties which is constructed from the marginal distribution of future lifetimes. Three insured parties were chosen from each party, namely husband, wife, and child. In this study, the marginal distribution of each insured party was constructed using the 2019 Indonesian Mortality Table (TMI IV). Furthermore, the actuarial margin rate uses BI-7-day data which is estimated using the Fuzzy interest rate. Furthermore, based on the joint distribution, it can also be calculated the value of benefit from an insurance product, namely a family annuity.

2. RESEARCH METHODS

2.1 Mortality Distribution

The distribution of future lifetime, $T(x)$, can be calculated if the mortality rate, $\mu(y)$, for each $y > x$ is known. The distribution of the remaining lifetime obtained by assuming a mathematical function for the death rate is called the mortality distribution. The empirical survival function of the discrete random variable T_x is written as follows [2],

$$S_{T(x)}(t) = \ {}_t p_x = \prod_{i=0}^{t-1} p_{x+i} = \prod_{i=0}^{t-1} (1 - q_{x+i}) \tag{1}$$

There are several mortality distributions used to model individual mortality rates as follows,

1. The Heligman-Pollard Model can be expressed as follows [10],

$$\frac{q_x}{p_x} = A_h^{(x+B_h)^{C_h}} + D_h \exp\left[-E_h \left(\ln \frac{x}{F_h}\right)^2\right] + G_h H_h^x \tag{2}$$

with parameter $A_h, B_h, C_h, D_h, E_h, F_h, G_h, H_h > 0$ dan $x \geq x_0$.

2. The Kostaki Model is a modification of the Heligman-Pollard model in which the parameter E is broken down into two parameters, E_1 and E_2, which are expressed as follows [11],

$$\frac{q_x}{p_x} = \begin{cases} A_k^{(x+B_k)^{C_k}} + D_k \exp\left[-E_{k1}\left(\ln\frac{x}{F_k}\right)^2\right] + G_k H_k^x, & x \leq F_k \\ A_k^{(x+B_k)^{C_k}} + D_k \exp\left[-E_{k1}\left(\ln\frac{x}{F_k}\right)^2\right] + G_k H_k^x, & x > F_k \end{cases} \tag{3}$$

3. The Kannisto-Makeham Model is best used for ages over 80 years old. The survival function of the Kannisto-Makeham model is [12],

$${}_tp_x = \exp\left(-C_m t - \frac{1}{B_m}\ln\left[\frac{1 + A_m \exp[B_m(x - x_0 + t)]}{1 + A_m \exp[B_m(x - x_0)]}\right]\right), \quad x \geq x_0 \tag{4}$$

Estimation of parameters in the above models by minimizing a loss function (LF). The LF used the log-likelihood function, which is defined as follows,

$$LF = \ln\left(\frac{\hat{o}}{o}\right)^2 \tag{5}$$

with $\hat{o}$ is the estimated value and o is the observed value [13].

2.2 Vine's Copula

The joint distribution of the insured parties will be modelled with the assumption of dependency using copula. This joint distribution model uses three values of Kendall's tau (τ) as follows:

$$\tau = \begin{cases} 0.25; & \text{weak} \\ 0.50; & \text{moderate} \\ 0.75; & \text{strong} \end{cases}$$

This joint distribution will describe a more complex dependency relationship. The copula model links the univariate marginal cumulative distribution function to a multivariate cumulative distribution [14]. This is stated in Sklar's Theorem as follows;

Theorem 1. *Let H is a joint distribution function with marginal distribution $F_1, F_2, \dots, F_d$, then there is a copula C such that for every $x_1, x_2, \dots, x_d \in \mathbb{R}$ holds,*

$$H_{X_1,X_2}(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big) = C_{U_1,U_2}(u_1, u_2)$$

with $u_1 = F_1(x_1)$, $u_2 = F_2(x_2)$.

It is clear that, $c(u_1, u_2) = \frac{\partial^2 C_{U_1,U_2}(u_1,u_2)}{\partial u_1 \partial u_2}$. One example of a copula family is the Archimedian copula in which there are several copulas such as Frank, Clayton Gumbel and Joe with different characteristics of the $C_{U_1,U_2}(u_1, u_2)$ function.

The concept of Vine's copula is to decompose the multivariate copula function into several bivariate copula functions [7] . Vine's Copula provides a more flexible way of constructing the joint distribution of multivariate variables used for at least three random variables for which the joint distribution function is to be known. In this study, the canonical vine copula (C-Vine) will be used with the density function, namely,

$$f(x_1, x_2, x_3) = c_{1,2}\big(F_1(x_1), F_2(x_2)\big) \times c_{1,3}\big(F_1(x_1), F_3(x_3)\big) \times c_{2,3|1}\big(F_2(x_2|x_1), F_3(x_3|x_1)\big) \times f_1(x_1) f_2(x_2) f_3(x_3) \tag{6}$$

with $F_2(x_2|x_1) = c_{1,2}\big(F(x_2), F(x_1)\big), F_3(x_3|x_1) = c_{1,3}(F(x_3), F(x_3))$.

2.3 Fuzzy Interest Rate

Fuzzification is the process of converting crisp values to fuzzy values. Values in the field are expressed in the form of Fuzzy data which has two aspects, namely the Fuzzy set with its membership value. Examples of fuzzification are triangular fuzzy numbers and trapezoidal fuzzy numbers.

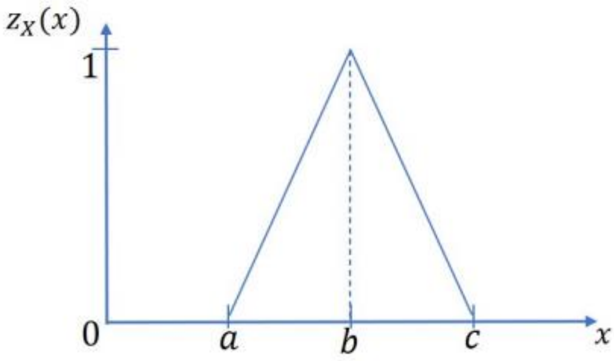


Figure 2. Triangular Fuzzy Number Graph

On **Figure 2**, the fuzzy triangular membership function of the Fuzzy number in the interval $[a, c]$ which is determined by three values (a, b, c) is defined where (a, b, c) are the lowest, the trusted, and the highest value respectively from the data. In triangular fuzzy numbers, the b value used is usually the average data **[15]**. Then, given that $\alpha - cut$ is the threshold level that changes the Fuzzy set to crisp. The process of converting Fuzzy sets to crisp is called defuzzification. The $\alpha - cut$ of the Fuzzy set X is defined as follows,

$$^{\alpha}X = \{x \in U : z_X(x) \geq \alpha\}, \qquad \alpha \in [0,1] \tag{7}$$

with $^{\alpha}X$ denotes an interval that contains all x that have a membership level greater than or equal to α **[16]**. Let $\tilde{\imath}$ represents Fuzzy interest rate with $\tilde{\imath} = (a, b, c)$, $a < b < c$. Fuzzy number can be defined uniquely with an interval by determining the $\alpha - cut$ value of $\tilde{\imath}$, namely:

$$^{\alpha}\tilde{\imath} = [a + (b - a)\alpha, c - (c - b)\alpha] = [\underline{i}(\alpha), \bar{i}(\alpha)], \qquad \alpha \in [0,1] \tag{8}$$

with $\underline{\boldsymbol{i}}(\boldsymbol{\alpha})$ and $\bar{\boldsymbol{i}}(\boldsymbol{\alpha})$ respectively are the lower and upper limits of $^{\boldsymbol{\alpha}}\tilde{\boldsymbol{\imath}}$ at a certain value of $\boldsymbol{\alpha}$ **[17]**.

2.4 Family Annuity

Life annuity is an annuity that is paid for life or for a certain period. In this study, the annuity is used for life and payments during life annuity. The annuity paid at the beginning and at the end of the period paid in 1 unit is stated by **[2]**,

$$a_{x_1 x_2 \dots x_n} = \sum_{t=1}^{\omega - \text{maks}\{x_1, x_2, \dots, x_n\}} v^t \;\; {}_tp_{x_1 x_1 x_2 \dots x_n} \tag{9}$$

with ${}_tp_{x_1 x_1 x_2 \dots x_n}$ is probability …. and ω is …. $(\omega = 112)$

The family annuity referred to here is a lifetime annuity where payments are made if the insured is still alive, payments can be made at the beginning or end of the policy period. The terms of the APV of this annuity are as follows:

a. Old Age Guarantee (OAG)

   The OAG benefits are paid to the insured every month according to the annuity payment date stated in the policy. This benefit will end after the insured dies.

b. Widow Guarantee (WG)

   The WG benefit is paid to the widower at p_1 of the OAG benefit every month, starting in the month following the last OAG payment. This benefit will end upon the death of the widower.

c. Child Guarantee (CG)

   The CG benefit is paid to the child at p_2 of the CG benefit every month, starting in the month following the last WG payment. This benefit ends when the child reaches the age of ω or dies before reaching the age of ω.

Suppose a family annuity contract with the insured parties being husband (x_1), wife (x_2), and child (x_3), then the APV of family annuity is,

$$\begin{aligned} APV(Y) &= a_{x_1x_2} + p_1\left(a_{x_1|x_2x_3} + a_{x_2|x_1x_3}\right) + p_2p_1 a_{\overline{x_1x_2}|x_3} \\ &= a_{x_1x_2} + p_1\left(a_{x_2x_3} - a_{x_1} + a_{x_1x_3} - a_{x_2}\right) + p_2p_1\left(a_{x_3} - a_{\overline{x_1x_2}:x_3}\right) \\ &= a_{x_1x_2} + p_1\left(a_{x_2x_3} - a_{x_1} + a_{x_1x_3} - a_{x_2}\right) + p_2p_1\left(a_{x_3} - \left(a_{x_1x_3} + a_{x_2x_3} - a_{x_1x_2x_3}\right)\right) \end{aligned} \quad (10)$$

In this section, we will determine the value of the annuity benefit obtained by the annuitant if pay a contribution of KT with an additional cost proportion $(0 < AC \leq 1)$. Additional cost consists of acquisition costs, general and administrative costs, policy maintenance costs, and margin. policy, and margin. From this proportion, the pure premium is obtained, which is formulated as follows,

$$\mathcal{P} = (1 - AC)KT \quad (11)$$

Then, we can get the benefit of family annuity as follow,

$$Benefit = \frac{\mathcal{P}}{APV(Y)} \quad (12)$$

3. RESULTS AND DISCUSSION

3.1 Determination of Marginal Distribution

Based on the characteristics of the existing mortality models, the Heligman-Pollard Model will be used for age $0 \leq x \leq 49$, the Kostaki Model for age $50 \leq x \leq 82$, and the Kannisto-Makeham Model for age $x > 82$. This model is called HKK which will be used in male and female mortality rates. The function of q_x is written as follows,

$$q_x = \begin{cases} A_h^{(x+B_h)^{C_h}} + D_h \exp\left[-E_h\left(\ln\frac{x}{F_h}\right)^2\right] + \frac{G_hH_h^x}{1+G_hH_h^x}, & 0 \leq x < 50 \\ \begin{cases} A_k^{(x+B_k)^{C_k}} + D_k \exp\left[-E_{k1}\left(\ln\frac{x}{F_k}\right)^2\right] + \frac{G_kH_k^x}{1+G_kH_k^x}, x \leq F_k \\ A_k^{(x+B_k)^{C_k}} + D_k \exp\left[-E_{k1}\left(\ln\frac{x}{F_k}\right)^2\right] + \frac{G_kH_k^x}{1+G_kH_k^x}, x > F_k \end{cases}, & 50 \leq x < 83 \\ 1 - \exp\left(-C_m t - \frac{1}{B_m}\ln\left[\frac{1 + A_m \exp[B_m(x-82)]}{1 + A_m \exp[B_m(x-82)]}\right]\right), & x \geq 83 \end{cases} \quad (13)$$

By minimizing **Equation 5**, the estimator of parameter values of the above models is obtained.

Table 1. The estimator of Mortality Rate Parameter for Male and Female

Gender	Age	Model	Parameters
Male	$0 \leq x < 50$	Heligman-Pollard	$A_h = 0.00057; B_h = 0.01397; C_h = 0.08207;$ $D_h = 0.00019; E_h = 13.18539; F_h = 20.45577;$ $G_h = 0.00003; H_h = 1.11094$
	$50 \leq x < 83$	Kostaki	$A_k = 0.00014; B_k = 0.01; C_k = 0.30742;$ $D_k = 0.0122; E_{k1} = 2.5401; E_{k2} = 0.05048;$ $F_k = 72.056; G_h = 2.268 \times 10^{-11}; H_k = 1.30272$
	$x \geq 83$	Kannisto-Mahekam	$A_m = 0.00885; B_m = 0.0968; C_k = 1.157 \times 10^{-6}$
Female	$0 \leq x < 50$	Heligman-Pollard	$A_h = 0.0003; B_h = 0.02671; C_h = 0.086;$ $D_h = 8.47 \times 10^{-5}; E_h = 1.1 \times 10^{23}; F_h = 1219.4;$ $G_h = 2.5 \times 10^{-5}; H_h = 1.09909$
	$50 \leq x < 83$	Kostaki	$A_k = 0.0003; B_k = 0.02671; C_k = 0.086;$ $D_k = 0.037; E_{k1} = 1.47091; E_{k2} = 0.02942;$ $F_k = 146.05; G_h = 3.487 \times 10^{-14}; H_k = 1.40083$
	$x \geq 83$	Kannisto-Mahekam	$A_m = 0.03502; B_m = 0.12467; C_k = 0.01888$

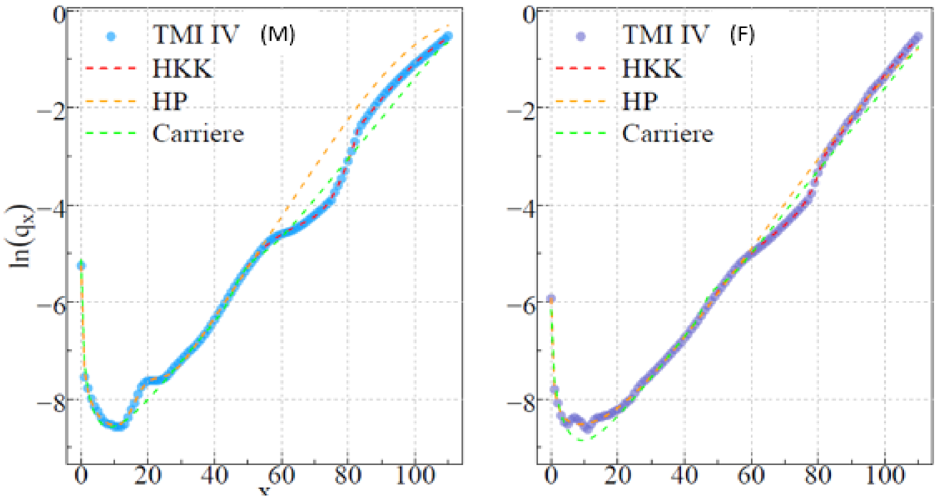
The selection of the HKK model is based on the smallest SSE value from several models. The Sum of Squared Error (SSE) criterion was used to compare the fit between models defined by [18],

$$SSE = \sum_{x=0}^{\omega}(q_x - \hat{q}_x)^2 \quad (14)$$

with $\hat{q}_x$ is the estimate of q_x. Based on **Table 2,** the HKK model has the smallest SSE value compared to the HP and Carriere models. So, it makes this model as the best model to draw mortality rates for male and female. This is also supported by the shape of the HKK model plot which is close to the observation value in **Figure 2**. It is very different with HP and Carriere models where they have several gaps with the observation value especially in the age of 60 to 100 for male and female also.

Table 2. SSE Values for the Mortality Rates

Gender	Model	SSE
Male	Heligman-Pollard, Kostaki, & Kannisto-Mahekam (HKK)	0.001384026*
	Heligman-Pollard (HP)	0.080343120
	Carriere	0.117778200
Female	Heligman-Pollard, Kostaki, & Kannisto-Mahekam (HKK)	0.002515042*
	Heligman-Pollard (HP)	0.004929184
	Carriere	0.098068130



(a) **(b)**

Figure 2. Estimated q_x TMI IV (a) Male and (b) Female

3.2 Determination of Joint Distribution

Let $(x_1),(x_2),(x_3)$ represent individuals aged x_1,x_2,x_3 respectively. Furthermore, the random variables T_{x_1},T_{x_2},T_{x_3} respectively represent the time remaining $(x_1),(x_2),(x_3)$ until death occurs. For 3 individuals consisting of father, mother, and son or daughter, the distribution function will be determined using Vine's copula. For example, (x_1,x_2,x_3) represents the father, mother, and child (boy or girl). Given $x_1=43, x_2=40, x_3=15$. For 3 individuals divided into 2 cases are as follows,

a. Case 1 (Let assumed: $\tau(T_{x_1},T_{x_2})=0.75, \tau(T_{x_1},T_{x_3})=0.25$)

b. Case 2 (Let assumed: $\tau(T_{x_1},T_{x_2})=0.75, \tau(T_{x_1},T_{x_3})=0.50$)

So, **Equation (6)** can write as follows,

$$
\begin{aligned}
f(t_{x_1},t_{x_2},t_{x_3}) &= c_{1,2}\left(F_1(t_{x_1}),F_2(t_{x_2})\right)\times c_{1,3}\left(F_1(t_{x_1}),F_3(t_{x_3})\right)\times c_{2,3|1}\left(F_2(t_{x_2}|t_{x_1}),F_3(t_{x_3}|t_{x_1})\right)\\
&\quad\times f_1(t_{x_1})f_2(t_{x_2})f_3(t_{x_3})\\
&= c_{1,2}(\,{}_tp_{x_1},\,{}_tp_{x_2})\times c_{1,3}(\,{}_tp_{x_1},\,{}_tp_{x_3})\times c_{2,3|1}\left(c_{1,2}(\,{}_tp_{x_1},\,{}_tp_{x_2}),c_{1,3}(\,{}_tp_{x_1},\,{}_tp_{x_3})\right)\\
&\quad\times f_1(t_{x_1})f_2(t_{x_2})f_3(t_{x_3})
\end{aligned}
$$

The best copula is then selected based on the loglikehood and Akaike Information Criteria (AIC) values **[19]**. Using Rstudio we can get the loglikehood and AIC values as follows,

Tabel 3. Loglikehood and AIC Values for $\tau = 0.75$

Annuitant	Copula	Loglikehood	AIC
Husband – Wife (T_{x_1}, T_{x_2})	**Frank**	**84533.351**	**-169065.702**
	Clayton	18813.747	-37626.494
	Gumbel	70768.338	-141535.677
	Joe	55254.436	-110507

Tabel 4. Loglikehood and AIC Values for $\tau = 0.5$

Annuitant	Copula	Loglikehood	AIC
Father – Son (T_{x_1}, T_3)	**Frank**	**16952.489**	**-33903.978**
	Clayton	8400.369	-16799.738
	Gumbel	11882.543	-23764.085
	Joe	5310.203	-10619.405
Father – Daughter (T_{x_1}, T_{x_3})	**Frank**	**12275.631**	**-24550.261**
	Clayton	7230.566	-14460.132
	Gumbel	7643.692	-15286.384
	Joe	2143.133	-4285.266

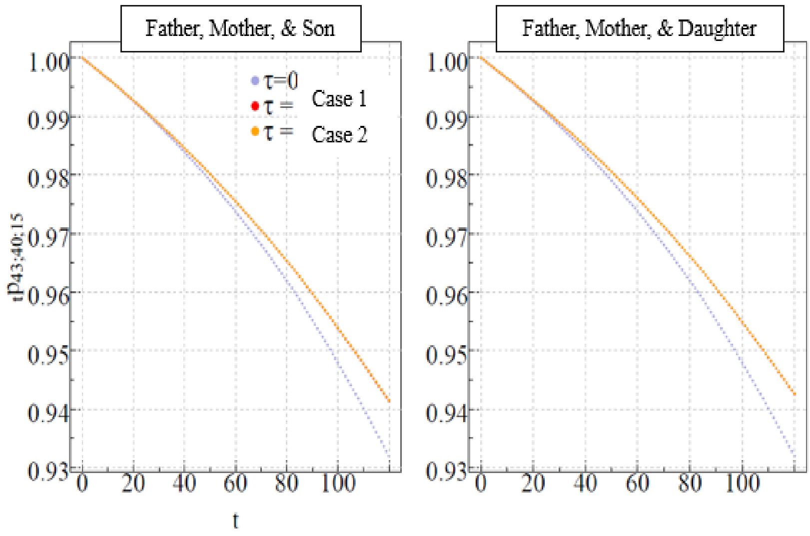
Tabel 5. Loglikehood and AIC Values for $\tau = 0.25$

Annuitant	Copula	Loglikehood	AIC
Father – Son (T_{x_1}, T_3)	**Frank**	**4416.466**	**-8831.932**
	Clayton	4040.253	-8079.507
	Gumbel	3629.833	-7258.666
	Joe	1603.280	-3205.560
Father – Daughter (T_{x_1}, T_{x_3})	**Frank**	**4416.918**	**-9209.913**
	Clayton	3451.951	-6902.902
	Gumbel	3352.052	-6703.104
	Joe	1153.144	-2305.287

Tabel 6. Loglikelihood and AIC Values of the Conditional Probability

Annuitant	Copula	Case 1		Case 2	
		Loglikehood	AIC	Loglikehood	AIC
Father – Mother – Son $(T_{x_2}, T_{x_3} \mid T_{x_1})$	Frank	56311.437	-112621.874	27642.213	-55283.425
	Clayton	**77984.225**	**-155967.450**	**39253.122**	**-78505.244**
	Gumbel	48183.058	-96365.115	27084.468	-54167.935
	Joe	31442.678	-62884.357	19597.978	-39194.957
Father – Mother – Daughter $(T_{x_2}, T_{x_3} \mid T_{x_1})$	Frank	31444.747	-62888.494	15006.399	-30011.798
	Clayton	**47875.749**	**-95750.499**	**23079.593**	**-46158.185**
	Gumbel	28067.232	-56133.465	15304.836	-30608.672
	Joe	16616.419	-33231.838	10811.668	-21622.336

Table 3- Table 5 conclude that the Frank copula is the best copula for the joint distribution with two individuals. This contrasts with the case of three individuals where **Table 6** shows that the best copula to choose is the Clayton copula. Furthermore, the value of the joint survival distribution or ${}_tp_{43:40:15}$ is simulated with the assumption of annuitant age as above. From **Figure 3**, with the value of $\tau = 0$, the resulting ${}_tp_{43:40:15}$ value is smaller than the ${}_tp_{43:40:15}$ value with the assumption of dependency.

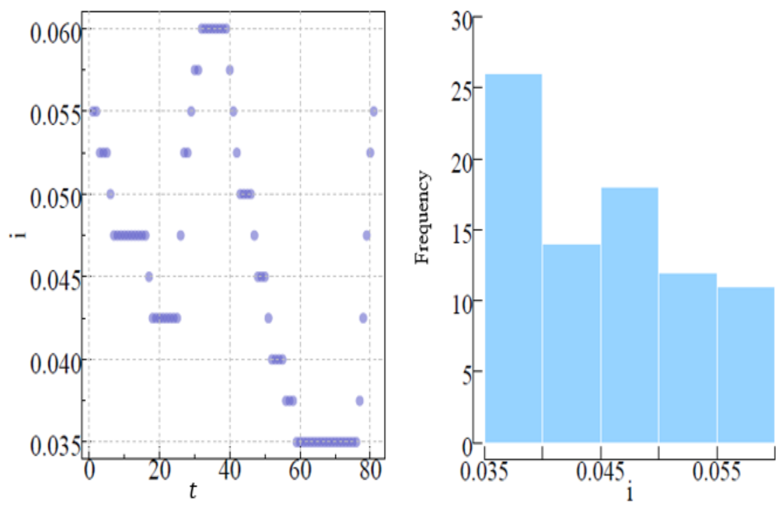


(a) (b)

Figure 3. The value of $_tp_{43:40:15}$ for some Cases
(a) Father, Mother and Son, (b) Father, Mother and Daughter

3.3 Fuzzy Interest Rate

In estimating the actuarial margin rate, BI data per year for the period April 2016 to December 2022. The descriptive statistics of this data are shown,



(a) (b)

Figure 4. BI Data per month (a) Scatter Plot, (b) Histogram

From **Figure 4**, it is known that the movement of the value of i has fluctuated significantly in the span of around 6 years with a mode value of 0.35. Then, the value $\alpha = 0.437$ is obtained, which is the average of fuzzy membership values using the median as a trusted value. It is also known that $a = 0.035$ (minimum), $b = 0.060$ (maximum), and $c = 0.048$ (median). By using **Equation (8)**, the actuarial margin level is obtained,

$$^{0.437}\tilde{i} = [0.035 + ((0.048 - 0.035) \times 0.437), 0.060 - ((0.060 - 0.048) \times 0.437)]$$
$$= [0.0404625, 0.0545375]$$

Since the margin rate is still in annual form, it is converted to monthly form through, $(1+i)^{\frac{1}{12}} - 1$. Thus, the actuarial margin rate of this data is written as follows,

$$^{0.437}\tilde{i} = [0.003310913; 0.003874685; 0.004434996]$$

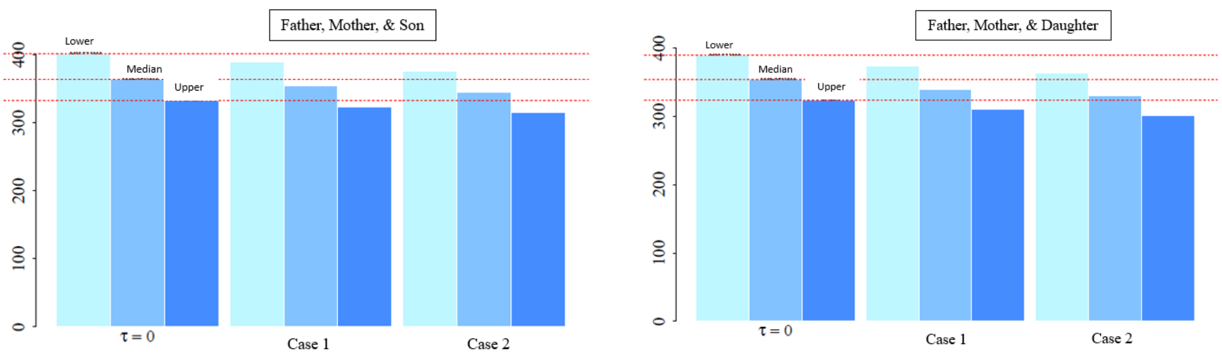
with the lower limit, median, and upper limit, respectively.

3.4 Determination of Family Annuity Benefit

The simulation results of marginal and joint distributions are used to calculate single life and multiple life annuity values. These annuity values are the components to simulate the family annuity calculation. The family annuity calculation uses the fuzzy interest rate and benefits are paid continuously to the beneficiary with a payment rate of 1 unit each month. Let assumed, $KT = 125{,}000{,}000$ and $AC = 20\%$. So, we can get the pure premium of family annuity $\mathcal{P} = 100{,}000{,}000$.

Table 7. The Benefits of Family Annuity

Annuitant	Case	Lower Limit	Median	Upper Limit
Father – Mother – Son $(T_{x_2}, T_{x_3} \| T_{x_1})$	$\tau = 0$	256,943	282,837	309,429
	Case 1	267,554	294,744	321,988
	Case 2	275,796	303,500	331,612
The change in benefits from $\tau = 0$		4.13%	4.21%	4.06%
		7.34%	7.31%	7.17%
Father – Mother – Daughter $(T_{x_2}, T_{x_3} \| T_{x_1})$	$\tau = 0$	249,923	274,352	300,146
	Case 1	256,711	282,348	308,807
	Case 2	265,747	289,697	316,602
The change in benefits from $\tau = 0$		3.00%	2.91%	2.89%
		6.63%	5.59%	5.48%



(a) **(b)**

Figure 5. The APV of Family annuity for some Cases
(a) Father, Mother and Son, (b) Father, Mother and Daughter

Based on Table 7, the value of annuity benefits on father, mother, and son with independence assumption ($\tau = 0$) is smaller than Case 1 and Case 2. The value of annuity benefits increases by about 4% as the value of τ increases. Then, the value of annuity benefits father, mother, and daughter with independence assumption ($\tau = 0$) is smaller than Case 1 and Case 2 too. The value of annuity benefits increases by around 3% as the value of τ increases. For both types of annuitants, the benefit value of Case 2 is also greater than Case 1. In contrast to the benefit value, the APV for the two types of annuitants shows that with the assumption of dependence, a smaller APV value is produced as shown in Figure 5. Overall, the value of annuity benefits on father, mother and son is greater than the annuity benefits on father, mother, and daughter.

4. CONCLUSIONS

Based on the analysis to calculate the annuity benefit, it can be concluded that,

1. The joint and marginal distribution of the random variables of one's survival. The Heligman-Pollard (HP) mortality model can capture the ages of infancy, childhood, and young adulthood well. Therefore, the Kostaki Model was used to capture old adults up to 83 years of age and after that, the Kannisto-Makeham Model was used up to 112 years of age. These three models are combined to get the best estimate of TMI IV.
2. The bivariate co-distribution is modeled with the Archimedian copula with the best copula being Frank's copula and the best co-conditional copula being modeled with Clayton copula. The trivariate joint distribution is modeled with the Vine's copula model which can capture dependencies more flexibly compared to multivariate copula which imposes the same correlation for all individuals.

3. The actuarial margin level is modeled using a fuzzy set with a triangular membership function to capture the minimum value, reliable value, and maximum value from the data. The fuzzy interest rate provides an interval of a number based on the degree of membership to a reliable value (average or median). Thus, the actuarial margin level value has a lower limit, middle value, and upper limit.

4. The annuity benefits calculation is assumed with some Kendall's tau values which are $\tau = (0.25; 0.50; 0.75)$. The value of annuity benefits on father, mother, and son and father, mother, and daughter increase by around 4% and 3% respectively.

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