BAREKENG: Journal of Mathematics and Its Applications
March 2024 Volume 18 Issue 1 Page 0507–0516
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss1pp0507-0516

CONSTRUCT THE TRIPLE ZERO GRAPH OF RING $\mathbb{Z}_n$ USING PYTHON

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Article History:

Received: 28th October 2023
Revised: 3rd January 2024
Accepted: 25th January 2024

Keywords:

Triple Zero Graph;
Python Algorithm.

ABSTRACT

Let R be a commutative ring with nonzero identity and $TZ(R) = \{a \in Z(R)^*$: there exists $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$ denotes the set of all triple zero elements of R. The triple zero graph of R, denoted by $TZ\Gamma(R)$, is an undirected graph with vertex set $TZ(R)$ where two distinct vertices a and b are adjacent if and only if $ab \neq 0$, and there exists a nonzero element c of R such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$. Python is a programming language with simple and easy-to-learn code that can be used to solve problems in algebra and graphs. In this paper, we construct the triple zero graph of ring $\mathbb{Z}_n$ using Python. Based on the output of the program, several properties of $TZ\Gamma(R)$ are obtained, such as if $n = 8$ and $n = 12$, then $TZ\Gamma(\mathbb{Z}_n)$ is a planar graph, if $n = p^3$ with p is prime numbers, then $TZ\Gamma(\mathbb{Z}_n)$ is a complete graph K_{p^2-p}, and if $n = 4p$ with p is prime numbers and $p \geq 3$, then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph.

How to cite this article:

P. Wulandari, V. Y. Kurniawan and N. A. Kurdhi., "CONSTRUCT THE TRIPLE ZERO GRAPH OF RING $\mathbb{Z}_n$ USING PYTHON," *BAREKENG: J. Math. & App.*, vol. 18, iss. 1, pp. 0507-0516, March, 2024.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*
Research Article · Open Access

1. INTRODUCTION

Graph theory in mathematics and computer science is a branch of science that studies the properties of graphs and is widely applied in solving various problems. Solving problems with graph theory will be easier because a problem is represented in the form of a graph, as well as problems in an algebraic structure will be more easily solved by using the concept of a graph, particularly in relation to connections between elements, as exemplified by Cayley graphs. Cayley graphs provide a clear and intuitive way to visualize the structure of a group. Each element in the group corresponds to a vertex in the graph, and group operations are represented by edges.

Graph G is a finite non-empty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called edges [1]. Meanwhile, the ring is a set R along with two operations on R, namely addition and multiplication, so that R with addition is an abelian group, is associative to multiplication operations, and multiplication operations are distributive to addition operations [2]. Given R is a commutative ring, which is a ring that has the commutative property of multiplication, and $Z(R)$ is the set of zero divisors of R. A ring R contains a zero divisor if there exists $x, y \in R$ such that $x.y = 0$. The zero-divisor graph of the commutative ring R, denoted by $G(R)$, is a graph whose vertices are all elements of R, and two vertices are adjacent if their multiplication are zero. This idea was introduced by Beck [3] in 1988. Then, Anderson and Livingston [4] developed the concept of a zero divisor graph by delimiting the set of vertices to be nonzero zero-divisors in R, that is $Z(R)^* = Z(R) - \{0\}$. Such a graph is denoted by $\Gamma(R)$. The concept of a zero divisor graph of a commutative ring according to Anderson-Livingston [4] has prompted much research afterward, as in [5], [6], [7], [8], and [9].

In 2021, Celikel [10] developed the concept of a triple zero graph $TZ\Gamma(R)$ of a commutative ring R. Let R be a commutative ring with nonzero identity and $TZ(R) = \{a \in Z(R)^*$: there exists $b, c \in R\backslash\{0\}$ such that $abc = 0, ab \neq 0, ac \neq 0, bc \neq 0\}$ denotes the set of all triple zero elements of R. The triple zero graph of R, denoted by $TZ\Gamma(R)$, is an undirected graph with vertex set $TZ(R)$ where two distinct vertices a and b are adjacent if and only if $ab \neq 0$, and there exists a nonzero element c of R such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$.

In this article, we focus on investigating the properties of the triple zero graph of ring $\mathbb{Z}_n$. To investigate the properties of $TZ\Gamma(\mathbb{Z}_n)$, the constructions of $TZ\Gamma(\mathbb{Z}_n)$ of various values of n is needed. If done manually, it will take a long time because there are several stages to be completed, namely determining all zero divisors in $\mathbb{Z}_n$, then identifying all triple zero elements in $\mathbb{Z}_n$ that will become its vertices, and finally investigating the adjacency between two vertices. Then, it will be faster if it is represented using a computer program like Python. Python is a programming language that is relatively easy to learn, has a rich set of features, and quite expressive, so it usually only takes a few lines of code to complete what would require many more lines of code in other languages [11]. Moreover, we choose Python for computational algebra due to its versatility, ease of learning, strong community support, seamless integration with other tools, and the availability of powerful numerical computing libraries like NumPy and SciPy, as well as graph plotting libraries like Matplotlib and NetworkX. In the concept of a graph of a ring, several researchers used the Python algorithm as in [12], [13], and [14]. In this paper, we discuss the algorithm to construct $TZ\Gamma(\mathbb{Z}_n)$ using Python. This paper is organized as follows. The first section of this paper is an introduction. The second section is research method used in this research. The third section is result and discussion including the algorithm that constructs the graph and an example of how the algorithm works. The last section is the conclusion of this discussion.

2. RESEARCH METHODS

The research method used in this research is a literature study by studying references in the form of books, journals, and theses regarding the structure of algebra, graph theory, and Python programming, especially the triple zero graph of a commutative ring.

The steps taken in this research are as follows.

1. Learning basic definitions and theorems related to algebraic structures and graph theory.
2. Learning the definition of the triple zero graph of a commutative ring.

3. Determining the algorithm for constructing the triple zero graph of the ring of integers modulo $\boldsymbol{n}$ with Python.
4. Investigating the properties of the triple zero graph of the ring of integers modulo $\boldsymbol{n}$.
5. Making conclusions.

This article aims to determine an algorithm for constructing $\boldsymbol{TZ\Gamma}(\mathbb{Z}_{\boldsymbol{n}})$ and to know the properties of the graphs. Therefore, several definitions underlying this research need to be outlined. Some of these definitions cover the basic concepts of graph, ring, and triple zero graph.

2.1 Basic Concepts of Graph

The following are some basic definitions of graph taken from Chartrand et al. [1] and Chartrand and Zhang [15].

Definition 1. [1] Graph G is a finite non-empty set $V(G)$ of objects which is called vertices together with the (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called edges. The number of vertices in a set $V(G)$ is called order, while the number of edges of a set $E(G)$ is called size.

As an example, below is an illustration of graph G.

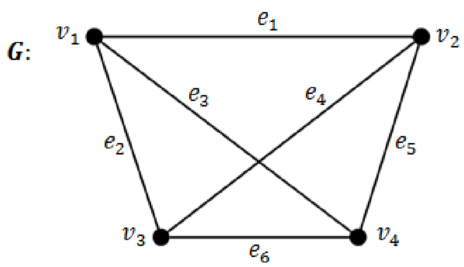


Figure 1. Graph G

Figure 1 is an illustration of graph G which has order 4 and size 6, with a vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and an edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Each graph must contain at least one vertex, but it is possible to have no edges.

Two distinct vertices, u and v, are said to be adjacent if there is an edge e connecting them. Then, edge e is called incident with vertices u and v. For example, graph G in **Figure 1**, vertices v_1 and v_2 are adjacent because they are connected by edge e_1.

Definition 2. [15] A $u - v$ walk of a graph G is an alternating sequence of vertices and edges starting at vertex u and ending at vertex v. A $u - v$ trail is a $u - v$ walk that does not repeat any edges. A $u - v$ path is a $u - v$ walk that does not repeat any vertices.

Below are examples of walk, trail, and path based on **Figure 1**. Example of $v_1 - v_4$ walk is $v_1 - v_4 : v_1, e_1, v_2, e_4, v_3, e_2, v_1, e_1, v_2, e_5, v_4$. Example of $v_1 - v_4$ trail is $v_1 - v_4 : v_1, e_1, v_2, e_4, v_3, e_2, v_1, e_3, v_4$. Example of $v_1 - v_4$ path is $v_1 - v_4 : v_1, e_1, v_2, e_4, v_3, e_6, v_4$.

Definition 3. [15] A circuit is a $u - v$ trail that begins and ends at the same vertex and consists of at least three vertices. A circuit that does not repeat any vertex except the start vertex and the end vertex is called a cycle. Cycle with n vertices is denoted by C_n.

The following is an example of a circuit and cycle based on **Figure 1**. Circuit in graph G is $v_1 - v_1 : v_1, e_1, v_2, e_4, v_3, e_2, v_1$. Cycle in graph G is $v_1 - v_1 : v_1, e_1, v_2, e_5, v_4, e_6, v_3, e_2, v_1$.

Definition 4. [1] A graph G is a connected graph if there is a $u - v$ path between any two vertices u and v in G.

Figure 1 is an example of a connected graph because there exists a $u - v$ path for any two vertices u and v.

Definition 5. [1] A graph G is a complete graph if every two different vertices in graph G are adjacent. The complete graph with order n is denoted by K_n.

Graph G in **Figure 1** is a complete graph because every two vertices are adjacent. Graph G is a complete graph K_4 because it has order 4.

Definition 6. [15] A graph G is a planar graph if G can be drawn in the plane with its edges not crossing each other.

Figure 1 is an example of a planar graph because it can be drawn on a plane with its edges not crossing each other.

Definition 7. [1] Given a graph G. A path in G that contains all vertices of G is called a Hamiltonian path of G, while a cycle in G that contains all the vertices of G is called a Hamiltonian cycle of G. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

In **Figure 1**, graph G is a Hamiltonian graph because it contains a Hamiltonian cycle. Hamiltonian cycle in graph G is $v_1, e_2, v_3, e_4, v_2, e_5, v_4, e_3, v_1$.

Definition 8. [15] The girth of graph G is the length of the shortest cycle contained in graph G, denoted by $gr(G)$.

In **Figure 1**, the girth of graph G is 3, in cycle $v_3 - v_3 : v_3, e_4, v_2, e_5, v_4, e_6, v_3$.

2.2 The Triple Zero Graph of Commutative Ring

Before we give the definition of the triple zero graph, we first recall the definition of zero divisor element by Durbin [2].

Definition 9. A nonzero element a in a ring R is a zero divisor element if there exists $b \in R$ such that $b \neq 0$ and $ab = 0$ or $ba = 0$.

The definition of the triple zero graph of a commutative ring is taken from Celikel [10].

Definition 10. The triple zero graph of R, denoted by $TZ\Gamma(R)$, is an undirected graph with vertex set $TZ(R) = \{a \in Z(R)^*$: there exists $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$ where two distinct vertices a and b are adjacent if and only if $ab \neq 0$, and there exists a nonzero element c of R such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$.

Example 1. The following is an example of $TZ\Gamma(\mathbb{Z}_8)$ where $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$.

Table 1. Cayley Table of $\mathbb{Z}_8$

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\bar{2}$	$\bar{5}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{2}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	$\bar{6}$	$\bar{3}$
$\bar{6}$	$\bar{0}$	$\bar{6}$	$\bar{4}$	$\bar{2}$	$\bar{0}$	$\bar{6}$	$\bar{4}$	$\bar{2}$
$\bar{7}$	$\bar{0}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

By definition, the vertex set of $TZ\Gamma(\mathbb{Z}_8)$ is a set of zero divisor elements in ring $\mathbb{Z}_8$ where there exists nonzero elements b, c in $\mathbb{Z}_8$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, and two distinct vertices a and b are adjacent if and only if $ab \neq 0$ and there exists a nonzero element c of $\mathbb{Z}_8$ such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$. A zero divisor element in ring $\mathbb{Z}_8$ are $Z(\mathbb{Z}_8)^* = \{\bar{2}, \bar{4}, \bar{6}\}$. If $a = \bar{2}$, then $b = \bar{2}$ and $c = \bar{6}$, if $a = \bar{4}$, then there exist no b and c that satisfy the condition above, and if $a = \bar{6}$, then $b = \bar{2}$ and $c = \bar{6}$. So, obtained $TZ(\mathbb{Z}_8) = \{\bar{2}, \bar{6}\}$. Then, $\bar{2}$ and $\bar{6}$ are adjacent because $ab \neq 0$ and there exists $c = \bar{2}$ such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$. So, obtained $E\big(TZ\Gamma(\mathbb{Z}_8)\big) = \{(\bar{2}, \bar{6})\}$. The $TZ\Gamma(\mathbb{Z}_8)$ is shown in **Figure 2**.

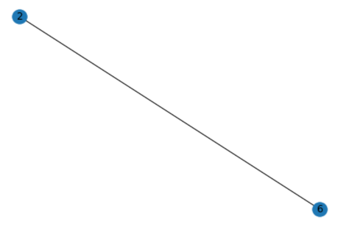


Figure 2. $\boldsymbol{TZ\Gamma(\mathbb{Z}_8)}$

Figure 2 is a triple zero graph of $\mathbb{Z}_8$ which has 2 vertices and 1 edge.

3. RESULTS AND DISCUSSION

In this article, we discuss how algorithm to construct the graphs $TZ\Gamma(\mathbb{Z}_n)$ using python. The results of that algorithm are as follows:

3.1 Algorithm for Constructing $TZ\Gamma(\mathbb{Z}_n)$

First, the algorithm for constructing $TZ\Gamma(\mathbb{Z}_n)$ is parameterized by the triple zero elements and derives the triple zero graph of those elements. Recall that $TZ(\mathbb{Z}_n)$ denoted the set of all triple zero elements of $\mathbb{Z}_n$ and $Z(\mathbb{Z}_n)^*$ denoted the set of nonzero zero divisors of $\mathbb{Z}_n$. The recursive algorithm for constructing $TZ\Gamma(\mathbb{Z}_n)$ is called TripleZeroGraph. Then, import the library to draw a graph in python.

TripleZeroGraph
Import library **networkx** and **matplotlib.pyplot**
Step 1:
for $a \in Z(\mathbb{Z}_n)^*$ and $b, c \in \mathbb{Z}_n \backslash \{0\}$ **do**
  if $a * b \neq 0$, $a * c \neq 0$, $b * c \neq 0$, $a * b * c = 0$ **then**
    add a to $TZ(\mathbb{Z}_n)$;
  end if;
end for;
Step 2:
for $i \in TZ(\mathbb{Z}_n)$ **do**
  add i to $V(TZ\Gamma(\mathbb{Z}_n))$;
  for $j \in TZ(\mathbb{Z}_n)$ and $k \in \mathbb{Z}_n \backslash \{0\}$ **do**
    if $i \neq j$ **then**
      if $i * j \neq 0$, $i * k \neq 0$, $j * k \neq 0$ and $i * j * k = 0$ **then**
        add $i \sim j$ to $E(TZ\Gamma(\mathbb{Z}_n))$;
      end if;
    end if;
  end for;
end for;
Step 3:
Draw $TZ\Gamma(\mathbb{Z}_n)$;
end;

In step 1, the algorithm checks for the triple zero elements of the ring $\mathbb{Z}_n$. For all elements zero divisors a, check that if there are $b, c \in \mathbb{Z}_n \backslash \{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$. If so, then a is a triple zero element of ring $\mathbb{Z}_n$, add a to $TZ(\mathbb{Z}_n)$. In step 2, the algorithm checks the adjacency between vertices, for $i \in TZ(\mathbb{Z}_n)$, add i to $V(TZ\Gamma(\mathbb{Z}_n))$. For $j \in TZ(\mathbb{Z}_n)$ and $k \in \mathbb{Z}_n \backslash \{0\}$, check that if $i \neq j$, $i \cdot j \neq 0$, $i \cdot k \neq 0$, and $j \cdot k \neq 0$. And then if $i \cdot j \cdot k = 0$. If so, i and j are adjacent. Otherwise, return to the loop process in step 2. Now, we have obtained vertex set and edge set of $TZ\Gamma(\mathbb{Z}_n)$. Finally, in step 3, the algorithm draws the triple zero graph of ring $\mathbb{Z}_n$.

3.2 Constructing $TZ\Gamma(\mathbb{Z}_n)$

In this section, an example explains how this algorithm runs in ring $\mathbb{Z}_n$.

Example 2. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 2, 3, 5, 7$.

For $n = 2$, we have $\mathbb{Z}_2$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_2)^*$ and $b, c \in \mathbb{Z}_2\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_2)$. We did not get the zero divisor elements and the triple zero elements, so the algorithm ended the loop process. So, in $\mathbb{Z}_2$, no graph is formed. The same case applies to $\mathbb{Z}_3, \mathbb{Z}_5$ and $\mathbb{Z}_7$ because the structure of $\mathbb{Z}_p$, where p is prime, is an integral domain. Therefore, no element $\mathbb{Z}_p$ is a zero divisor or a triple zero element.

Example 3. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 4,9,25$.

For $n = 4$, we have $\mathbb{Z}_4$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_4)^*$ and $b, c \in \mathbb{Z}_4\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_4)$. We get $Z(\mathbb{Z}_4)^* = \{2\}$, but we did not get the triple zero elements, so the algorithm ended the loop process. So, in $\mathbb{Z}_4$, no graph is formed. The same case applies to $\mathbb{Z}_9$ and $\mathbb{Z}_{25}$, where no triple zero element are found in them.

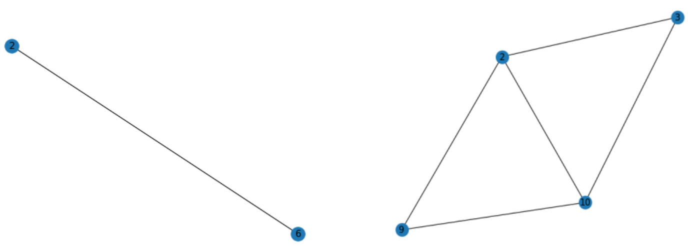
Example 4. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 6,10,15$.

For $n = 6$, we have $\mathbb{Z}_6$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_6)^*$ and $b, c \in \mathbb{Z}_6\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_6)$. We get $Z(\mathbb{Z}_6)^* = \{2,3,4\}$, but we did not get the triple zero elements, so the algorithm ended the loop process. So, in $\mathbb{Z}_6$, no graph is formed. The same case applies to $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$, where no triple zero element are found in them.

Example 5. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 8, 12$.

For $n = 8$, we have $\mathbb{Z}_8$. Step 1, the algorithm checks with the looping process for each element $a \in Z(\mathbb{Z}_8)^*$ and $b, c \in \mathbb{Z}_8\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_8)$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_8) = \{2, 6\}$. The triple zero elements are the vertices of $TZ\Gamma(\mathbb{Z}_8)$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_8)^*$, $k \in \mathbb{Z}_8\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2$, $j = 6$, and there exist $k = 2$, so that $2 \cdot 6 = 4 \neq 0$, $2 \cdot 2 = 4 \neq 0$, $6 \cdot 2 = 4 \neq 0$ and $2 \cdot 6 \cdot 2 = 0$. Then, 2 and 6 are adjacent. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_8)$.

For $n = 12$, we have $\mathbb{Z}_{12}$. Step 1, the algorithm checks with the looping process for each element $a \in Z(\mathbb{Z}_{12})^*$ and $b, c \in \mathbb{Z}_{12}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{12})$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_{12}) = \{2, 3, 9, 10\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{12})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{12})^*$, $k \in \mathbb{Z}_{12}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2$, $j = 3$, and there exist $k = 2$, so that $2 \cdot 3 = 6 \neq 0$, $2 \cdot 2 = 4 \neq 0$, $3 \cdot 2 = 6 \neq 0$ and $2 \cdot 3 \cdot 2 = 0$. Then, 2 and 3 are adjacent. By the same process, an $i - j$ path is found between any two vertices i and j in $TZ(\mathbb{Z}_{12})^*$. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{12})$.



(a) **(b)**

Figure 3. The Triple Zero Graph of $\mathbb{Z}_n$ for (a) $n = 8$, (b) $n = 12$

Figure 3 is the result of the construction of the triple zero graph of $\mathbb{Z}_n$ for $n = 8, 12$. The triple zero graph of $\mathbb{Z}_8$ consists of 2 vertices and 1 edge, while the triple zero graph of $\mathbb{Z}_{12}$ consists of 4 vertices and 5 edges.

Example 6. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 20, 28, 44, 52$.

For $n = 20$, we have $\mathbb{Z}_{20}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{20})^*$ and $b, c \in \mathbb{Z}_{20}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{20})$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_{20}) = \{2, 5, 6, 14, 15, 18\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{20})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{20})^*$, $k \in \mathbb{Z}_{20}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2, j = 5$, and there exist $k = 2$, so that $2 \cdot 5 = 10 \neq 0$, $2 \cdot 2 = 4 \neq 0$, $5 \cdot 2 = 10 \neq 0$ and $2 \cdot 5 \cdot 2 = 0$. Then, 2 and 5 are adjacent. By the same process, an $i - j$ path is found between any two vertices i and j in $TZ(\mathbb{Z}_{20})^*$. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{20})$.

For $n = 28$, we have $\mathbb{Z}_{28}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{28})^*$ and $b, c \in \mathbb{Z}_{28}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{28})$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_{28}) = \{2, 6, 7, 10, 18, 21, 22, 26\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{28})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{28})^*$, $k \in \mathbb{Z}_{28}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2$, $j = 6$, and there exist $k = 7$, so that $2 \cdot 6 = 12 \neq 0$, $2 \cdot 7 = 14 \neq 0$, $6 \cdot 7 = 14 \neq 0$ and $2 \cdot 6 \cdot 7 = 0$. Then, 2 and 6 are adjacent. By the same process, an $i - j$ path is found between any two vertices i and j in $TZ(\mathbb{Z}_{28})^*$. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{28})$.

For $n = 44$, we have $\mathbb{Z}_{44}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{44})^*$ and $b, c \in \mathbb{Z}_{44}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{44})$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_{44}) = \{2, 6, 10, 11, 14, 18, 26, 30, 33, 34, 38, 42\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{44})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{44})^*$, $k \in \mathbb{Z}_{44}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2$, $j = 6$, and there exist $k = 11$, so that $2 \cdot 6 = 12 \neq 0$, $2 \cdot 11 = 22 \neq 0$, $6 \cdot 11 = 22 \neq 0$ and $2 \cdot 6 \cdot 11 = 0$. Then, 2 and 6 are adjacent. By the same process, an $i - j$ path is found between any two vertices i and j in $TZ(\mathbb{Z}_{44})^*$. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{44})$.

For $n = 52$, we have $\mathbb{Z}_{52}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{52})^*$ and $b, c \in \mathbb{Z}_{52}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{52})$. In this step, we get the triple zero elements, $Z(\mathbb{Z}_{52}) = \{2, 6, 10, 13, 14, 18, 22, 30, 34, 38, 39, 42, 46, 50\}$. The triple zero element is the vertices of the $TZ\Gamma(\mathbb{Z}_{52})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{52})^*$, $k \in \mathbb{Z}_{52}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 2$, $j = 6$, and there exist $k = 13$, so that $2 \cdot 6 = 12 \neq 0$, $2 \cdot 13 = 26 \neq 0$, $6 \cdot 13 = 26 \neq 0$ and $2 \cdot 6 \cdot 13 = 0$. Then, 2 and 6 are adjacent. By the same process, an $i - j$ path is found between any two vertices i and j in $TZ(\mathbb{Z}_{52})^*$. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{52})$.

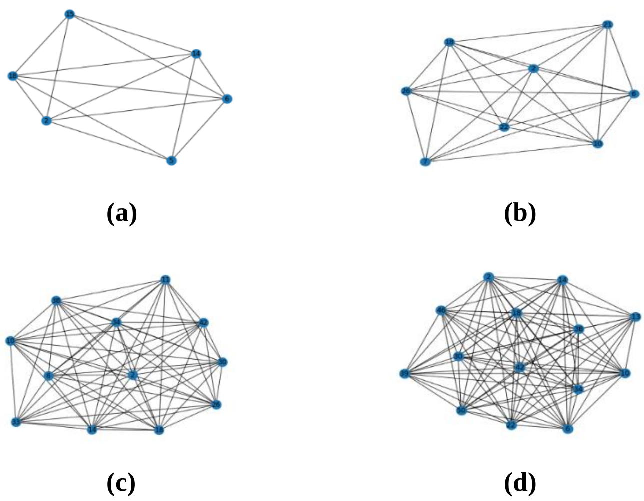


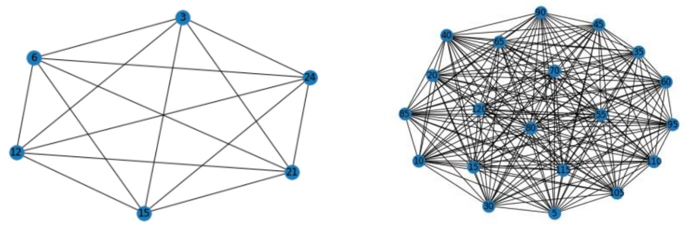
Figure 4. The Triple Zero Graph of $\mathbb{Z}_n$ for (a) $n = 20$, (b) $n = 28$, (c) $n = 44$, (b) $n = 52$

Figure 4 is the result of the construction of the triple zero graph of $\mathbb{Z}_n$ for $n = 20,28,44,52$. The triple zero graph of $\mathbb{Z}_{20}$ consists of 6 vertices and 14 edges, the triple zero graph of $\mathbb{Z}_{28}$ consists of 8 vertices and 27 edges, the triple zero graph of $\mathbb{Z}_{44}$ consists of 12 vertices and 65 edges, while the triple zero graph of $\mathbb{Z}_{52}$ consists of 14 vertices and 90 edges.

Example 7. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 27, 125$.

For $n = 27$, we have $\mathbb{Z}_{27}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{27})^*$ and $b, c \in \mathbb{Z}_{27}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{27})$. In this step, we get the triple zero elements, $TZ(\mathbb{Z}_{27}) = \{3, 6, 12, 15, 21, 24\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{27})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{27})^*$, $k \in \mathbb{Z}_{27}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 3$, $j = 6$, and there exist $k = 3$, so that $3 \cdot 6 = 18 \neq 0$, $3 \cdot 3 = 9 \neq 0$, $6 \cdot 3 = 18 \neq 0$ and $3 \cdot 6 \cdot 3 = 0$. Then, 3 and 6 are adjacent. By the same process, all vertices in $TZ(\mathbb{Z}_{27})$ are found to be adjacent to one another. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{27})$.

For $n = 125$, we have $\mathbb{Z}_{125}$. Step 1, algorithm checking with the looping process for each element $a \in Z(\mathbb{Z}_{125})^*$ and $b, c \in \mathbb{Z}_{125}\backslash\{0\}$, if $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$, then add a to $TZ(\mathbb{Z}_{125})$. In this step, we get the triple zero elements, $(\mathbb{Z}_{125}) = \{5, 10, 15, 20, 30, 35, 40, 45, 55, 60, 65, 70, 80, 85, 90, 95, 105, 110, 115, 120\}$. The triple zero elements are the vertices of the $TZ\Gamma(\mathbb{Z}_{125})$. Then, in step 2, the algorithm checks the adjacency between vertices. Based on the conditions for $i, j \in TZ(\mathbb{Z}_{125})^*$, $k \in \mathbb{Z}_{125}\backslash\{0\}$, and $i \neq j$. If $i \cdot j \neq 0$, $i \cdot k \neq 0$, $j \cdot k \neq 0$ and $i \cdot j \cdot k = 0$. When $i = 5$, $j = 10$, and there exist $k = 5$, so that $5 \cdot 10 = 50 \neq 0$, $5 \cdot 5 = 25 \neq 0$, $10 \cdot 5 = 50 \neq 0$ and $5 \cdot 10 \cdot 5 = 0$. Then, 5 and 10 are adjacent. By the same process, all vertices in $TZ(\mathbb{Z}_{125})$ are found to be adjacent to one another. There are no more vertices in the loop process, so the algorithm ends the loop process. In step 3, the algorithm draws $TZ\Gamma(\mathbb{Z}_{125})$.



(a) **(b)**

Figure 5. The Triple Zero Graph of $\mathbb{Z}_n$ for (a) $n = 27$, (b) $n = 125$

Figure 5 is the result of the construction of the triple zero graph of $\mathbb{Z}_n$ for $n = 27,125$. The triple zero graph of $\mathbb{Z}_{27}$ consists of 6 vertices and 15 edges, while the triple zero graph of $\mathbb{Z}_{125}$ consists of 20 vertices and 190 edges.

Example 8. Construct the triple zero graph of $\mathbb{Z}_n$ for $n = 60$.

The last is an example for the case of the higher order of n with $n = 60$. It can be seen how easly is it to construct the triple zero graph by considering the processing time of the program. By running the program that has been made, below are the result of the triple zero graph of $\mathbb{Z}_{60}$.

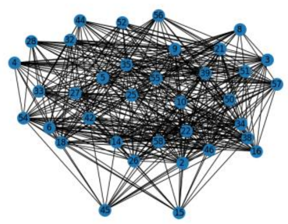


Figure 6. The Triple Zero Graph of $\mathbb{Z}_n$ for $n = 60$

Figure 6 is the result of the construction of the triple zero graph of $\mathbb{Z}_n$ for $n = 60$. The triple zero graph of $\mathbb{Z}_{60}$ consists of 36 vertices and 415 edges. Then, the time to process this graph is less than 5 seconds. Of course, this is faster to solve than manually.

Observation results have been obtained for the properties of the triple zero graph of the ring $\mathbb{Z}_n$ based on the program outputs. These observations contribute to identifying properties of the triple zero graph of the ring $\mathbb{Z}_n$, which still require algebraic proof.

Observation results:

(i) If $n = 8$ and $n = 12$, then $TZ\Gamma(\mathbb{Z}_n)$ is a planar graph.

(ii) If $n = p$ or $n = p^2$ or $n = pq$ with p and q are two distinct prime numbers, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$.

(iii) If $n = p^3$ with p is prime numbers, then $TZ\Gamma(\mathbb{Z}_n)$ is a complete graph K_{p^2-p}.

(iv) If $n = 4p$ with p is prime numbers and $p \geq 3$, then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph.

4. CONCLUSIONS

In this paper, we explain an algorithm for constructing the triple zero graph of ring $\mathbb{Z}_n$ using Python. We provide examples to illustrate how the algorithm operates within the ring $\mathbb{Z}_n$. The Python program designed for constructing the triple zero graph proves highly efficient, serving as a valuable tool for exploring properties within the triple zero graph of the ring $\mathbb{Z}_n$. This efficiency becomes evident as the program quickly generates graphs for different variations of the ring. For instance, constructing $TZ\Gamma(\mathbb{Z}_{60})$ in Example 8 takes less than 5 seconds. Consequently, the properties of the triple zero graph of the ring $\mathbb{Z}_n$ can be easily observed.

ACKNOWLEDGMENT

The authors would like to thank the Research and Community Services Institute of Universitas Sebelas Maret for funding this research in the 2023-2024 academic year.

REFERENCES

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 6th ed. New York: CRC Pers Taylor and Francis Group, 2016. doi: 10.1201/b19731.

[2] J. R. Durbin, *Modern Algebra: An Introduction*, 6th ed. New York: John Wiley and Sons, 2015.

[3] I. Beck, "Coloring of Commutative," vol. 226, no. 1, pp. 208–226, 1988.

[4] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebr.*, vol. 217, no. 2, pp. 434–447, 1999, doi: 10.1006/jabr.1998.7840.s

[5] A. Cherrabi, H. Essannouni, E. Jabbouri, and A. Ouadfel, "On a new extension of the zero-divisor graph (II)," *Beitrage zur Algebr. und Geom.*, vol. 62, no. 4, pp. 945–953, 2021, doi: 10.1007/s13366-020-00559-8.

[6] S. Redmond and S. Szabo, "When metric and upper dimensions differ in zero divisor graphs of commutative rings," *Discret. Math. Lett.*, vol. 5, pp. 34–40, 2021, doi: 10.47443/dml.2021.0005.

[7] A. M. Alanazi, M. Nazim, and N. Ur Rehman, "Planar, Outerplanar, and Toroidal Graphs of the Generalized Zero-Divisor Graph of Commutative Rings," *J. Math.*, vol. 2021, 2021, doi: 10.1155/2021/4828579.

[8] H. Mariiaa, V. Y. Kurniawan, and Sutrima, "Zero annihilator graph of semiring of matrices over Boolean semiring," *AIP Conf. Proc.*, vol. 2326, 2021, doi: 10.1063/5.0039557.

[9] R. Yudatama, V. Y. Kurniawan, and S. B. Wiyono, "Annihilator graph of semiring of matrices over Boolean semiring," *J. Phys. Conf. Ser.*, vol. 1494, no. 1, 2020, doi: 10.1088/1742-6596/1494/1/012009.

[10] E. Y. Çelikel, "The Triple Zero Graph of a Commutative Ring," *Fac. Sci. Univ. Ankara Ser. A1 Math. Stat.*, vol. 70, no. 2, pp. 653–663, 2021, doi: 10.31801/cfsuasmas.786804.

[11] J. B. Schneider, S. L. Broschat, and J. Dahmen, *Algorithmic problem solving with Python*. United States: Washington State University, 2014.

[12] W. Tapanyo, T. Tongpikul, and S. Kaewpradit, "gcd-Pairs in Zn and their graph representations," pp. 1–11, 2022, [Online]. Available: http://arxiv.org/abs/2206.01847

[13] Y. Tian and L. Lixiang, "Comments on the Clique Number of Zero-Divisor Graphs of Zn," *J. Math.*, vol. 2022, 2022.

[14] T. Hartati and V. Y. Kurniawan, "Construct The Triple Nilpotent Graph of Ring Using Python," vol. 020004, no. 2886, 2023.

[15] G. Chartrand and P. Zhang, *A First Course in Graph Theory*. New York: Dover Publication, 2012.