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A PROPERLY EVEN HARMONIOUS LABELING OF SOME WHEEL GRAPH W_n FOR n IS EVEN

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ABSTRACT

A properly even harmonious labeling of a graph G with q edges is an injective mapping f from the vertices of graph G to the integers from 0 to $2q-1$ such that induces a bijective mapping f^* from the edges of G to $\{0,2,\dots,2q-2\}$ defined by $f^*(v_iv_j)=(f(v_i)+f(v_j))(mod 2q)$. A graph that has a properly even harmonious labeling is called a properly even harmonious graph. In this research, we will show the existence of a properly even harmonious labeling of some wheel graph W_n for n is even.

Keywords:

Properly Even Harmonious Labeling;
Properly Even Harmonious Graph;
Wheel graph.

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1. INTRODUCTION

A Graph G is a pair of sets (V, E) such that V is a nonempty set of vertices and E is a set (which can be empty) of unordered pairs of vertices that represent the edges [1]. An edge connecting a vertex u to a vertex v is denoted by uv or vu [2]. If each pair of vertices u and v of a graph has at least one single path which joins them, then this graph is called a connected graph, otherwise, it is called a disconnected graph. A finite consecutive sequence of vertices and edges of graph G is called walk, which can be written as $W = v_0e_1v_1e_2 \dots e_kv_k$ for $k \in \mathbb{Z}$. A cycle graph with n vertices is a closed walk which begins and ends at the same vertex but repeats no edges and denoted by C_n [3]. Moreover, a graph with n vertices where each pair of vertices joined by an edge is called a complete graph and denoted by K_n. A graph that is formed by adding a new vertex which is joined by an edge to each vertices of a cycle graph C_n is called a wheel graph and is denoted by W_n [4]. The graph that will be studied in this research is a connected wheel graph W_n.

Before discussing about graph labeling, we need to know the basic definition of function. A set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$ is called a function f from A to B or a mapping f of A into B. The function f is said to be injective if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ for $x_1, x_2 \in A$. Besides, the function f is said to be surjective if $f(A) = B$ which means the value of f is all of the elements of the set B. If function f is injective and surjective, then f is a bijective function [5].

Graph labeling is a function from the elements (vertices or edges, or both) of a graph to a set of elements (generally a non-negative or positive integer) by satisfying certain rules [6] and it was first introduced by Sedláček in 1963. The methods of graph labeling began with Rossa in 1967 and various methods continue to develop such as harmonious labeling which was introduced by Graham and Sloane in connection with error-connecting codes and channel assignment problems. A graph G with q edges is said to be harmonious if there is an injective function $f: V(G) \to \{0,1,2, \dots, q-1\}$ such that induces a bijective function $f^*: E(G) \to \{0,1,2, \dots, q-1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod q)$ for every $uv \in E(G)$ [7]. Many variants of harmonious labeling were developed, two of them are odd harmonious and even harmonious labeling. Liang and Bai defined a function f to be an odd harmonious labeling of graph G with q edges if f is an injection from the vertices of graph G to the integers from 0 to $2q - 1$ such that the induced mapping from the edges of G to the odd integers between 1 to $2q - 1$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection [8]. Whereas, an even harmonious labeling of graph G with q edges is an injective function $f: V(G) \to \{0,1,2, \dots, 2q\}$ such that induces a bijective function $f^*: E(G) \to \{0,2,4, \dots, 2q-2\}$ defined by $f^*(uv) = (f(u) + f(v))(mod 2q)$ for every $uv \in E(G)$ [9]. Many results relevant to harmonious, odd harmonious, even harmonious, and other variants of harmonious labeling had been surveyed by Gallian [10]. Besides, Lasim et al. found a function to build new labelings based on existing labelings [11]. In this study, we focus on a variant of even harmonious labeling, that is a properly even harmonious labeling. A function f is said to be a properly even harmonious labeling of a graph G with q edges if f is an injection from the vertices of graph G to the integers from 0 to $2q - 1$ which induces a bijective function f^* from the edges of G to $0, 2, \dots, 2(q - 1)$ defined by $f^*(v_iv_j) = (f(v_i) + f(v_j))(mod 2q)$ [12].

There are several properly even harmonious graphs that have been proven, such as graph $K_{1,m,n}$ and union of two coconut trees [13] , generalized Petersen graphs [14], and there were also some previous studies that relevant to wheel graph W_n, such as Gallian and Schoenhard that proved that wheel graph W_n, written as $K_1 + C_n$, for n is odd are properly even harmonious graphs [15]. Besides, according to the research of Gallian and Stewart about properly even harmonious labelings of some union certain graphs [12], Olivia stated a theorem that the union of a wheel graph W_n for $n \geq 3$ is odd with certain graphs having a properly even harmonious labeling [16]. Moreover, Diyanatut explained that the wheel graph W_n for n is even is not a properly even harmonious graph by labeling the center vertex $v_0 = 2$ and the other vertices $v_i = 4(i - 1)$ for every $i \geq 1$ [17]. Apart from this given label, there were no more explanation about wheel graph W_n for n is even. Therefore, in this research we will determine another label that satisfies a properly even harmonious labeling of some wheel graph W_n for n is even. In particular, for $4 \leq n \leq 14$.

2. RESEARCH METHODS

The method used in this research is the literature study method that studies references such as books or journal which are relevant to the research. Here are the steps to do the research:

1. Learning about the characterization of a properly even harmonious labeling of a graph
2. Determining a properly even harmonious labeling of wheel graph W_n for n is even using the algorithm shown in **Figure 1**

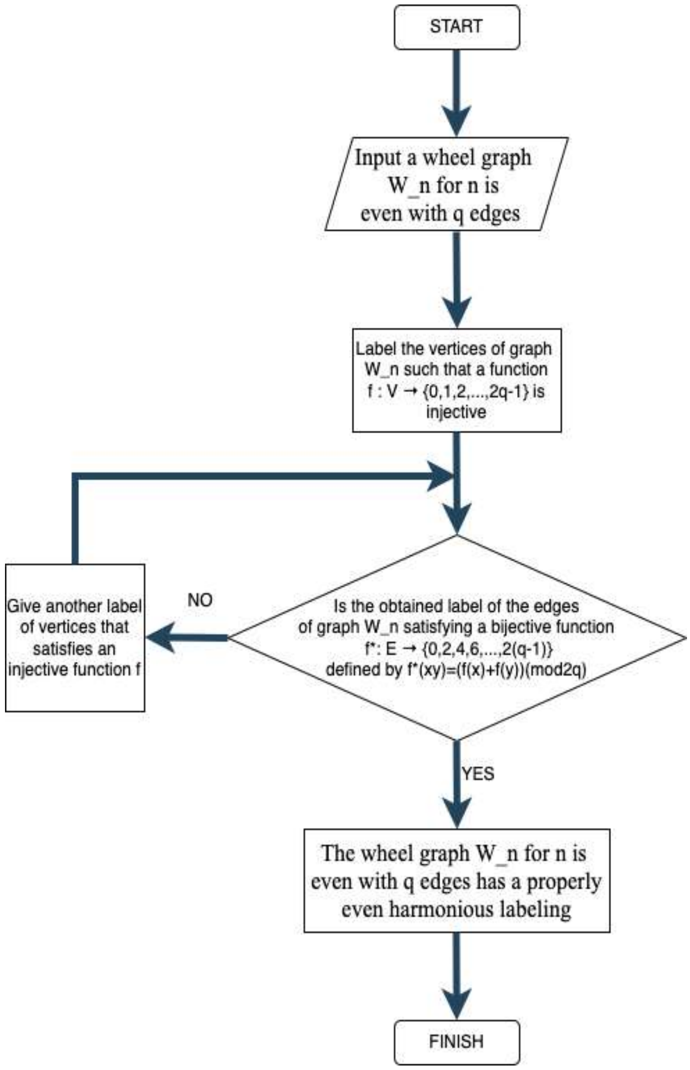


Figure 1. The algorithm to determine a properly even harmonious labeling

3. RESULTS AND DISCUSSION

Diyanatut stated that wheel graph W_n for n is even is not a properly even harmonious graph if a certain labeling is given as shown in **Figure 2 [17]**.

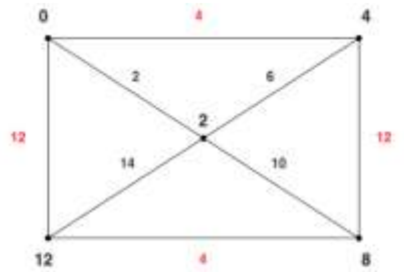


Figure 2. A vertices and edges labeling of graph W_4

In this research, we will show the existence of another labeling that satisfies a properly even harmonious labeling of some wheel graph W_n for n is even. In particular, for $4 \leq n \leq 14$. According to the algorithm shown in **Figure 1**, first we need to determine the vertices and edges labels for each of wheel graph W_n for $4 \leq n \leq 14$ is even as explained in the following steps.

3.1 Giving Notations of Vertices and Edges

The notations of vertices and edges of wheel graph W_n can be shown in **Figure 3**

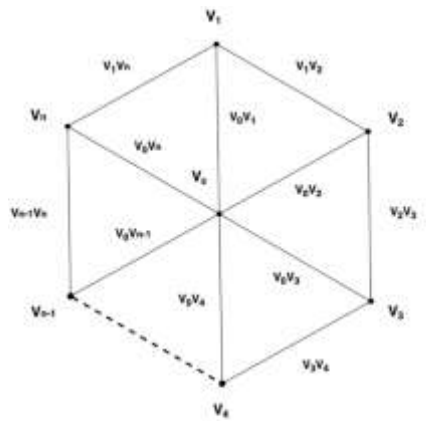
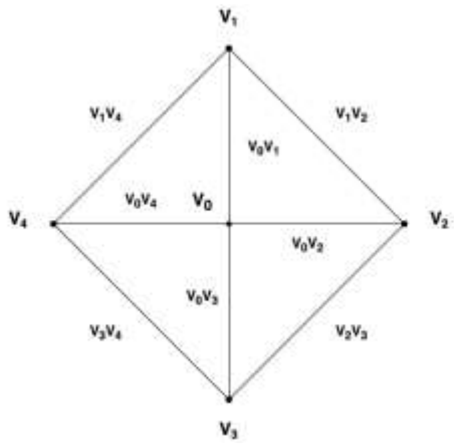
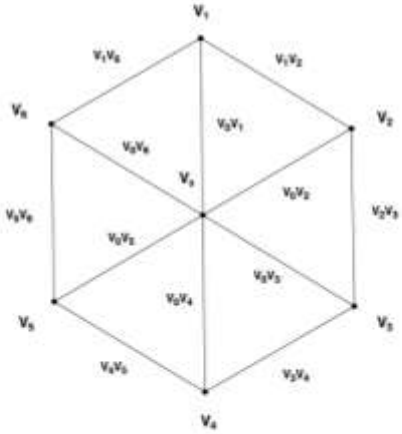


Figure 3. Notations of vertices and edges of graph W_n

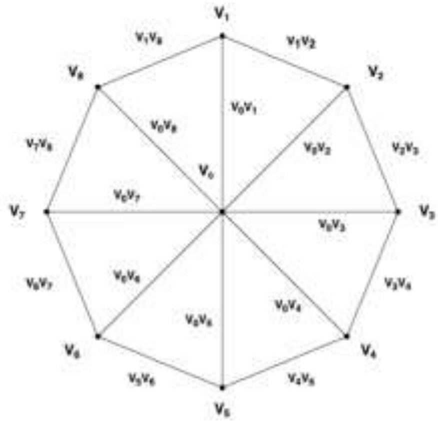
Example 1. The notations of vertices and edges for each of wheel graph W_n for $n = 4,6,8$ are shown in **Figure 4**



(a)



(b)



(c)

Figure 4. Notations of vertices and edges of graph (a) W_4, (b) W_6, and (c) W_8

Therefore, the notations of vertices and edges of wheel graph W_n can be written as

$$V(W_n) = \{v_0, v_1, v_2, v_3, v_4, \dots, v_{n-1}, v_n\} \quad (1)$$

$$E(W_n) = \{v_0v_1, v_0v_2, v_0v_3, v_0v_4, \dots, v_0v_{n-1}, v_0v_n, v_1v_2, v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_1v_n\}$$

It can be seen that the number of edges of wheel graph W_n is $q = 2n$.

3.2 Determining the Labels of Vertices and Edges

According to the notations written in **Equation 1**, the vertices labels of wheel graph W_n can be determined by defining a function $f: V(W_n) \to \{0,1,2,3, \dots, 2q-1\}$ or it can be written as $f: V(W_n) \to \{0,1,2,3, \dots, 4n-1\}$ which induces a function $f^*: E(W_n) \to \{0,2,4,6, \dots, 2(q-1)\}$ or it can be written as $f^*: E(W_n) \to \{0,1,2,3, \dots, 4n-2\}$ such that $f^*(v_iv_j) = (f(v_i) + f(v_j))(mod 4n)$ for every $v_iv_j \in E(W_n)$ and $i \neq j$.

Example 2. Assume that the vertices labels of wheel graph W_n for $4 \leq n \leq 14$ is even defined as a function $f: V(W_n) \to \{0,1,2,3, \dots, 4n-1\}$ such that for every $v_i \in V(W_n)$

$$f(v_0) = \begin{cases} 0 & for\ n = 4,6,8 \\ 14 & for\ n = 10,12,14 \end{cases}$$

$$f(v_1) = \begin{cases} 4 & for\ n = 4,6,8 \\ 0 & for\ n = 10,12,14 \end{cases}$$

$$f(v_2) = \begin{cases} 12 & for\ n = 4,6,8 \\ 4 & for\ n = 10,12,14 \end{cases}$$

$$f(v_3)=\begin{cases}10 \ for\ n=4,6,8\\ 2 \ for\ n=10,12,14\end{cases}$$
$$f(v_4)=\begin{cases}14 \ for\ n=4,6,8\\ 10 \ for\ n=10,12,14\end{cases}$$
$$f(v_5)=\begin{cases}18 \ for\ n=4,6,8\\ 20 \ for\ n=10,12,14\end{cases}$$
$$f(v_6)=\begin{cases}2 \ for\ n=6\\ 20\ for\ n=8\\ 12\ for\ n=10,12,14\end{cases} \quad (2)$$
$$f(v_7)=\begin{cases}8 \ for\ n=8\\ 24\ for\ n=10\\ 44\ for\ n=12,14\end{cases}$$
$$f(v_8)=\begin{cases}26 \ for\ n=8,10\\ 32\ for\ n=12\\ 34\ for\ n=14\end{cases}$$
$$f(v_9)=\begin{cases}34 \ for\ n=10\\ 6\ for\ n=12,14\end{cases}$$
$$f(v_{10})=\begin{cases}28 \ for\ n=10\\ 34\ for\ n=12\\ 32\ for\ n=14\end{cases}$$
$$f(v_{11})=\begin{cases}8 \ for\ n=12\\ 22\ for\ n=14\end{cases}$$
$$f(v_{12})=\begin{cases}36 \ for\ n=12\\ 30\ for\ n=14\end{cases}$$
$$f(v_{13})=36$$
$$f(v_{14})=28$$

which induces a function $f^*: E(W_n) \to \{0,1,2,3,\dots,4n-2\}$ such that for every $v_iv_j \in E(W_n)$ and $i \neq j$

$$f^*(v_0v_1)=\big(f(v_0)+f(v_1)\big)(mod\ 4n)=\begin{cases}(0+4)(mod\ 4n)\ for\ n=4,6,8\\ (14+0)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}4\ for\ n=4,6,8\\ 14\ for\ n=10,12,14\end{cases}$$
$$f^*(v_0v_2)=\big(f(v_0)+f(v_2)\big)(mod\ 4n)=\begin{cases}(0+12)(mod\ 4n)\ for\ n=4,6,8\\ (14+4)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}12\ for\ n=4,6,8\\ 18\ for\ n=10,12,14\end{cases}$$
$$f^*(v_0v_3)=\big(f(v_0)+f(v_3)\big)(mod\ 4n)=\begin{cases}(0+10)(mod\ 4n)\ for\ n=4,6,8\\ (14+2)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}10\ for\ n=4,6,8\\ 16\ for\ n=10,12,14\end{cases}$$
$$f^*(v_0v_4)=\big(f(v_0)+f(v_4)\big)(mod\ 4n)=\begin{cases}(0+14)(mod\ 4n)\ for\ n=4,6,8\\ (14+10)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}14\ for\ n=4,6,8\\ 24\ for\ n=10,12,14\end{cases}$$
$$f^*(v_0v_5)=\big(f(v_0)+f(v_5)\big)(mod\ 4n)=\begin{cases}(0+18)(mod\ 4n)\ for\ n=6,8\\ (14+20)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}18\ for\ n=6,8\\ 34\ for\ n=10,12,14\end{cases}$$
$$f^*(v_0v_6)=\big(f(v_0)+f(v_6)\big)(mod\ 4n)=\begin{cases}(0+2)(mod\ 4n)\ for\ n=6\\ (0+20)(mod\ 4n)\ for\ n=8\\ (14+12)(mod\ 4n)\ for\ n=10,12,14\end{cases}$$
$$=\begin{cases}2\ for\ n=6\\ 20\ for\ n=8\\ 26\ for\ n=10,12,14\end{cases}$$

$$f^*(v_0v_7) = \left(f(v_0) + f(v_7)\right)(mod\ 4n) = \begin{cases} (0+8)(mod\ 4n)\ for\ n=8 \\ (14+24)(mod\ 4n)\ for\ n=10 \\ (14+44)(mod\ 4n)\ for\ n=12,14 \end{cases}$$
$$= \begin{cases} 8\ for\ n=8 \\ 38\ for\ n=10 \\ 10\ for\ n=12 \\ 2\ for\ n=14 \end{cases}$$
$$f^*(v_0v_8) = \left(f(v_0) + f(v_8)\right)(mod\ 4n) = \begin{cases} (0+26)(mod\ 4n)\ for\ n=8 \\ (14+26)(mod\ 4n)\ for\ n=10 \\ (14+32)(mod\ 4n)\ for\ n=12 \\ (14+34)(mod\ 4n)\ for\ n=14 \end{cases}$$
$$= \begin{cases} 26\ for\ n=8 \\ 0\ for\ n=10 \\ 46\ for\ n=12 \\ 48\ for\ n=14 \end{cases} \tag{3}$$
$$f^*(v_0v_9) = \left(f(v_0) + f(v_9)\right)(mod\ 4n) = \begin{cases} (14+34)(mod\ 4n)\ for\ n=10 \\ (14+6)(mod\ 4n)\ for\ n=12,14 \end{cases}$$
$$= \begin{cases} 8\ for\ n=10 \\ 20\ for\ n=12,14 \end{cases}$$
$$f^*(v_0v_{10}) = \left(f(v_0) + f(v_{10})\right)(mod\ 4n) = \begin{cases} (14+28)(mod\ 4n)\ for\ n=10 \\ (14+34)(mod\ 4n)\ for\ n=12 \\ (14+32)(mod\ 4n)\ for\ n=14 \end{cases}$$
$$= \begin{cases} 2\ for\ n=10 \\ 0\ for\ n=12 \\ 46\ for\ n=14 \end{cases}$$
$$f^*(v_0v_{11}) = \left(f(v_0) + f(v_{11})\right)(mod\ 4n) = \begin{cases} (14+8)(mod\ 4n)\ for\ n=12 \\ (14+22)(mod\ 4n)\ for\ n=14 \end{cases}$$
$$= \begin{cases} 22\ for\ n=12 \\ 36\ for\ n=14 \end{cases}$$
$$f^*(v_0v_{12}) = \left(f(v_0) + f(v_{12})\right)(mod\ 4n) = \begin{cases} (14+36)(mod\ 4n)\ for\ n=12 \\ (14+30)(mod\ 4n)\ for\ n=14 \end{cases}$$
$$= \begin{cases} 2\ for\ n=12 \\ 44\ for\ n=14 \end{cases}$$
$$f^*(v_0v_{13}) = \left(f(v_0) + f(v_{13})\right)(mod\ 4n) = (14+36)(mod\ 4n) = 50$$
$$f^*(v_0v_{14}) = \left(f(v_0) + f(v_{14})\right)(mod\ 4n) = (14+28)(mod\ 4n) = 42$$
$$f^*(v_1v_2) = \left(f(v_1) + f(v_2)\right)(mod\ 4n) = \begin{cases} (4+12)(mod\ 4n)\ for\ n=4,6,8 \\ (0+4)(mod\ 4n)\ for\ n=10,12,14 \end{cases}$$
$$= \begin{cases} 0\ for\ n=4 \\ 16\ for\ n=6,8 \\ 4\ for\ n=10,12,14 \end{cases}$$
$$f^*(v_2v_3) = \left(f(v_2) + f(v_3)\right)(mod\ 4n) = \begin{cases} (12+10)(mod\ 4n)\ for\ n=4,6,8 \\ (4+2)(mod\ 4n)\ for\ n=10,12,14 \end{cases}$$
$$= \begin{cases} 6\ for\ n=4 \\ 22\ for\ n=6,8 \\ 6\ for\ n=10,12,14 \end{cases}$$
$$f^*(v_3v_4) = \left(f(v_3) + f(v_4)\right)(mod\ 4n) = \begin{cases} (10+14)(mod\ 4n)\ for\ n=4,6,8 \\ (2+10)(mod\ 4n)\ for\ n=10,12,14 \end{cases}$$
$$= \begin{cases} 8\ for\ n=4 \\ 0\ for\ n=6 \\ 24\ for\ n=8 \\ 12\ for\ n=10,12,14 \end{cases}$$
$$f^*(v_4v_5) = \left(f(v_4) + f(v_5)\right)(mod\ 4n) = \begin{cases} (14+18)(mod\ 4n)\ for\ n=6,8 \\ (10+20)(mod\ 4n)\ for\ n=10,12,14 \end{cases}$$

$$= \begin{cases} 8 \; for \; n = 6 \\ 0 \; for \; n = 8 \\ 30 \; for \; n = 10,12,14 \end{cases}$$

$$f^*(v_5v_6) = \big(f(v_5) + f(v_6)\big)(mod\; 4n) = \begin{cases} (18+2)(mod\; 4n) \; for \; n = 6 \\ (18+20)(mod\; 4n) \; for \; n = 8 \\ (20+12)(mod\; 4n) for \; n = 10,12,14 \end{cases}$$

$$= \begin{cases} 20 \; for \; n = 6 \\ 6 \; for \; n = 8 \\ 32 \; for \; n = 10,12,14 \end{cases}$$

$$f^*(v_6v_7) = \big(f(v_6) + f(v_7)\big)(mod\; 4n) = \begin{cases} (20+8)(mod\; 4n) \; for \; n = 8 \\ (12+24)(mod\; 4n) for \; n = 10 \\ (12+44)(mod\; 4n) for \; n = 12,14 \end{cases}$$

$$= \begin{cases} 28 \; for \; n = 8 \\ 36 \; for \; n = 10 \\ 8 \; for \; n = 12 \\ 0 \; for \; n = 14 \end{cases}$$

$$f^*(v_7v_8) = \big(f(v_7) + f(v_8)\big)(mod\; 4n) = \begin{cases} (8+26)(mod\; 4n) \; for \; n = 8 \\ (24+26)(mod\; 4n) for \; n = 10 \\ (44+32)(mod\; 4n) for \; n = 12 \\ (44+34)(mod\; 4n) for \; n = 14 \end{cases}$$

$$= \begin{cases} 2 \; for \; n = 8 \\ 10 \; for \; n = 10 \\ 28 \; for \; n = 12 \\ 22 \; for \; n = 14 \end{cases}$$

$$f^*(v_8v_9) = \big(f(v_8) + f(v_9)\big)(mod\; 4n) = \begin{cases} (26+34)(mod\; 4n) for \; n = 10 \\ (32+6)(mod\; 4n) for \; n = 12 \\ (34+6)(mod\; 4n) for \; n = 14 \end{cases}$$

$$= \begin{cases} 20 \; for \; n = 10 \\ 38 \; for \; n = 12 \\ 40 \; for \; n = 14 \end{cases}$$

$$f^*(v_9v_{10}) = \big(f(v_9) + f(v_{10})\big)(mod\; 4n) = \begin{cases} (34+28)(mod\; 4n) for \; n = 10 \\ (6+34)(mod\; 4n) for \; n = 12 \\ (6+32)(mod\; 4n) for \; n = 14 \end{cases}$$

$$= \begin{cases} 22 \; for \; n = 10 \\ 40 \; for \; n = 12 \\ 38 \; for \; n = 14 \end{cases}$$

$$f^*(v_{10}v_{11}) = \big(f(v_{10}) + f(v_{11})\big)(mod\; 4n) = \begin{cases} (34+8)(mod\; 4n) for \; n = 12 \\ (32+22)(mod\; 4n) for \; n = 14 \end{cases}$$

$$= \begin{cases} 42 \; for \; n = 12 \\ 54 \; for \; n = 14 \end{cases}$$

$$f^*(v_{11}v_{12}) = \big(f(v_{11}) + f(v_{12})\big)(mod\; 4n) = \begin{cases} (8+36)(mod\; 4n) for \; n = 12 \\ (22+30)(mod\; 4n) for \; n = 14 \end{cases}$$

$$= \begin{cases} 44 \; for \; n = 12 \\ 52 \; for \; n = 14 \end{cases}$$

$$f^*(v_{12}v_{13}) = \big(f(v_{12}) + f(v_{13})\big)(mod\; 4n) = (30+36)(mod\; 4n) = 10$$
$$f^*(v_{13}v_{14}) = \big(f(v_{13}) + f(v_{14})\big)(mod\; 4n) = (36+26)(mod\; 4n) = 8$$
$$f^*(v_1v_4) = \big(f(v_1) + f(v_4)\big)(mod\; 4n) = (4+14)(mod\; 4n) = 2$$
$$f^*(v_1v_6) = \big(f(v_1) + f(v_6)\big)(mod\; 4n) = (4+2)(mod\; 4n) = 6$$
$$f^*(v_1v_8) = \big(f(v_1) + f(v_8)\big)(mod\; 4n) = (4+26)(mod\; 4n) = 30$$
$$f^*(v_1v_{10}) = \big(f(v_1) + f(v_{10})\big)(mod\; 4n) = (0+28)(mod\; 4n) = 28$$
$$f^*(v_1v_{12}) = \big(f(v_1) + f(v_{12})\big)(mod\; 4n) = (0+36)(mod\; 4n) = 36$$
$$f^*(v_1v_{14}) = \big(f(v_1) + f(v_{14})\big)(mod\; 4n) = (0+28)(mod\; 4n) = 28$$

Therefore, the obtained vertices labels in **Equation (2)** and edges labels in **Equation (3)** for each of wheel graph W_n for $4 \le n \le 14$ is even can be shown in **Figure 5**

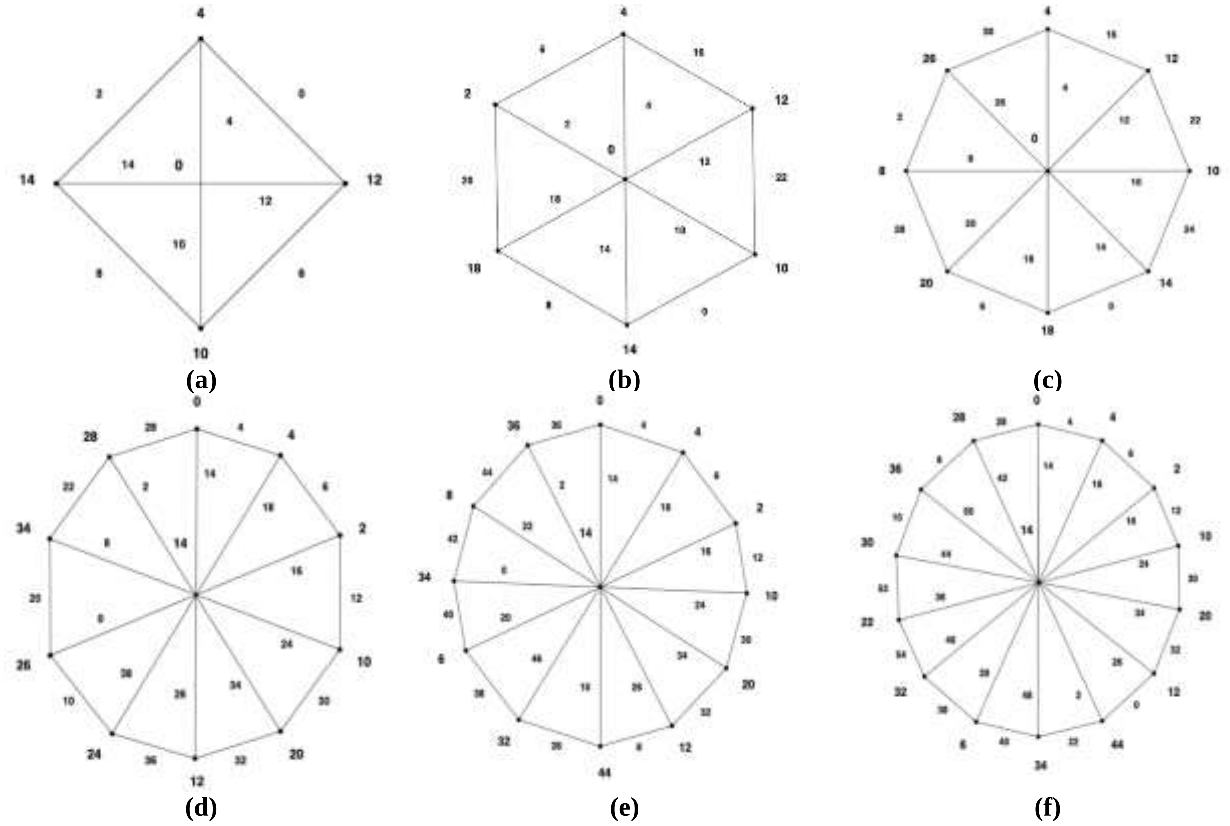


Figure 5. A Vertices and Edges Labeling of graph (a) W_4, (b) W_6, (c) W_8, (d) W_{10}, (e) W_{12}, and (f) W_{14}

Next, the following theorem proves that the vertices and edges labeling for each of wheel graphs in **Figure 5** is a properly even harmonious labeling.

Theorem 1. *Each of wheel graph W_n for $4 \le n \le 14$ is even is a properly even harmonious graph.*

Proof. The vertices labeling of wheel graph W_n in **Equation (2)**, for every $v_i \in V(W_n)$ and $4 \le n \le 14$ is even, satisfies a function $f: V(W_n) \to \{0,1,2,3,\dots,4n-1\}$ which can be defined as

$$f(v_i) = \begin{cases} 4i\,;\, i = 0,1 \; and \; n = 4,6,8 \\ 12\,;\, i = 2 \; and \; n = 4,6,8 \\ 4i-2\,;\, i = 3,4,5 \; and \; n = 4,6,8 \\ 2\,;\, i = 6 \; and \; n = 6;\; i = 3 \; and \; n = 10,12,14 \\ 8\,;\, i = 7 \; and \; n = 8 \\ 26\,;\, i = 8 \; and \; n = 8 \\ 14\,;\, i = 0 \; and \; n = 10,14 \\ 0\,;\, i = 1 \; and \; n = 10,12,14 \\ 2i\,;\, i = 2,6,14 \; and \; n = 10,12,14 \\ 3i-2\,;\, i = 4,10 \; and \; n = 10 \\ 2i+10\,;\, i = 5,7,8 \; and \; n = 10 \\ 34\,;\, i = 8 \; and \; n = 14;\; i = 9 \; and \; n = 10 \\ 10\,;\, i = 4 \; and \; n = 12,14 \\ 20\,;\, i = 6 \; and \; n = 8\,;\; i = 5 \; and \; n = 12,14 \\ 44\,;\, i = 7 \; and \; n = 12,14 \\ 32\,;\, i = 8 \; and \; n = 12\,;\, i = 10 \; and \; n = 14 \\ i-3\,;\, i = 9,11 \; and \; n = 12 \\ 2i+14\,;\, i = 0,10 \; and \; n = 12 \\ 36\,;\, i = 12 \; and \; n = 12;\, i = 13 \; and \; n = 14 \\ 6;\, i = 9 \; and \; n = 14 \\ 22\,;\, i = 11 \; and \; n = 14 \\ 30\,;\, i = 12 \; and \; n = 14 \end{cases} \quad (4)$$

such that for every $v_1v_n, v_0v_i, v_iv_{i+1} \in E(W_n)$ satisfies the induced function $f^*: E(W_n) \to \{0,2,4,6, \dots, 4n-2)\}$ defined as

$$f^*(v_1v_n) = \begin{cases} 2n-6\,; n = 4,6 \\ 30\,; n = 8 \\ 28\,; n = 10,14 \\ 36\,; n = 12 \end{cases} \quad (5)$$

$$f^*(v_0v_i) = \begin{cases} 4\,; i = 1 \text{ and } n = 4,6,8 \\ 12\,; i = 2 \text{ and } n = 4,6,8 \\ 4i-2\,; i = 3,4,5 \text{ and } n = 4,6,8 \\ 2\,; i = 6 \text{ and } n = 6; i = 7 \text{ and } n = 14; i = 12 \text{ and } n = 12 \\ 20\,; i = 6 \text{ and } n = 8; i = 9 \text{ and } n = 12,14 \\ 8\,; i = 7 \text{ and } n = 8; i = 9 \text{ and } n = 10 \\ 26\,; i = 8 \text{ and } n = 8; i = 8 \text{ and } n = 8 \\ 14\,; i = 1 \text{ and } n = 10,12,14 \\ 3i+12\,; i = 2,4 \text{ and } n = 10,12,14; i = 10 \text{ and } n = 10 \\ 16\,; i = 3 \text{ and } n = 10,12,14 \\ 2i+24\,; i = 5,7,8 \text{ and } n = 10; i = 13 \text{ and } n = 14 \\ 2i+14\,; i = 6,11,14 \text{ and } n = 10,14; i = 6 \text{ and } n = 12 \\ 34; i = 5 \text{ and } n = 12,14 \\ 10\,; i = 7 \text{ and } n = 12 \\ 46\,; i = 8 \text{ and } n = 12; i = 10 \text{ and } n = 14 \\ 48\,; i = 8 \text{ and } n = 14 \\ 22\,; i = 11 \text{ and } n = 12 \\ 0\,; i = 10 \text{ and } n = 12 \\ 44\,; i = 12 \text{ and } n = 14 \end{cases} \quad (6)$$

$$f^*(v_iv_{i+1}) = \begin{cases} 16(mod\ 4n); i = 1 \text{ and } n = 4,6,8 \\ 22(mod\ 4n); i = 2 \text{ and } n = 4,6,8 \\ 8i(mod\ 4n)\,; i = 3,4 \text{ and } n = 4,6,8 \\ 20\,; i = 5 \text{ and } n = 6 \\ 28\,; i = 6 \text{ and } n = 8; i = 7 \text{ and } n = 12 \\ 2\,; i = 7 \text{ and } n = 8 \\ 2i+2\,; i = 1,2 \text{ and } n = 10,12,14 \\ 12\,; i = 3 \text{ and } n = 10,12,14 \\ 2i+22\,; i = 4,5 \text{ and } n = 10,12,14\,; i = 9,10,11 \text{ and } n = 12 \\ 36\,; i = 6 \text{ and } n = 10 \\ 10\,; i = 7 \text{ and } n = 10; i = 12 \text{ and } n = 14 \\ (2i+44)(mod\ 4n)\,; i = 8,9 \text{ and } n = 10 \\ 56(mod\ 4n); i = 6 \text{ and } n = 12,14 \\ 38\,; i = 8 \text{ and } n = 12; i = 9 \text{ and } n = 14 \\ 22\,; i = 7 \text{ and } n = 14 \\ 40\,;\ i = 8\ \text{ and } n = 14 \\ 54\,; i = 10 \text{ and } n = 14 \\ 52\,; i = 11 \text{ and } n = 14 \\ 6\,; i = 5 \text{ and } n = 8 \\ 8\,; i = 13 \text{ and } n = 14 \end{cases} \quad (7)$$

It can be seen from **Equation (4),** that each vertex of wheel graph W_n for $4 \le n \le 14$ is even has a different label, which means that for every $v_i, v_j \in V(W_n)$, if $v_i \neq v_j$, then $f(v_i) \neq f(v_j)$. Therefore, f is an injective function. The **Equation (5) - Equation (7)** show that each edge of wheel graph W_n for $4 \le n \le 14$ is even has a different label, which means that for every edge $e_i, e_j \in E(W_n)$ and $e_i \neq e_j$, then $f^*(e_i) \neq f^*(e_j)$. Therefore, f^* is an injective function. Besides, for every $y \in \{0,2,4,6, \dots, 2(q-1)\}$ is the label of an edge $e \in E(W_n)$ such that $y = f^*(e) = f^*(v_i v_j) = (f(v_i) + f(v_j))(mod 2q)$. It means that f^* is a surjective and so a bijective function.

According to the definition of a properly even harmonious labeling [12], the vertices and edges labeling of wheel graph W_n for $4 \leq n \leq 14$ is even as shown in Figure 5 satisfies a properly even harmonious labeling. Therefore, each of wheel graph W_n for $4 \leq n \leq 14$ is even is a properly even harmonious graph. ■

4.CONCLUSIONS

In this research, we proved that each of wheel graph W_n for $4 \leq n \leq 14$ is even is a properly even harmonious graph, which means that there exists a properly even harmonious labeling in each of these graphs as shown in Figure 5. The general formula of the properly even harmonious labeling of wheel graph W_n for n is even can be studied for further research.

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