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BOUNDEDNESS AND EXISTENCE ANALYSIS SOLUTION OF AN OPTIMAL CONTROL PROBLEMS ON MATHEMATICAL COVID-19 MODEL

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ABSTRACT

The research conducted is a part of a literature review on mathematical models that apply analytical mathematics. The research focuses on the COVID-19 model, which incorporates optimum control variables previously investigated and interpreted by Hakim. Depending on the current model, we will further develop the analysis and demonstrate the non-negativity condition as well as the boundedness criteria for the solutions. Additionally, we conduct several supplementary analyses by applying the Lipschitz function to examine the uniqueness of the solutions and the existence of the solution are hold on the autonomous system. The purpose of this research is to support the previous findings that incorporated an optimal control into the model to reduce public COVID-19 treatment. Finally, the research verifies that the control variables used satisfy all of the existence criteria, as outlined in Theorem 5 of this research.

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1. INTRODUCTION

Mathematical analysis is one of a discipline part of mathematics that is especially beneficial in describing biological or social behaviors currently happening in the world [1]. One of the ideas is the mathematical analysis provides a significant role in preventing the outbreak of diseases or pandemics by understanding and predicting how the disease will spread in population, and how we can develop the effective ways to reduce the impact [2], [3]. The pandemic is remains one of humanity's greatest challenges. The infectious spread of disease, like the viruses that requires significant planning, controlling, and knowledge in arranging and organizing, which including the mathematical analysis. The COVID-19 disease became a pandemic that shook the earth towards the end of 2019[4].

The SARS-CoV-2 Coronavirus causes COVID-19, which typically appears as a pulmonary dysfunction disease. This virus is frequently transmitted by respiratory droplets, that occur when an individual or object relates within one to two meters of an infected individual. Coughing and sneezing might also contribute to the outbreak [5]. The World Health Organization identified the first case of the COVID-19 pandemic on December 2019, in China. Additionally, in March 2020, the World Health Organization announced that the virus had affected approximately 118,000 people globally, and rapidly spreading across nearly 110 countries. Consequently, the WHO classified COVID-19 as a worldwide concern, and although this disease is a one-time event, it gives suffered protection against the next infections [6].

Mathematical epidemiology holds a significant role to improve our comprehension of how infectious diseases spread, their impact, and the potential forecasting strategies for mitigating their treat. Mathematical models have been used to compare, plan, implement, and evaluate many initiatives targeting at identifying, preventing, and regulating the spread of illnesses such as Coronavirus [7]. Mathematical and non-mathematical researchers have made significant contributions to this topic. Their collaborative efforts have significantly assisted public health professionals in understanding infectious disease behavior and developing effective intervention methods to prevent disease transmission in populations worldwide.

We know that some noteworthy contributors to mathematical modeling include McKendrick (1927), and Kermack (1932). The SIR diagram compartment of mathematical model was introduced by Kermack and McKendrick, which indicated the beginning of epidemic modeling. [2]. It subsequently became a fundamental model in COVID-19 research, involving mathematical principles used in studies by Rangasamy et al. [8], Khajanchi et al. [4], Malinzi et al. [7], Huy et al. [9], Peter et al. [5], and DarAssi et al. [10]. Afterwards, the implementation of quarantine regulations and isolation methods have inspired to construct mathematical models by Trisilowati et al. [11], Hakim [12], Rois et al. [13], and all of these models were the foundation for analyzing the spread of COVID-19 disease. An extension on the research presented a compartments diagram of COVID-19 incorporating the vaccination into mathematical model [7], [14]. Furthermore, the mathematical model was further developed through adding the use of masks as a control for decreasing the COVID-19 outbreak [10].

As illustrated by the above facts, numerous disease, such as Measles [15], [16], HIV [17], Monkeypox [18], and Tuberculosis [19] need an optimal control strategy to manage their transmission. Optimal control was also used in managing Zika disease [20], Type 2 Diabetes [21], Hepatitis B [22], Cholera [23], and COVID-19 [6], [24]. Earlier investigations have effectively used the optimum control methodology as a tool for controlling disease. Furthermore, analytical mathematics is used to figure out the boundedness and existence of solution for depicting the model. Some studies investigate the existence and boundedness of the transmission of the measles model. In addition, the research conducted by Nainggolan et al. [25] and Hakim [6] discusses about the existence and boundedness theorem of an optimal control into mathematical COVID-19 compartment.

The benefit and highlight of the current research is to strengthen the previously performed research by Hakim [12] regarding an optimal control of the spreading COVID-19 disease, which is only discussed numerically without demonstrating an analysis of the positivity and boundedness of the solution. As a result, it is important to conduct this research to give fundamental evidence that the following numerical results are suitable and feasible. This work is organized into several sections. The outline of section 2 develops several methods. Section 3 has three subsections that address in depth into the results and discussions. Section 3.1 concentrates on the fundamental concepts of solution non-negativity, whereas Section 3.2 describes the solution's boundedness. Sections 3.3 and 3.4 clarify the terms uniqueness and existence of solution systems, respectively. Finally, Section 3.5 discusses the existence of control theorem and its relevance to disease control. In conclusion, Section 4 provides some final remarks.

2. RESEARCH METHODS

In the current research, we establish an optimal control COVID-19 transmission epidemic system by Hakim [12]. The model is divided into several classes with different criteria, namely people who are susceptible without masks (S_1), people who are susceptible and using masks (S_2), people who are infected without masks (I_1), people who are infected and using masks (I_2), people in quarantine (Q), and the total population denoted by $N = S_1 + S_2 + I_1 + I_2 + Q$. These categories also can be expressed mathematically through a nonlinear differential equation, by following term bellow

$$\begin{aligned}
\frac{dS_1(t)}{dt} &= \mu + \eta_2 S_2(t) - (\mu + \eta_1)S_1(t) - \big(1 - z_1(t)\big)\beta S_1(t)I_1(t) - z_2(t)S_1(t)\\
\frac{dS_2(t)}{dt} &= \eta_1 S_1(t) + z_2 S_1(t) - (\mu + \eta_2)S_2(t)\\
\frac{dI_1(t)}{dt} &= \big(1 - z_1(t)\big)\beta S_1(t)I_1(t) + \eta_2 I_2(t) - (\eta_1 + \mu + \gamma + \alpha)I_1(t) - z_3(t)I_1(t)\\
\frac{dI_2(t)}{dt} &= \eta_1 I_1(t) - (\eta_2 + \mu + \gamma + \alpha)I_2(t) + z_3(t)I_1(t)\\
\frac{dQ(t)}{dt} &= \alpha\big(I_1(t) + I_2(t)\big) - (\mu + \gamma + \theta)Q(t),
\end{aligned} \tag{1}$$

Based on the Equation (1) above, we adapt to the model that was carried out by three control variables. As an illustration, $z_1\,(t)$ was used to manage the direct interactions between susceptible and infected individuals who are not wearing masks. As a preventive step to against COVID-19, $z_2(t)$ was introduced to encourage the use masks among infected individuals. Lastly, the control variable $z_3(t)$ was implemented to set up a requirement for susceptible individuals to consistently wear masks. Additionally, the following approaches can be employed to examine several conditions of an optimal control problem in mathematical modelling of COVID-19:

1. Evaluate the positivity and boundedness of solution through the application of standard methods for solving ordinary differential equations.
2. Assess the uniqueness of the solution to the control system by utilizing the Lipschitz condition approach.
3. Investigate the existence of solutions to the control system by employing a method in the theory of partial derivatives.
4. Examine the persistence of control variables within the system, demonstrating that the utilization of these control can effectively mitigate to COVID-19 disease.

3. RESULTS AND DISCUSSION

In this passage, we investigate some analysis of an optimal control properties, namely the uniqueness of solution, the non-negativity or positivity of solution, the boundedness, and the existence of control variable on the autonomous system of the COVID-19 model.

3.1 The Uniqueness of Solution

In this subsection, we determine that the autonomous Equation (1) has one solution (uniqueness) if the Lipschitz condition is hold, and this condition is figured in theorem below.

Theorem 1. *Let $\vec{h}(t, \vec{\omega})$ in Lipschitz condition*

$$\left|\vec{h}(t, \vec{\omega}^*) - \vec{h}(t, \vec{\omega}^{**})\right| \leq k|\vec{\omega}^* - \vec{\omega}^{**}|,$$

with the $(t, \vec{\omega}^*)$ and $(t, \vec{\omega}^{**})$ into the feasible region $\Gamma = \{(S_1, S_2, I_1, I_2, Q)\}$, and k is a non-negative number, such that the function $\vec{h}(t, \vec{\omega})$ has one solution (uniqueness).

Proof. In this part, we start from the first equation on Equation (1) to determine that the pair state (t, S_1^*) and (t, S_1^{**}) holds on the Lipschitz condition. For detailing, we present

$$\begin{aligned}
|h(t,S_1^*)-h(t,S_1^{**})| &= |(\mu+\eta_1)(S_1^{**}-S_1^*)+(1-z_1)\beta I_1(S_1^{**}-S_1^*)+z_2(S_1^{**}-S_1^*)| \\
&= |\mu+\eta_1+(1-z_1)\beta I_1+z_2||S_1^{**}-S_1^*| \\
&\le (|\mu|+|\eta_1|+|(1-z_1)\beta I_1|+|z_2|)|S_1^{**}-S_1^*| \\
&\le \left(\mu+\eta_1+(1-z_1)\beta \sup_{t\in D_{I_1}}|I_1|+z_2\right)|S_1^{**}-S_1^*| \\
&\le \left(\mu+\eta_1+(1-z_1)\beta M_{I_1}+z_2\right)|S_1^*-S_1^{**}| \\
&\le k_{S_1}|S_1^*-S_1^{**}|,
\end{aligned}$$

if $k_{S_1}=\mu+\eta_1+(1-z_1)\beta M_{I_1}+z_2$, then it implies that the Lipschitz condition is satisfied by the term $|h(t,S_1^*)-h(t,S_1^{**})|\le k_{S_1}|S_1^*-S_1^{**}|$. Secondly, to other subpopulations in **Equation (1)**, it can be shown that the following equation satisfy the Lipschitz conditions, namely

$$\begin{aligned}
|h(t,S_2^*)-h(t,S_2^{**})| &= |(\mu+\eta_2)(S_2^{**}-S_2^*)| \\
&= |\mu+\eta_2||S_2^{**}-S_2^*| \\
&\le (|\mu|+|\eta_2|)|S_2^{**}-S_2^*| \\
&\le (\mu+\eta_2)|S_2^*-S_2^{**}| \\
&\le k_{S_2}|S_2^*-S_2^{**}|,
\end{aligned}$$

with the value $k_{S_2}=\mu+\eta_2$, and the condition applies $|h(t,S_2^*)-h(t,S_2^{**})|\le k_{S_2}|S_2^*-S_2^{**}|$. Infected subpopulation without medical masks can be demonstrated as

$$\begin{aligned}
|h(t,I_1^*)-h(t,I_1^{**})| &= |(1-z_1)\beta S_1(I_1^*-I_1^{**})+(\eta_1+\mu+\gamma+\alpha+z_3)(I_1^{**}-I_1^*)| \\
&\le |(1-z_1)\beta S_1(I_1^*-I_1^{**})|+|(\eta_1+\mu+\gamma+\alpha+z_3)(I_1^{**}-I_1^*)| \\
&\le |(1-z_1)\beta S_1||I_1^*-I_1^{**}|+|(\eta_1+\mu+\gamma+\alpha+z_3)||I_1^*-I_1^{**}| \\
&\le (|(1-z_1)\beta S_1|+|\eta_1|+|\mu|+|\gamma|+|\alpha|+|z_3|)|I_1^*-I_1^{**}| \\
&\le \left((1-z_1)\beta \sup_{t\in D_{S_1}}|S_1|+\eta_1+\mu+\gamma+\alpha+z_3\right)|I_1^*-I_1^{**}| \\
&\le \left((1-z_1)\beta M_{S_1}+\eta_1+\mu+\gamma+\alpha+z_3\right)|I_1^*-I_1^{**}| \\
&\le k_{I_1}|I_1^*-I_1^{**}|,
\end{aligned}$$

such that we get the value $k_{I_1}=(1-z_1)\beta M_{S_1}+\eta_1+\mu+\gamma+\alpha+z_3$. While, the infected subpopulation with masks shall be shown by following term

$$\begin{aligned}
|h(t,I_2^*)-h(t,I_2^{**})| &= |(\eta_2+\mu+\gamma+\alpha)(I_2^{**}-I_2^*)| \\
&= |\eta_2+\mu+\gamma+\alpha||I_2^{**}-I_2^*| \\
&\le (|\eta_2|+|\mu|+|\gamma|+|\alpha|)|I_2^{**}-I_2^*| \\
&\le (\eta_2+\mu+\gamma+\alpha)|I_2^*-I_2^{**}| \\
&\le k_{I_2}|I_2^*-I_2^{**}|,
\end{aligned}$$

so that we obtain the value $k_{I_2}=\eta_2+\mu+\gamma+\alpha$. The final part of the equation also satisfies Lipschitz condition as well,

$$\begin{aligned}
|h(t,Q^*)-h(t,Q^{**})| &= |(\mu+\gamma+\theta)(Q^{**}-Q^*)| \\
&= |\mu+\gamma+\theta||Q^{**}-Q^*| \\
&\le (|\mu|+|\gamma|+|\theta|)|Q^{**}-Q^*|
\end{aligned}$$

$$\leq (\mu+\gamma+\theta)|Q^* - Q^{**}|$$
$$\leq k_Q|Q^* - Q^{**}|,$$

with a positive constant is related to the variable Q, namely $k_Q = \mu+\gamma+\theta$. Therefore, using the Lipschitz condition analysis, we could conclude that the autonomous **Equation (1)** has a unique solution.

3.2 The Non-negativity of Solution

In this section, to describe the **Equation (1)** has the positivity of solution on epidemiological importance. As a result, it is crucial for proving that all variables remain non-negative at all time t. To fix this concern, we investigated the theorem listed below.

Theorem 2. *The solution of* ***Equation (1)*** *remains non-negative for all* $t > 0$, *provided that the initial conditions for* $S_1(0)$, $S_2(0)$, $I_1(0)$, $I_2(0)$, *and* $Q(0)$ *are all greater than or equal to zero and belong to the region of* Γ.

Proof. By taking the equation that relate with susceptible people without mask in **Equation (1)**, and arranging algebra we get

$$\begin{aligned}\frac{dS_1}{dt} &= \mu + \eta_2 S_2 - (\mu+\eta_1)S_1 - (1-z_1)\beta S_1 I_1 - z_2 S_1\\ &= \mu + \eta_2 S_2 - (\mu+\eta_1+(1-z_1)\beta I_1 + z_2)S_1\\ &\geq -(\mu+\eta_1+(1-z_1)\beta I_1 + z_2)S_1\end{aligned}$$

Therefore, we have the new term that the derivation of S_1 is

$$\frac{dS_1}{dt} \geq -(\mu+\eta_1+(1-z_1)\beta I_1 + z_2)S_1. \tag{2}$$

Integrating **Equation (2)** on both sides and using separation method, we obtain the following expression

$$\int_{S_1(0)}^{S_1(t)} \frac{dS_1}{S_1} \geq \int_0^t -(\mu+\eta_1+(1-z_1)\beta I_1 + z_2)dt,$$

and the trivial solution of **Equation (2)** is

$$S_1(t) \geq S_1(0)e^{-(\mu+\eta_1+(1-z_1)\beta I_1+z_2)t} \geq 0. \tag{3}$$

Similarly, the solution of the other equations in the **Equation (1)** are obtained as follows

$$\frac{dS_2}{dt} = \eta_1 S_1 + z_2 S_1 - (\mu+\eta_2)S_2 \geq -(\mu+\eta_2)S_2,$$

and the representation of susceptible wear mask as below

$$\frac{dS_2}{dt} \geq -(\mu+\eta_2)S_2. \tag{4}$$

Integrating Equation (4), such that the trivial solution of **Equation (4)** is

$$S_2(t) \geq S_2(0)e^{-(\mu+\eta_2)t} \geq 0. \tag{5}$$

Whereas the infected population without masks is given below

$$\begin{aligned}\frac{dI_1}{dt} &= (1-z_1)\beta S_1 I_1 + \eta_2 I_2 - (\eta_1+\mu+\gamma+\alpha)I_1 - z_3 I_1\\ &= 1 + \eta_2 I_2 - (\eta_1+\mu+\gamma+\alpha+z_1\beta S_1 + z_3)I_1\\ &\geq -(\eta_1+\mu+\gamma+\alpha+z_1\beta S_1+z_3)I_1,\end{aligned}$$

and then the new form as adheres

$$\frac{dI_1}{dt} \geq -(\eta_1 + \mu + \gamma + \alpha + z_1\beta S_1 + z_3)I_1. \tag{6}$$

Using the separation method to **Equation (6)**, the solution obtained is

$$I_1(t) \geq I_1(0)e^{-(\eta_1+\mu+\gamma+\alpha+z_1\beta S_1+z_3)t} \geq 0. \tag{7}$$

The infected population use masks are expressed by

$$\frac{dI_2}{dt} = \eta_1 I_1 - (\eta_2 + \mu + \gamma + \alpha)I_2 + z_3 I_1 \geq -(\eta_2 + \mu + \gamma + \alpha)I_2,$$

such that

$$\frac{dI_2}{dt} \geq -(\eta_2 + \mu + \gamma + \alpha)I_2. \tag{8}$$

By integrating the **Equation (8)**, we have the trivial solution of

$$I_2(t) \geq I_2(0)e^{-(\eta_2+\mu+\gamma+\alpha)t} \geq 0. \tag{9}$$

Finally, the last equation of the **Equation (1)** is

$$\frac{dQ}{dt} = \alpha(I_1 + I_2) - (\mu + \gamma + \theta)Q \geq -(\mu + \gamma + \theta)Q,$$

then we generate

$$\frac{dQ}{dt} \geq -(\mu + \gamma + \theta)Q. \tag{10}$$

Integrating the **Equation (10)** by using separation technique, we get the trivial solution of

$$Q(t) \geq Q(0)e^{-(\mu+\gamma+\theta)t} \geq 0. \tag{11}$$

Therefore, the consequence of **Equations (3), (5), (7), (9)**, and **(11)** show that the all variables in **Equation (1)** are positive for all $t > 0$, and hence the theorem of non-negativity is proven.

3.3 The Boundedness of Solution

The boundedness of solution defines of specific region to the solutions of the **Equation (1)**, and this idea is enhanced by the theorem following.

Theorem 3. *Assuming the* ***Equation (1)*** *remains valid, and considering every solution of the model with the initial condition in positive in* $N = S_1 + S_2 + I_1 + I_2 + Q \in \Re^5_+$*, and the set* Γ *as* $t \to \infty$*, such that the region of the feasible solution is bounded by*

$$\Gamma = \{(S_1, S_2, I_1, I_2, Q) \in \Re^5_+ : N \leq 1\}.$$

Proof. From **Equation (1)**, we know that notation N leads to the total of population, such as

$$\frac{dN}{dt} = \frac{dS_1}{dt} + \frac{dS_2}{dt} + \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dQ}{dt} = \mu - \mu N - \gamma(I_1 + I_2 + Q) - \theta Q \leq \mu(1 - N). \tag{12}$$

Based on **Equation (12)** we get the first order ordinary differential equation, and by separating variables we obtain

$$\frac{dN}{dt} \leq \mu(1 - N),$$

then by integrating both sides of the inequality above from 0 to t, we generate the new form

$$\int_{N(0)}^{N(t)} \frac{dN}{1 - N} \leq \int_0^t \mu dt,$$

$$-\ln\big(1-N(t)\big)+\ln\big(1-N(0)\big)\le \mu t\,,$$

and by arranging the natural logarithm properties, we get the equation become

$$\ln\left(\frac{1-N(0)}{1-N(t)}\right)\le \mu t. \tag{13}$$

Manipulating **Equation (13)** with the natural logarithm properties, we have the changed form

$$\frac{1-N(0)}{1-N(t)}\le e^{\mu t}. \tag{14}$$

Arranging and operating **Equation (14)** as an ordinary differential equation, we obtain the solution as follows

$$N(t)\le 1-\big(1-N(0)\big)e^{-\mu t}.$$

Hence at the time $t\to\infty$, then implies that $N(t)\le 1$, and the solutions are bounded in the region Γ.

3.4 The Existence of a Solution

This section, we explain about the existence of **Equation (1)** that has been studied by Hakim **[12]** previously. Before delivering the theorem to establish the existence of a solution into **Equation (1)**, we modify the right-hand side of **Equation (1)** to reconfigure into an alternative expression.

$$\begin{aligned}
h_{S_1}(t,\omega)&=\mu+\eta_2S_2-(\mu+\eta_1)S_1-(1-z_1)\beta S_1I_1-z_2S_1\\
h_{S_2}(t,\omega)&=\eta_1S_1+z_2S_1-(\mu+\eta_2)S_2\\
h_{I_1}(t,\omega)&=(1-z_1)\beta S_1I_1+\eta_2I_2-(\eta_1+\mu+\gamma+\alpha)I_1-z_3I_1\\
h_{I_2}(t,\omega)&=\eta_1I_1-(\eta_2+\mu+\gamma+\alpha)I_2+z_3(t)I_1\\
h_Q(t,\omega)&=\alpha(I_1+I_2)-(\mu+\gamma+\theta)Q(t),
\end{aligned} \tag{15}$$

with $\omega=(S_1,S_2,I_1,I_2,Q)$.

Theorem 4. *Suppose the function $h(t,\omega)$ have a partial derivative $\frac{\partial h}{\partial\omega}$ dan $\left|\frac{\partial h}{\partial\omega}\right|<\infty$, and satisfies the Lipschitz criteria, then the function $h(t,\omega)$ has an existence and boundedness of solution.*

Proof. Attend to the right side of **Equation (15)**, it is obvious to obtain the partial derivative and its absolute value. In detail, we can show in susceptible subpopulations without mask can be partially derivation, namely

$$\frac{\partial h_{S_1}(t,\omega)}{\partial S_1}=-(\mu+\eta_1)-(1-z_1)\beta I_1-z_2,$$

then we obtain

$$\left|\frac{\partial h_{S_1}(t,\omega)}{\partial S_1}\right|=|-(\mu+\eta_1)-(1-z_1)\beta I_1-z_2|<\infty.$$

The next step is differentiating to the variable S_2 , which yields

$$\frac{\partial h_{S_1}(t,\omega)}{\partial S_2}=\eta_2, then \left|\frac{\partial h_{S_1}(t,\omega)}{\partial S_2}\right|=|\eta_2|<\infty.$$

If it is differentiating to the variable I_1, we submit

$$\frac{\partial h_{S_1}(t,\omega)}{\partial I_1}=-(1-z_1)\beta S_1,\text{then }\left|\frac{\partial h_{S_1}(t,\omega)}{\partial I_1(t)}\right|=|-(1-z_1)\beta S_1|<\infty.$$

Then, the derivation into variable I_2, we have

$$\frac{\partial h_{S_1}(t,\omega)}{\partial I_2} = 0, \text{then } \left|\frac{\partial h_{S_1}(t,\omega)}{\partial I_2}\right| = 0 < \infty,$$

and while differentiating into variable Q, obviously

$$\frac{\partial h_{S_1}(t,\omega)}{\partial Q} = 0, \text{then } \left|\frac{\partial h_{S_1}(t,\omega)}{\partial Q(t)}\right| = 0 < \infty.$$

Analogous with the partially derivation process above, it is clear to find all the equations in **Equation (15)** are continuous and boundedness.

3.5 The Existence of Control Variable for Controlling the Systems

The existence analysis of control variable is conducted to depict and support the expressed of control variables provide a beneficial and deeper meaning. We determine the theorem 5 below based on the research previously by Hakim **[12]**.

Theorem 5. *Suppose the exists of a control variable* $Z = (z_1(t), z_2(t), z_3(t))$ *within* **Equation (1)**, *such that the following condition is satisfied*

$$\min_{z\in Z} J(z_1(t), z_2(t), z_3(t), z_4(t)) = J(z_1^*(t), z_2^*(t), z_3^*(t)).$$

Proof. Following the analysis conducted in **[6], [26]**, It is established that the optimal control through the Equation (1) will exist when several of the conditions explained below are realized.

1. The control Z is a non-empty set.

   It is evident that by using this control, the desired function is able to be performed. Using a demonstration by contradiction, suppose we define the objective function as follows:

$$\max J(\vec{z}) = \int_{t_0}^{t_f} \left(AI_1(t) + BI_2(t) + Cz_1^2(t) + Dz_2^2(t) + Ez_3^2(t)\right) dt.$$

   This implies that the crucial purpose of the functional objective is to maximize the infected subpopulations with and without masks. However, considering the time interval $t = [t_0, t_f]$ is bounded. To indicate the existence of disease prevention methods, the control variable should be minimized as feasible bellow

$$\min J(\vec{z}) = \int_{t_0}^{t_f} \left(AI_1(t) + BI_2(t) + Cz_1^2(t) + Dz_2^2(t) + Ez_3^2(t)\right) dt,$$

   and demonstrated that the control set is non-empty of elements.

2. The control set in Z is convex and closed.

   a. For any $z \in$ Z, and $z' \in$ Z, we will demonstrate that $r = \theta z + (1-\theta)z' \in$ Z, for all $\theta \in [0,1]$. Clearly, if $\theta z \le \theta$ and $(1-\theta)z' \le (1-\theta)$, then we can deduce that $\theta z + (1-\theta)z' \le \theta + (1-\theta) = 1$. Ultimately, we have $0 \le \theta z + (1-\theta)z' \le 1$, for all $u \in$ Z, and for all $\theta \in [0,1]$. Therefore, the set of control Z is a convex.

   b. Consider any control variable $z \notin [a,b]$, it implies that $z < a$ or $z > b$. Now, if $z < a$, it follows that there exists $\epsilon_z = |z - a| > 0$, which the results in the intersection of the set and the neighborhood of the control being null set, denoted as $[a,b] \cap V_\epsilon(z) = \emptyset$. Similarly, if $z > b$, it means there exists $\epsilon_z = |z - b| > 0$, and the intersection of the set and the neighborhood of the control also becomes an null set, $[a,b] \cap V_\epsilon(z) = \emptyset$. Consequently, it can be concluded that the control variable z is a closed, where $z \in Z$.

3. The equation on the right-hand side of the **Equation (1)** is bounded by control configuration and a linear function.

In the first phase, we use **Equation (1)** as a fundamental term to change it into a matrix, namely

$$\begin{bmatrix}\frac{dS_1}{dt}\\\frac{dS_2}{dt}\\\frac{dI_1}{dt}\\\frac{dI_2}{dt}\\\frac{dQ}{dt}\end{bmatrix}=\begin{bmatrix}\eta_2S_2-(\mu+\eta_1)S_1-\beta S_1I_1\\\eta_1S_1-(\mu+\eta_2)S_2\\\beta S_1I_1+\eta_2I_2-(\eta_1+\mu+\gamma+\alpha)I_1\\\eta_1I_1-(\eta_2+\mu+\gamma+\alpha)I_2\\\alpha(I_1+I_2)-(\mu+\gamma+\theta)Q\end{bmatrix}+\begin{bmatrix}z_1\beta S_1I_1-z_2S_1\\z_2S_2\\-z_1\beta S_1I_1-z_3I_1\\z_3I_1\\0\end{bmatrix}+\begin{bmatrix}\mu\\0\\0\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}\eta_2S_2\\\eta_1S_1\\\beta S_1I_1+\eta_2I_2\\\eta_1I_1\\\alpha(I_1+I_2)\end{bmatrix}-\begin{bmatrix}(\mu+\eta_1)S_1+\beta S_1I_1\\(\mu+\eta_2)S_2\\(\eta_1+\mu+\gamma+\alpha)I_1\\(\eta_2+\mu+\gamma+\alpha)I_2\\(\mu+\gamma+\theta)Q\end{bmatrix}+\begin{bmatrix}z_1\beta S_1I_1\\z_2S_2\\0\\z_3I_1\\0\end{bmatrix}-\begin{bmatrix}z_2S_1\\0\\z_1\beta S_1I_1+z_3I_1\\0\\0\end{bmatrix}+\begin{bmatrix}\mu\\0\\0\\0\\0\end{bmatrix}$$

$$<\begin{bmatrix}\eta_2S_2\\\eta_1S_1\\\beta S_1I_1+\eta_2I_2\\\eta_1I_1\\\alpha(I_1+I_2)\end{bmatrix}+\begin{bmatrix}z_1\beta S_1I_1\\z_2S_2\\0\\z_3I_1\\0\end{bmatrix}+\begin{bmatrix}\mu\\0\\0\\0\\0\end{bmatrix}$$

$$\leq\left\|\begin{bmatrix}\eta_2S_2\\\eta_1S_1\\\beta S_1I_1+\eta_2I_2\\\eta_1I_1\\\alpha(I_1+I_2)\end{bmatrix}\right\|+\left\|\begin{bmatrix}z_1\beta S_1I_1\\z_2S_2\\0\\z_3I_1\\0\end{bmatrix}\right\|+\left\|\begin{bmatrix}\mu\\0\\0\\0\\0\end{bmatrix}\right\|$$

$$\leq\begin{bmatrix}M_1\\M_2\\M_3\\M_4\\M_5\end{bmatrix}=\vec{M}.$$

It's clear that the right-hand side of the autonomous **Equation (1)** is bounded by a control variable and a linear function.

4. The functional objective is convex with the Z region

Consider any variables x_i and y_j, where $i,j=1,2,3$, and a domain of $0\leq\theta\leq 1$. In this section, we will demonstrate that

$$J\big((1-\theta)\vec{x}(t)+\theta\vec{y}(t)\big)\leq(1-\theta)J\big(\vec{x}(t)\big)+\theta J\big(\vec{y}(t)\big), \tag{16}$$

with $\vec{x}=\big(x_1(t),x_2(t),x_3(t)\big)^T$, and $\vec{y}=\big(y_1(t),y(t),y_3(t)\big)^T$. Next, by incorporating the objective function in Hakim **[12]** into **Equation (16)**, resulting in

$$\begin{aligned}AI_1(t)+BI_2(t)+C\big((1-\theta)x_1(t)+\theta y_1(t)\big)^2+D\big((1-\theta)x_2(t)+\theta y_2(t)\big)^2\\+E\big((1-\theta)x_3(t)+\theta y_3(t)\big)^2\\\leq(1-\theta)\left(AI_1(t)+BI_2(t)+Cx_1^2(t)+Dx_2^2(t)+Ex_3^2(t)\right)\\+\theta\left(AI_1(t)+BI_2(t)+Cy_1^2(t)+Dy_2^2(t)+Ey_3^2(t)\right).\end{aligned} \tag{17}$$

Subsequently, through manipulation and organization of **Equation (17)**, we obtain a detailed and equivalent expression as follows

$$(1-\theta)Cx_1^2(t)+\theta Cy_1^2(t)-C(1-\theta)^2x_1^2(t)-2C(1-\theta)\theta x_1(t)y_1(t)-C\theta^2y_1^2+(1-\theta)Dx_2^2(t)$$
$$+\theta Dy_2^2(t)-D(1-\theta)^2x_2^2(t)-2D(1-\theta)\theta x_2(t)y_2(t)-D\theta^2y_2^2+(1-\theta)Ex_3^2(t)$$
$$+\theta Dy_3^2(t)-D(1-\theta)^2y_3^2(t)-2D(1-\theta)\theta x_3(t)y_3(t)-D\theta^2y_1^2\geq 0,$$

$$Cx_1^2(t)\big((1-\theta)-(1-\theta)^2\big)+C\theta y_1^2(t)(1-\theta)-2C(1-\theta)\theta x_1(t)y_1(t)$$
$$+Dx_2^2(t)\big((1-\theta)-(1-\theta)^2\big)+D\theta y_2^2(t)(1-\theta)-2D(1-\theta)\theta x_2(t)y_2(t)$$
$$+Ex_3^2(t)\big((1-\theta)-(1-\theta)^2\big)+E\theta y_3^2(t)(1-\theta)-2E(1-\theta)\theta x_3(t)y_3(t)\geq 0,$$

$$C\theta\left((1-\theta)x_1^2(t)-2(1-\theta)x_1(t)y_1(t)+(1-\theta)x_1^2(t)\right)$$
$$+D\theta\left((1-\theta)x_2^2(t)-2(1-\theta)x_2(t)y_2(t)+(1-\theta)y_2^2(t)\right)$$
$$+E\theta\left((1-\theta)x_3^2(t)-2(1-\theta)x_3(t)y_3(t)+(1-\theta)y_3^2(t)\right)\geq 0,$$

$$C\theta\left(\sqrt{(1-\theta)}x_1(t)-\sqrt{(1-\theta)}y_1(t)\right)^2+D\theta\left(\sqrt{(1-\theta)}x_2(t)-\sqrt{(1-\theta)}y_2(t)\right)^2$$
$$+E\theta\left(\sqrt{(1-\theta)}x_3(t)-\sqrt{(1-\theta)}y_3(t)\right)^2\geq 0.$$

As a result, the value of the integrand for the goal function is convex.

5. The integrand of the objective function is bounded

Given parameters $\upsilon_1>C$, $\upsilon_2>D$, $\upsilon_3>E$, as well as variables $I_1(t)$ and $I_2(t)$ constrained within the interval $[t_0,t_f]$, such that the population size of $I_1(t)$ remains below or equal to $I_1(t_f)$, and $I_2(t)$ is limited to $I_2(t_f)$, the objective function

$$AI_1(t)+BI_2(t)+Cz_1^2(t)+Dz_2^2(t)+Ez_3^2(t)\leq AI_1(t)+BI_2(t)+\upsilon_1z_1^2(t)+\upsilon_2z_2^2(t)+\upsilon_3z_3^2(t)$$
$$\leq A(t_f)+B(t_f)+\upsilon_1|z_1^2|(t)+\upsilon_2|z_2^2|(t)+\upsilon_3|z_3^2|(t)=\mathcal{M}.$$

It is obvious that the functional objective is bounded by the $\mathcal{M}=A(t_f)+B(t_f)+\upsilon_1|z_1^2|(t)+\upsilon_2|z_2^2|(t)+\upsilon_3|z_3^2|(t)$.

4. CONCLUSIONS

This research is an extension of the optimal control problem investigated by Hakim. The fundamental goal of this study is to evaluate the boundedness and confirm the existence of solutions for the proposed control system on the mathematical modelling of COVID-19. The analytical results show that the control system designed adheres to the positivity and boundedness criteria, respectively the existence of control variable on the COVID-19 model is satisfied the all criteria. As a consequence, the following requirements for mathematical modeling of COVID-19 with several control are achieved: the control variable is not an empty set, and it is convex and closed. Furthermore, the right-hand side of the nonlinear equation is bounded by control variable and linear functions, whereas the functional objective is convex and bounded by a constant value, and theorem 5 contains a detailed proof of the existence of control variable in optimal problems.

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