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BAYES ESTIMATION OF EXPONENTIALLY DISTRIBUTED SURVIVAL DATA UNDER SYMMETRIC AND ASYMMETRIC LOSS FUNCTIONS

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ABSTRACT

In a research study, population data are often not available, so the population parameter is unknown. Meanwhile, knowledge about the population parameter is needed to know the characteristics of the studied population. Therefore, it is needed to estimate the parameter of the population which can be estimated by sample data. There are several methods of parameter estimation which are generally classified into classical and Bayesian method. This research studied the Bayesian parameter estimation method to determine the parameters of the exponentially distributed survival data associated with the reliability measure of the estimates under symmetric and asymmetric loss functions for complete sample data in a closed form. The symmetric loss functions used in this research are Squared Error Loss Function (SELF) and Minimum Expected Loss Function (MELF). The asymmetric loss functions used are the General Entropy Loss Function (GELF) and Linex Loss Function (LLF). Performance of some loss functions used in this research are then compared through numerical simulation to select the best loss function in determining the parameter estimation of the exponentially distributed survival data. We also studied which loss function is best for underestimation and overestimation modeling. Based on simulation results, the Bayes estimates using MELF is the best method to estimate population parameters of the exponentially distributed survival data for the overestimation modeling, while LLF is the best for the underestimation modeling. We provided direct application in a case study of fluorescence lamp survival data. The results show that the best method to estimate the parameter θ *of the standard fluorescence life data is using LLF for underestimation with* $\hat{\theta}_L = 0.480173$ *and MELF for overestimation with* $\hat{\theta}_M = 0.410872$.

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1. INTRODUCTION

In statistics, the expected conclusion of a study is the conclusion for a wide scope, i.e., population. However, it isn't easy to obtain the population data, so we use sample data as the population subset. Based on that fact, the characteristic of the population called parameter is also unknown; therefore, it can be estimated using sample data. In estimation theory, the characteristics of the sample data that can describe population characteristics are known as the estimator of a population parameter.

Estimation of population parameters can be done using some methods; the simplest method is a method of moment introduced by Karl Pearson in 1800 [1]. The second method is the maximum likelihood estimation method. The method is used to estimate the value of the population parameter if the distribution of the population is known. The other method is the Bayesian method, which is fundamentally different from the previous methods. Bayesian method has a prior distribution as the subjective distribution, which is then adjusted to the sample information. The result is referred to as posterior distribution.

One of the data whose population parameters are interesting to estimate is lifetime/survival data. Lifetime is the observed time interval of an individual when first entering into observation until it is out of observation. Survival function distribution based on a certain knowledge or assumption about its population distribution is included in the parametric function. Some population distributions that can be used to describe lifetime data are Exponential distribution, Weibull distribution, Gamma distribution, Rayleigh distribution, etc [2]. Furthermore, the Exponential model is one of the most important models in lifetime data analysis [3]– [5]. The exponential distribution is crucial in survival data modeling due to its constant hazard rate, lack of memory property, ease of mathematical analysis, and widespread use in survival analysis methods like Cox proportional hazards regression. Its constant hazard rate makes it suitable for scenarios where the risk of an event remains constant over time. Moreover, its memoryless property simplifies modeling where events occur randomly and independently of past occurrences. Mathematically, it offers simple analysis with readily computable probability density and cumulative distribution functions. Despite not always perfectly fitting survival data, it serves as a foundational assumption in survival analysis, providing valuable insights and a starting point for more complex models [21], [22].

The loss function in Bayesian inference measures how well a statistical model or parameter estimation produced by Bayesian approaches predicts or reconstructs the observed data. This loss function allows for a qualitative assessment of the estimation quality by considering the deviation between the observed values and those predicted by the model. Thus, the optimal model selection or parameter estimation can be achieved by minimizing the resulting loss [23]. Some loss functions are popular in modeling survival analysis using the Bayesian method, such as squared error loss function [6], minimum expected loss function [7]–[9], weighted minimum expected loss function [10], general entropy loss function [9], [11]–[13], linex loss function [9], [12], [14]–[17]. Various results have been derived from the estimation using the previously mentioned loss functions. In general, squared error loss function is the simplest loss function in Bayesian modeling. It has been shown that the asymmetric loss functions defined by [6] are more appropriate than the typical squared error loss function. Another study showed that minimum expected loss function estimator has smaller root mean squared errors than ordinary Bayes and maximum likelihood estimators [7], [8]. Many studies also showed that Bayes estimators perform well under asymmetric loss functions, such as general entropy loss function and linex loss function, when underestimation is more serious than overestimation [9], [12], [13]. Each of the previous studies conducts a partial comparison to compare the loss function, such as only comparing the squared error loss function and general entropy loss function, squared error loss function, and linex loss function, etc. Therefore, in this research, we try to compare two symmetric and two asymmetric loss functions for complete sample data in a closed form and study the best estimator for each underestimation and overestimation modeling. We also provide the direct application using fluorescence lamp survival data.

2. RESEARCH METHODS

2.1 Exponentially Distributed Survival Data

The survival data function, often referred to as the survival function, represents the probability that an event of interest (such as death, failure, or any other outcome) has not occurred by a certain time point. It gives an estimate of the proportion of individuals or units in a population that has survived up to a given time.

The survival function is a fundamental concept in survival analysis, a branch of statistics dealing with time-to-event data [21], [22]. In this study, we used exponentially distributed survival data. Exponential distribution is widely used in the field of Statistics, especially in the reliability and queue theory. Exponential distribution is a special form of Gamma distribution with $\alpha = 1$ and $\beta = \frac{1}{\theta}$. Its probability density function is defined by

$$f(x,\theta) = \theta\, e^{-x\,\theta}, \qquad x \geq 0 \tag{1}$$

where θ is the rate of exponential distribution or the hazard value [18]. The hazard function itself is a fundamental concept in survival analysis that describes the instantaneous rate of occurrence of an event (such as death or failure) at time t, given that the individual or unit has survived up to time t [21], [22].

Furthermore, parameter θ of the exponentially distributed survival data is assumed to follow Gamma distribution with parameter α and β. The distribution of the parameter in such kind of Bayesian method is called a conjugate prior distribution defined by

$$\pi(\theta) = \frac{1}{\Gamma\,(\alpha)\beta^{\alpha}}\, \theta^{\alpha-1}\, e^{-\frac{\theta}{\beta}} \tag{2}$$

where α is a complex number whose real part is positive and $\Gamma\,(\alpha)$ is a gamma function defined by

$$\Gamma\,(\alpha) = \int_0^{\infty} x^{\alpha-1}\, e^{-x}\; dx, \qquad \alpha > 0 \tag{3}$$

2.2 Bayesian Estimation Method

In classical approach, parameter θ is a fixed quantities whose value is unknown. In the Bayesian approach, it is considered a quantity whose variation is described by a probability distribution (prior distribution). This is a subjective distribution based on the researcher's beliefs and is formulated before the sample data is taken. The adjustment of the prior distribution is called the posterior distribution. This adjustment is made using Bayes rules [1] given by

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} \tag{4}$$

The population parameter can be estimated in Bayesian estimation by minimizing the risk factor using a certain loss function. In this study, we use two types of loss functions: symmetric and asymmetric. The symmetric loss functions used in this study are the Squared Error Loss Function (SELF) and Minimum Expected Loss Function (MELF), while the asymmetric loss functions used are the General Entropy Loss Function (GELF) and Linex Loss Function (LLF). The four loss functions are given by

$$L(\theta,\hat{\theta}_S) = \left(\hat{\theta}_S - \theta\right)^2, \quad 0 < \theta < \infty \tag{5}$$

$$L(\theta,\hat{\theta}_M) = w\left(\hat{\theta}_M - \theta\right)^2, \quad 0 < \theta < \infty \tag{6}$$

$$L(\theta,\hat{\theta}_G) = \left(\frac{\hat{\theta}_G}{\theta}\right) - \alpha_1 \ln\left(\frac{\hat{\theta}_G}{\theta}\right) - 1, \quad 0 < \theta < \infty, \alpha_1 \neq 0 \tag{7}$$

$$L(\theta,\hat{\theta}_L) = \exp\left[\alpha_2\left(\hat{\theta}_L - \theta\right)\right] - \alpha_2\left(\hat{\theta}_L - \theta\right) - 1, \quad 0 < \theta < \infty, \alpha_2 \neq 0 \tag{8}$$

3. RESULTS AND DISCUSSION

3.1 Posterior Distribution

Let X is a survival data follows Exponential distribution with parameter θ. In Bayesian analysis, parameter θ is considered to follow a certain distribution known as prior distribution. In this case, Gamma distribution is chosen as the conjugate prior distribution for Exponential distribution with parameter α and β. By matching the mean and variance of Gamma distribution with the mean and variance of Exponential distribution, we obtain $\alpha = 1$ and $\beta = \bar{x}$. The posterior distribution can be determined by multiplying the prior distribution with the sample information obtained from its likelihood, and the prior is independent of the likelihood [19]. The likelihood function is obtained as below

$$f(\tilde{x}|\theta) = \Pi_{i=1}^{n} f(x,\theta) \ = \theta^{n}\, e^{-\sum_{i=1}^{n} x_i\, \theta} \tag{9}$$

The joint probability function of $\tilde{X}$ and θ is obtained by multiplying Equation (9) and the prior distribution as in Equation (2) so that

$$f(\tilde{x},\theta) = \ f(\tilde{x}|\theta)\cdot \pi(\theta) \ = \ e^{-\theta\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)} \frac{\theta^{n+\alpha-1}}{\Gamma\,(\alpha)\beta^{\alpha}} \tag{10}$$

While the probability density function of the marginal distribution of $\tilde{X}$ is

$$m(\tilde{x}) \ = \int_{-\infty}^{\infty} f(\tilde{x},\theta)\, d\theta \ = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\beta^{\alpha}\left(\sum_{i=1}^{n} x_i \ + \frac{1}{\beta}\right)^{n+\alpha}} \tag{11}$$

By using Equation (10) and the Bayes rules from Equation (4), the posterior distribution is obtained as follows

$$\pi(\theta|\tilde{x}) \ = \frac{f(\tilde{x}|\theta)\pi(\theta)}{m(\tilde{x})} \ = \frac{e^{-\left(\sum_{i=1}^{n} x_i + \frac{1}{\beta}\right)}\,\theta^{n+\alpha-1}}{\Gamma(n+\alpha)\left(\frac{1}{\sum_{i=1}^{n} x_i \ + \frac{1}{\beta}}\right)}, \qquad \theta > 0 \tag{12}$$

Therefore, the posterior distribution of $\theta|\tilde{x}$ follows Gamma distribution with parameter $\alpha^{*} \ = n+\alpha$ and $\beta^{*} \ = \frac{1}{\sum_{i=1}^{n} x_i + \frac{1}{\beta}}$.

3.2 Bayesian Estimates using Symmetric and Asymmetric Loss Functions

In this section, we determined Bayesian estimates of exponentially distributed survival data for complete sample data using symmetric and asymmetric loss functions as in Equation (5) to Equation (8). Unlike any other research which solves the estimation using numerical method, we obtained the estimation in a closed form. Bayesian estimator of θ can be obtained by minimizing the expectation of the loss function with respect to the estimator $\hat{\theta}$ as follows

$$\frac{d}{d\,\hat{\theta}}\left(E\left(L(\theta,\hat{\theta})\right)\right) = 0 \tag{13}$$

3.2.1 Bayesian Estimates using Symmetric Loss Functions

We used two types of symmetric loss functions, i.e., Squared Error Loss Function (SELF) and Minimum Expected Loss Function (MELF). Squared Error Loss Function (SELF) is a symmetric loss function used to minimize the risk factor in Bayesian estimates, where the formula of SELF is referred to Equation (5). By assuming that both positive and negative errors are serious and the Bayesian estimator of θ using this function is notated by $\hat{\theta}_S$, the estimator is obtained by minimizing the expectation of the SELF with respect to $\hat{\theta}_S$, using Equation (13). With some calculations, the Bayesian estimator for θ using the SELF approach is obtained as

$$\hat{\theta}_S = E(\theta) \tag{14}$$

Therefore, the closed form of $\hat{\theta}_S$ is

$$\hat{\theta}_S = E(\theta) = \int_0^\infty \theta\, f(\theta)\, d\theta = \frac{n+\alpha}{\sum_{i=1}^n x_i + \frac{1}{\beta}} \tag{15}$$

The second type of symmetric loss function is MELF. Minimum Error Loss Function (MELF) is a symmetric loss function used to minimize the risk factor in Bayesian estimates, where the formula of MELF is referred to **Equation (6)** where w is a weighting function. For $w = 1$, the loss function is referred to the SELF formula. In this study, we used $w = \theta^{-2}$ [7], so that **Equation (6)** become

$$L(\theta, \hat{\theta}_M) = \frac{\left(\hat{\theta}_M - \theta\right)^2}{\theta^2} \tag{16}$$

$\hat{\theta}_M$ is Bayesian estimator of θ using MELF approach. By using **Equation (13)**, Bayesian estimator $\hat{\theta}_M$ is obtained

$$\hat{\theta}_M = \frac{E\left(\frac{1}{\theta}\right)}{E\left(\frac{1}{\theta^2}\right)} \tag{17}$$

From **Equation (17)**, we determined the value of the numerator and denominator as follows

$$E\left(\frac{1}{\theta}\right) = \int_0^\infty \frac{1}{\theta} f(\theta)\, d\theta = \frac{\left(\sum_{i=1}^n x_i + \frac{1}{\beta}\right)}{(n+\alpha-1)} \tag{18}$$

$$E\left(\frac{1}{\theta^2}\right) = \int_0^\infty \frac{1}{\theta^2} f(\theta)\, d\theta = \frac{\left(\sum_{i=1}^n x_i + \frac{1}{\beta}\right)^2}{(n+\alpha-1)(n+\alpha-2)} \tag{19}$$

Therefore, substitute **Equation (18)** and **Equation (19)** into **Equation (17)**, we obtained the closed form of $\hat{\theta}_M$, i.e.

$$\hat{\theta}_M = \frac{n+\alpha-2}{\sum_{i=1}^n x_i + \frac{1}{\beta}} \tag{20}$$

3.2.2 Bayesian Estimates using Asymmetric Loss Functions

We used the General Entropy Loss Function (GELF) and Linex Loss Function (LLF) for the asymmetric loss functions. In some estimations using the symmetric loss function, it was found that some matters, like the assumption, are unsuitable. Over-estimation may be more serious than under-estimation otherwise. In this situation, the asymmetric loss function is considered more accurate. General Entropy Loss Function (GELF) is one of the types of asymmetric loss functions [20] with the formula given by **Equation (7)** where parameter α_1 represents asymmetric deviation. By using **Equation (13)**, the Bayesian estimator $\hat{\theta}_G$ is obtained as follows

$$\hat{\theta}_G = \left(E(\theta^{-\alpha_1})\right)^{-\frac{1}{\alpha_1}} \tag{21}$$

The value of $E(\theta^{-\alpha_1})$ in **Equation (21)** is given by

$$E(\theta^{-\alpha_1}) = \int_0^\infty \theta^{-\alpha_1} f(\theta)\, d\theta = \frac{\Gamma(n+\alpha-\alpha_1)}{\Gamma(n+\alpha)\left(\sum_{i=1}^n x_1 + \frac{1}{\beta}\right)^{-\alpha_1}} \tag{22}$$

Therefore, the closed form of Bayesian estimator $\hat{\theta}_G$ is obtained by substituting **Equation (22)** into **Equation (21)**

$$\hat{\theta}_G = \left(\frac{\Gamma(n+\alpha-\alpha_1)}{\Gamma(n+\alpha)}\right)^{-\frac{1}{\alpha_1}} \left(\frac{1}{\sum_{i=1}^n x_i + \frac{1}{\beta}}\right) \tag{23}$$

The second type of the asymmetric loss function used in this paper is LLF. Linex Loss Function (LLF) is the abbreviation of Linear Exponential Loss Function and the other type of asymmetric loss function. The formula of LLF is given by **Equation (8)**, where parameter α_2 is used to determine the form of the linex loss function. When the estimation error $\hat{\theta} = \theta$ is in interval -1 to 1, it can be seen that for $|\alpha_2| = 1$, the function is asymmetric. By using **Equation (13)**, Bayesian estimator of $\hat{\theta}_L$ is obtained as

$$\hat{\theta}_L = \frac{-\ln E\left(e^{-\alpha_2\theta}\right)}{\alpha_2} \tag{24}$$

The value of $E\left(e^{-\alpha_2\theta}\right)$ in **Equation (24)** is given by

$$E\left(e^{-\alpha_2\theta}\right) = \int_0^\infty e^{-\alpha_2\theta} f(\theta)\, d\theta = \left[\left(\frac{1}{\sum_{i=1}^n x_i + \frac{1}{\beta}}\right)^{n+\alpha} \left(\sum_{i=1}^n x_i + \frac{1}{\beta} + \alpha_2\right)^{n+\alpha}\right]^{-1} \tag{25}$$

Therefore, the closed form of Bayesian estimator $\hat{\theta}_L$ is obtained by substituting **Equation (25)** into **Equation (24)**, i.e.,

$$\hat{\theta}_L = -\frac{n+\alpha}{\alpha_2} \ln\left(\frac{\sum_{i=1}^n x_i + \frac{1}{\beta}}{\sum_{i=1}^n x_i + \frac{1}{\beta} + \alpha_2}\right) \tag{26}$$

3.3 Simulation

In this section, we conducted a simulation for parameter estimation using the Bayesian method under symmetric and asymmetric loss functions and compared the results of the two approaches to obtain the characteristics of each approach.

3.3.1 Simulation of Bayesian Estimates using Symmetric Loss Functions

In the simulation using SELF and MELF approach, we generated exponentially distributed data with $\theta = 1$ and $\alpha = 1$ with various number of samples n. The result of the simulation is given by **Figure 1**.

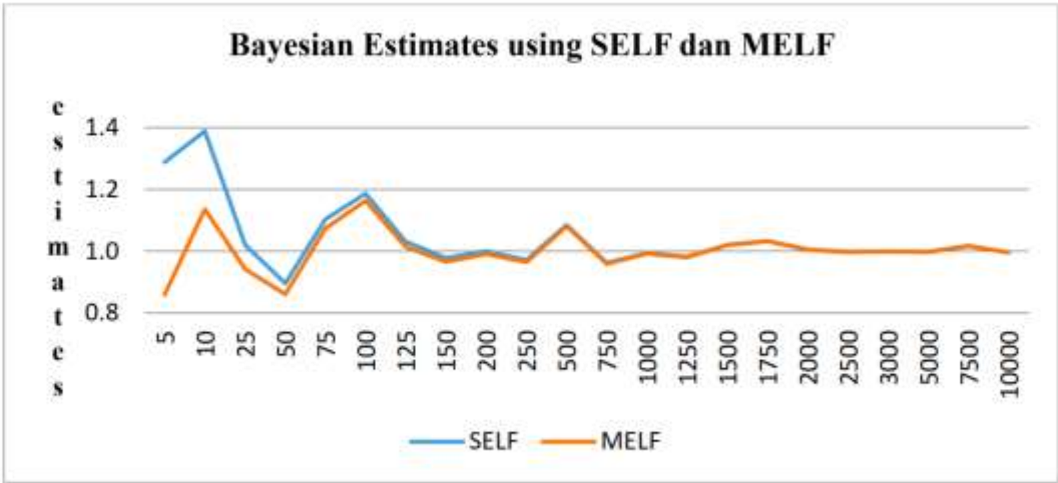

Figure 1. Bayesian Estimates using SELF and MELS Approaches

Figure 1 shows that as the number of n increases, the value of the two estimators $\hat{\theta}$ will get closer to the actual θ. This means that the larger the sample size used, the better the estimated parameter will get and the closer it is to the actual value of the parameter. In other words, the loss value is close to zero. **Figure 2** shows the loss value of SELF and MELF.

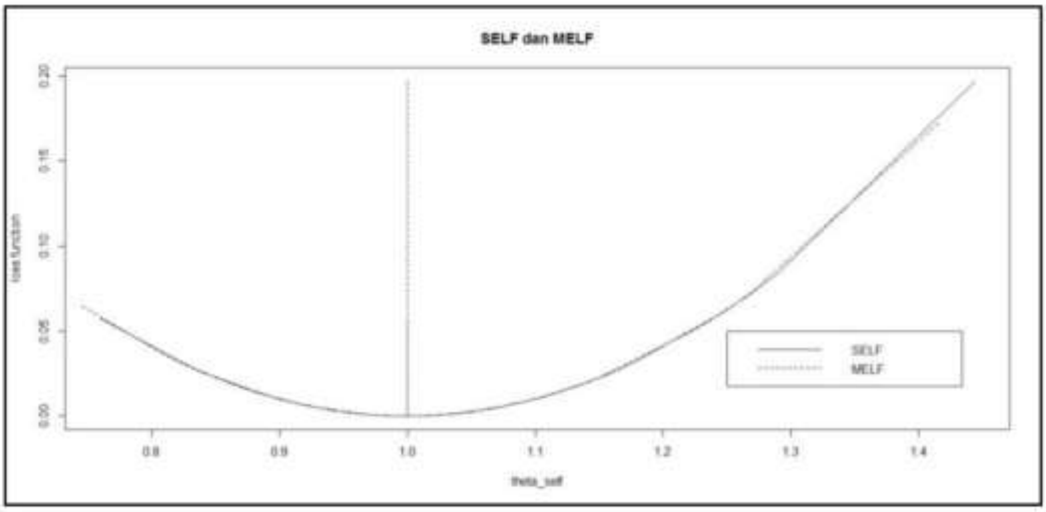

Figure 2. Loss Value of SELF and MELF

Based on Figure 1 and Figure 2, we concluded that MELF is good for over-estimation, while SELF is good for under-estimation.

3.3.2 Simulation of Bayesian Estimates using Asymmetric Loss Function

In the simulation using GELF, we generated exponentially distributed data with $\theta = 1, \alpha_1 = -2, -1, 1, 2$ with various number of n. The weakness of this method is the limited size of n which can be used, this is because there is factorial operation in the GELF formula which makes the calculation limited in the number of samples used. The result of the estimation is shown in Figure 3.

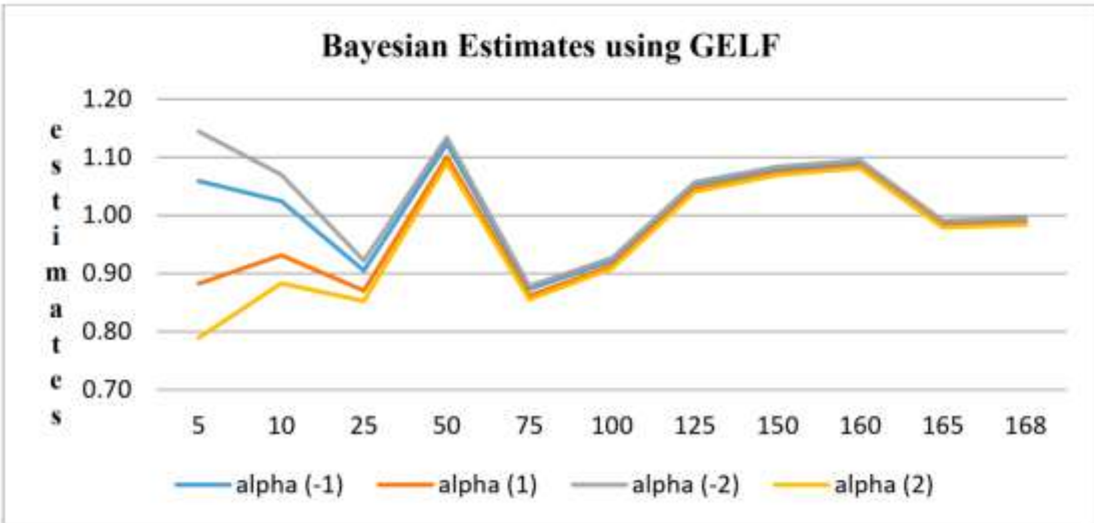

Figure 3. Bayesian Estimates using GELF

Based on Figure 3, the larger the value of n, the better the estimation of θ. The estimation using $\alpha_1 = -1$ or $\alpha_1 = 1$ gives smaller loss value than using $\alpha_1 = -2$ or $\alpha_1 = 2$. This is supported by the loss value of GELF approach for various value of α_1 which is given by Figure 4.

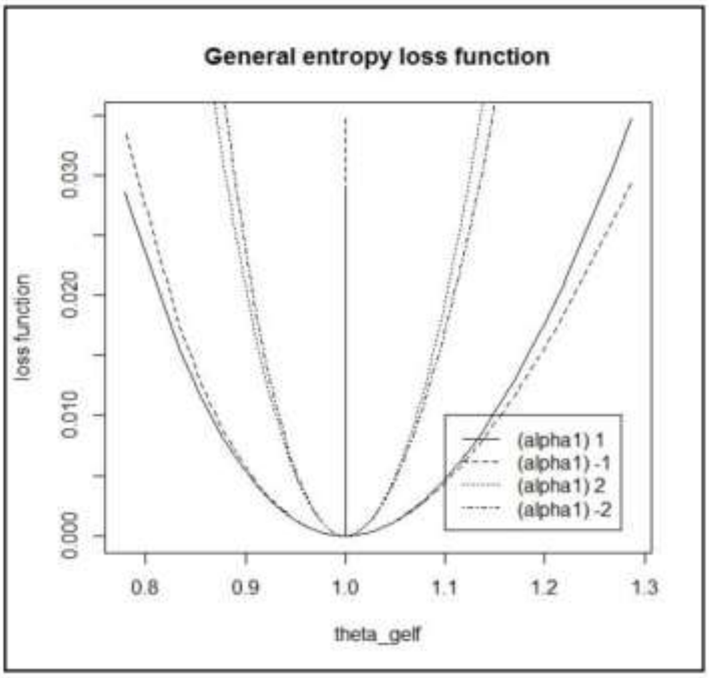

Figure 4. Loss Value of GELF

The value of $\alpha_1 > 0$ gives good results for over-estimation and $\alpha_1 < 0$ gives good results for under-estimation.

The second simulation using asymmetric loss function is the simulation using LLF approach. In this simulation, we generated exponentially distributed data with $\theta = 1, \alpha_2 = -0.25, -0.1, 0.1, 0.25$ with various number of n. Unlike GELF approach, the sample size of n in this approach is unlimited. The result of the estimation is given by **Figure 5**.

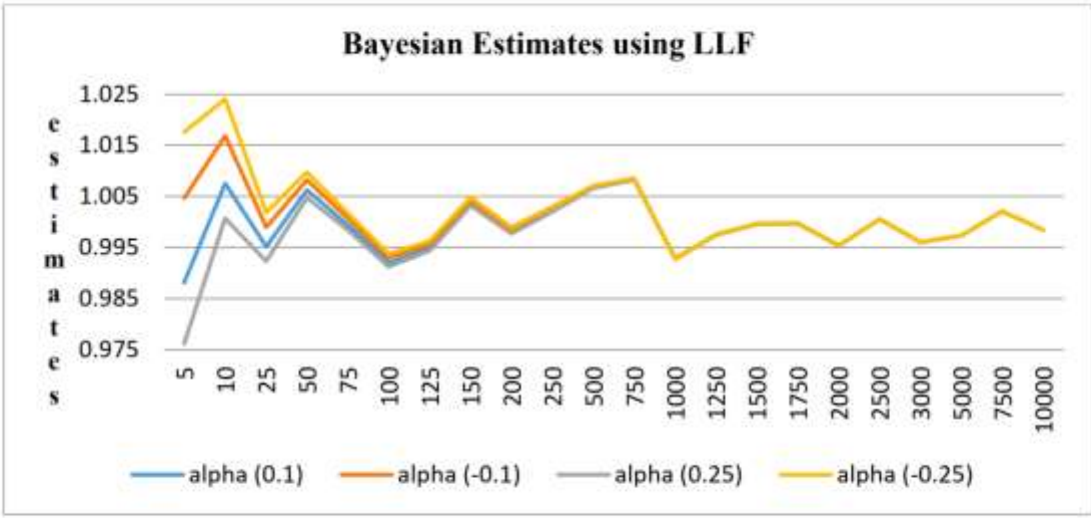

Figure 5. Bayesian Estimates of LLF

Based on **Figure 5**, we can see that the larger the sample size n, the closer the estimation value to the actual value of θ. The estimation using this approach tend to give a small interval loss value compared to the previous approach, that are SELF, MELF, and GELF. The loss value of LLF is given by **Figure 6**.

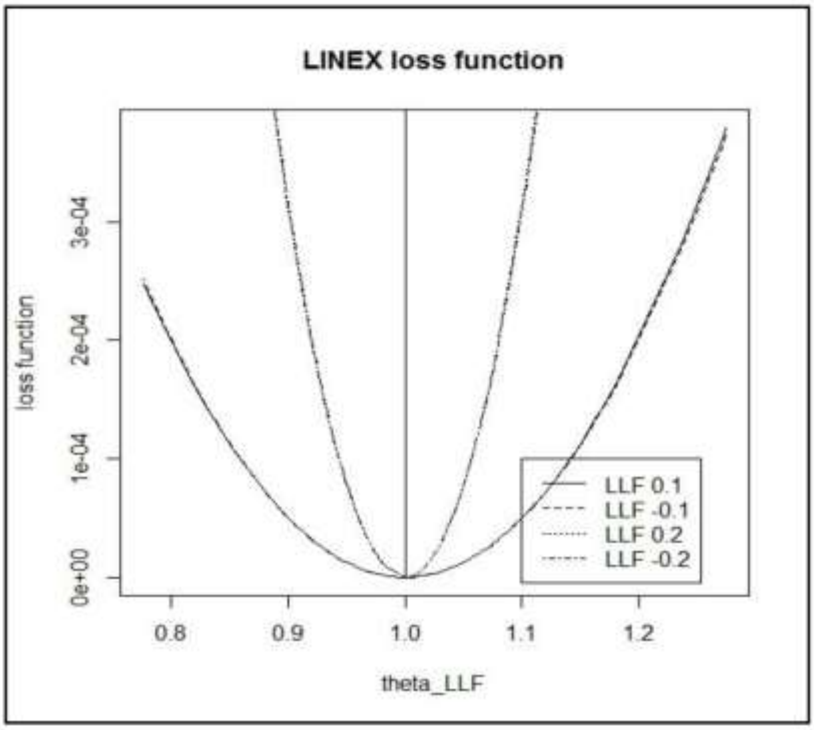

Figure 6. Loss Value of LLF

The best estimation is when the loss value is close to zero. When the loss function approaches zero, it indicates that the statistical model or parameter estimation generated by the Bayesian approach predicts or reconstructs the observed data very well. In other words, the smaller the value of the loss function, the closer the model's predictions are to the observed data, and therefore, the parameter estimation is considered better. In this context, the primary goal of Bayesian inference is to find model parameters that minimize the loss function [24]. Based on the simulation, the value of $\alpha_2 = -0.1$ or $\alpha_2 = 0.1$ gives the smaller loss value than $\alpha_2 = -0.25$ or $\alpha_2 = 0.25$. The value $\alpha_2 > 0$ gives good estimation for over estimation and $\alpha_2 < 0$ gives good estimation for under estimation.

Furthermore, we compared the simulation result between GELF and LLF by generated exponentially distributed data with $\theta = 1, \alpha_1 = -1,1, \alpha_2 = -0.1,0.1$ (the best value of α_1 and α_2 in each simulation) with the sample size of $n = 100$. The result of the comparison between GELF and LLF is given by **Figure 7**.

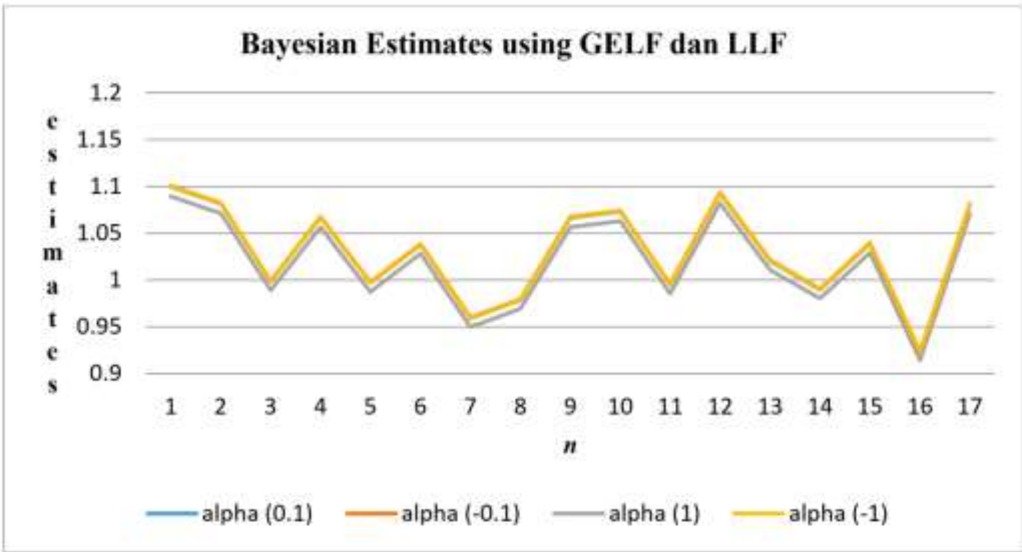

Figure 7. Bayesian Estimates using GELF and LLF

In comparing performance between GELF and LLF, we used loss value as the reference, where the best estimation is when the loss value is close to zero. From the estimation result, we obtained the smallest loss value when we used the LLF approach with $\alpha_2 = -0.1$ and $\alpha_2 = 0.1$. The result showed in **Figure 8**.

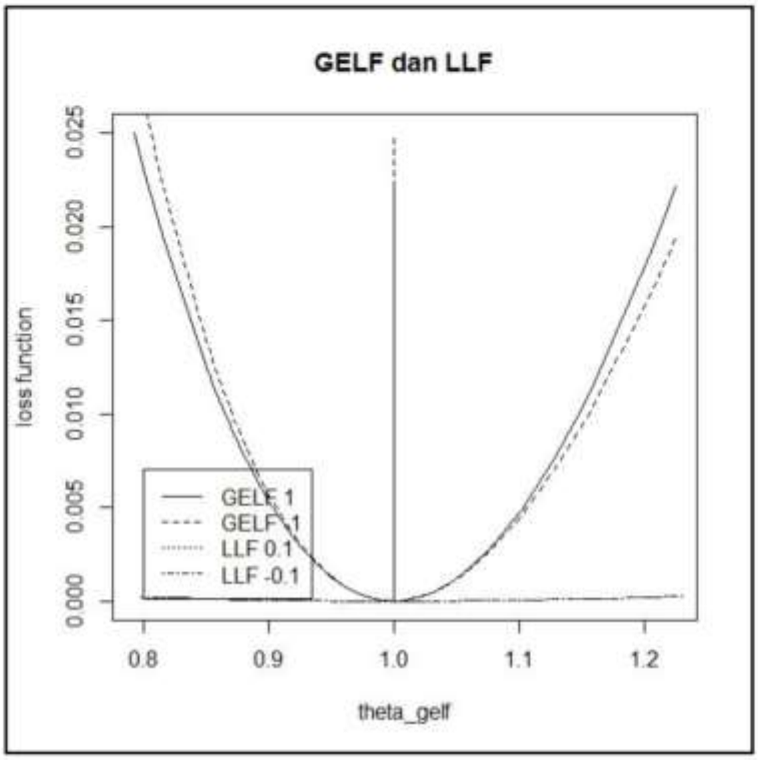

Figure 8. Loss Value of GELF and LLF

3.3.3 Comparison Between SELF, MELF, GELF, and LLF

In this section, we compared the best Bayesian estimation between SELF, MELF, GELF, and LLF by generating exponentially distributed data with $\theta = 1, \alpha_1 = -1{,}1,\ \alpha_2 = -0.1{,}0.1$ with sample size $n = 100$. The result of the four estimation is given by

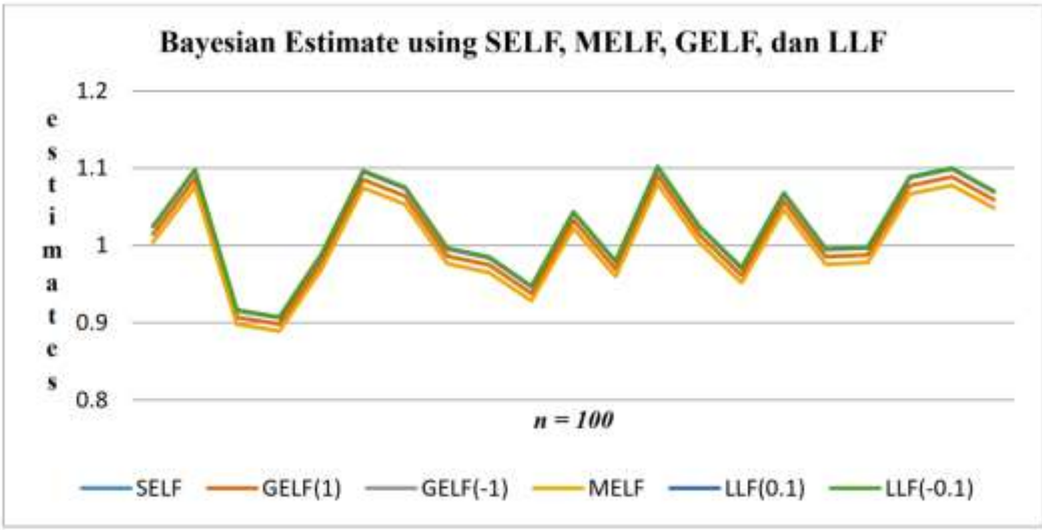

Figure 9. Bayesian Estimates using SELF, MELF, GELF, and LLF

Based on the result shown in **Figure 9**, we obtained that under symmetric loss function, MELF is the best approach for over-estimation, while SELF is the best approach for under-estimation. Furthermore, under asymmetric loss function, GELF with $\alpha_1 > 0$ is the best approach for over estimation and LLF with $\alpha_2 < 0$ is the best approach for under estimation. Those result is shown by

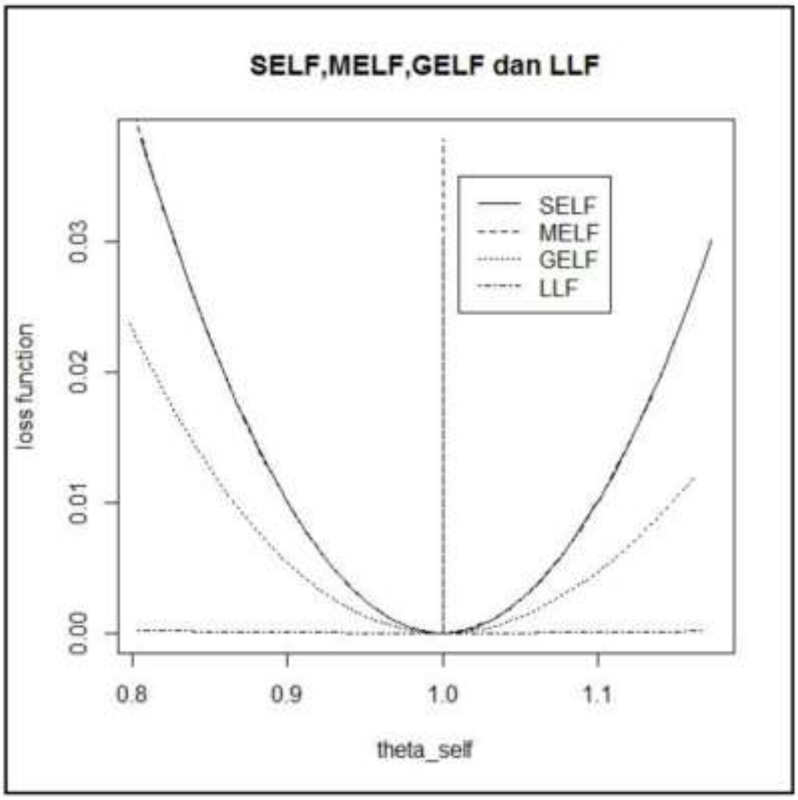

Figure 10. Loss Value of SELF, MELF, GELF, and LLF

To determine the best loss function approach, we compared all approaches with the best α_1 and α_2 in each approach (for asymmetric loss function) with sample size $n = 100$. Based on **Figure 10**, we can see that LLF has the smallest loss value compared to SELF, MELF, and GELF. It is supported by the loss value of LLF, which is in the bottom near zero.

In addition to the loss value, we choose the best loss function approach using the value of each bias for under-estimation and over-estimation. The results are shown in **Table 1** and **Table 2** for under-estimation, **Table 3** and **Table 4** for over-estimation.

Table 1. Estimation Results of All Loss Functions for Under Estimation

No.	$\hat{\theta}_S$	$\hat{\theta}_M$	$\hat{\theta}_G(1)$	$\hat{\theta}_G(-1)$	$\hat{\theta}_L(0.1)$	$\hat{\theta}_L(-0.1)$
1.	0.9162	0.8981	0.9071	0.9162	0.9158	0.9166
2.	0.9072	0.8892	0.8982	0.9072	0.9068	0.9076
3.	0.9882	0.9687	0.9784	0.9882	0.9877	0.9887
4.	0.9961	0.9763	0.9862	0.9961	0.9956	0.9966
5.	0.9845	0.9650	0.9748	0.9845	0.9840	0.9850
6.	0.9472	0.9284	0.9378	0.9472	0.9467	0.9476
7.	0.9795	0.9601	0.9698	0.9795	0.9790	0.9800
8.	0.9710	0.9517	0.9614	0.9710	0.9705	0.9714
9.	0.9953	0.9756	0.9854	0.9953	0.9948	0.9958
10.	0.9976	0.9779	0.9878	0.9976	0.9971	0.9981

Table 2. Bias Value of All Loss Functions for Under Estimation

No.	$\hat{\theta}_S$	$\hat{\theta}_M$	$\hat{\theta}_G(1)$	$\hat{\theta}_G(-1)$	$\hat{\theta}_L(0.1)$	$\hat{\theta}_L(-0.1)$
1.	-0.0838	-0.1019	-0.0929	-0.0838	-0.0842	-0.0834
2.	-0.0928	-0.1108	-0.1018	-0.0928	-0.0932	-0.0924
3.	-0.0118	-0.0313	-0.0216	-0.0118	-0.0123	-0.0113
4.	-0.0039	-0.0237	-0.0138	-0.0039	-0.0044	-0.0034
5.	-0.0155	-0.0350	-0.0252	-0.0155	-0.0160	-0.0150
6.	-0.0528	-0.0716	-0.0622	-0.0528	-0.0533	-0.0524

No.	$\hat{\theta}_S$	$\hat{\theta}_M$	$\hat{\theta}_G(1)$	$\hat{\theta}_G(-1)$	$\hat{\theta}_L(0.1)$	$\hat{\theta}_L(-0.1)$
7.	-0.0205	-0.0399	-0.0302	-0.0205	-0.0210	-0.0200
8.	-0.0290	-0.0483	-0.0386	-0.0290	-0.0295	-0.0286
9.	-0.0047	-0.0244	-0.0146	-0.0047	-0.0052	-0.0042
10.	-0.0024	-0.0221	-0.0122	-0.0024	-0.0029	-0.0019
BIAS	-0.0317	-0.0509	-0.0413	-0.0317	-0.0322	**-0.0313**

Based on Table 1 and Table 2, LLF approach with $\alpha_2 < 0$ has the smallest value of bias compared to the SELF, MELF, and GELF approaches.

Table 3. Estimation Results of All Loss Functions for Over Estimation

No.	$\hat{\theta}_S$	$\hat{\theta}_M$	$\hat{\theta}_G(1)$	$\hat{\theta}_G(-1)$	$\hat{\theta}_L(0.1)$	$\hat{\theta}_L(-0.1)$
1.	1.0247	1.0044	1.0145	1.0247	1.0241	1.0252
2.	1.0977	1.0759	1.0868	1.0977	1.0971	1.0983
3.	1.0962	1.0745	1.0863	1.0962	1.0956	1.0986
4.	1.0746	1.0533	1.0639	1.0746	1.0740	1.0752
5.	1.0431	1.0224	1.0327	1.0430	1.0425	1.0436
6.	1.1016	1.0797	1.0906	1.1016	1.1009	1.1022
7.	1.0239	1.0036	1.0138	1.0239	1.0234	1.0244
8.	1.0676	1.0465	1.0570	1.0676	1.0760	1.0682
9.	1.0883	1.0667	1.0775	1.0883	1.0877	1.0888
10.	1.0995	1.0778	1.0886	1.0995	1.0989	1.1001

Table 4. Bias Value of All Loss Functions for Over Estimation

No.	$\hat{\theta}_S$	$\hat{\theta}_M$	$\hat{\theta}_G(1)$	$\hat{\theta}_G(-1)$	$\hat{\theta}_L(0.1)$	$\hat{\theta}_L(-0.1)$
1.	0.0247	0.0044	0.0145	0.0247	0.0241	0.0252
2.	0.0977	0.0759	0.0868	0.0977	0.0971	0.0983
3.	0.0962	0.0745	0.0863	0.0962	0.0956	0.0986
4.	0.0746	0.0533	0.0639	0.0746	0.0740	0.0752
5.	0.0431	0.0224	0.0327	0.0430	0.0425	0.0436
6.	0.1016	0.0797	0.0906	0.1016	0.1009	0.1022
7.	0.0239	0.0036	0.0138	0.0239	0.0234	0.0244
8.	0.0676	0.0465	0.0570	0.0676	0.0760	0.0682
9.	0.0883	0.0667	0.0775	0.0883	0.0877	0.0888
10.	0.0995	0.0778	0.0886	0.0995	0.0989	0.1001
BIAS	0.0717	**0.0505**	0.0611	0.0717	0.0711	0.0723

Based on Table 3 and Table 4, MELF has the smallest bias value compared to SELF, GELF, and LLF approaches. Overall, we concluded the best loss function of Bayesian estimation by grouping it in Figure 11.

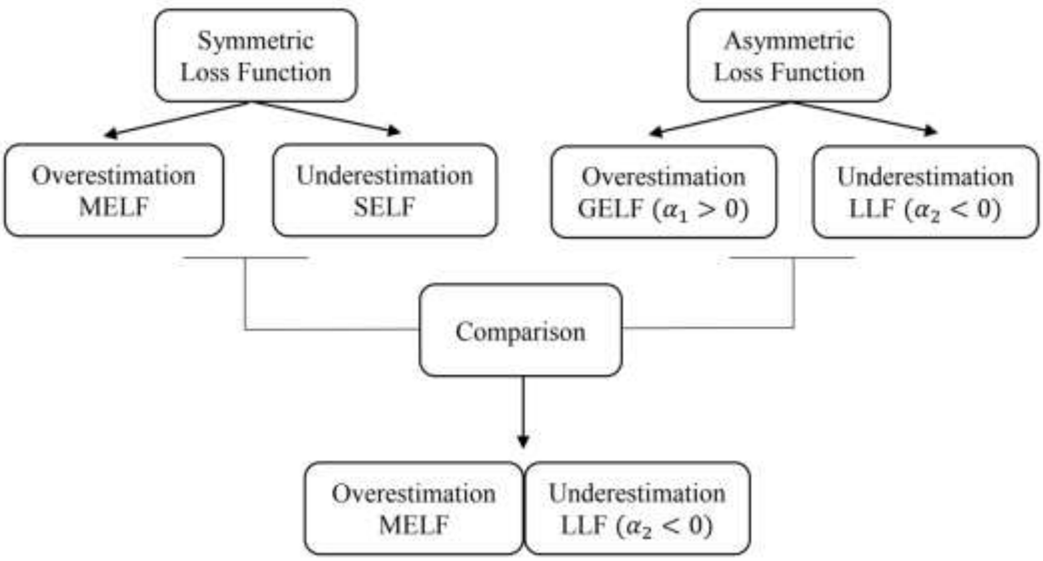

Figure 11. The Best Loss Function for Bayesian Estimation

Figure 11 shows that the loss function for the best Bayesian estimation is divided into two, namely MELF for overestimation and LLF with $\alpha_2 < 0$ for underestimation. Thus, the selection of the loss function can be done in a more tolerant manner regarding which estimation results, whether overestimation or underestimation.

3.4 Application

In this section, we implemented the result of the simulation by estimating the value of parameter θ of the real data, which follows Exponential distribution Data used in this study is secondary data, that is, lifetime data, also called time-to-event data, which means the duration required for fluorescence to persist, from the standard fluorescence life recommended to use with LEDs and laser diodes. The data is given in Table 5.

Table 5. Lifetime Data of Fluorescence

Standard	Lifetime (ns)
Dimethyl-POPOP	1.45
BBO	1.24
Coumarin 6	2.5
Dimethyl-POPOP	1.45
Coumarin 6	2.5
Fluorescein	4
Fluorescein	4
BodipyFL	5.8
Cy5	1
Alexa Fluor 647	1
Alexa Fluor 700	1
Alexa Fluor 750	0.66
Indocynanine Green	0.52

Source: http://www.iss.com/resources/reference/data_tables/StandardsLEDsLaserDiodes.html

Before estimating the parameter, we check the distribution of the standard fluorescence life data using the Anderson-Darling (AD) test to determine whether it is exponentially distributed data. The output of the AD test is given in Table 6.

Table 6. Anderson Darling Test for Standard Fluorescence Life Data

Distribution	AD	p-value
Exponential	0.779	0.199

The null hypothesis (H_0) states that the standard fluorescence life data follows Exponential distribution otherwise and using the significant level of 5%, we obtained that the decision is failed to reject H_0. It gives the conclusion that the standard fluorescence life data is exponentially distributed data. Furthermore, we estimated the parameter of θ of the standard fluorescence life data by using the Bayesian method under symmetric and asymmetric loss functions, and the result is given in Table 7.

Table 7. Parameter Estimation of Standard Fluorescence Life Data using Bayesian Method Under Symmetric and Asymmetric Loss Functions

Loss Function	Bayesian Estimates
Squared Error Loss Function (SELF)	0.479351
Minimum Expected Loss Function (MELF)	**0.410872**
General Entropy Loss Function (GELF) with $\alpha_1 = 1$	0.445111
General Entropy Loss Function (GELF) with $\alpha_1 = -1$	0.479351
Linex Loss Function (LLF) with $\alpha_2 = 0.1$	0.478532
Linex Loss Function (LLF) with $\alpha_2 = -0.1$	**0.480173**

Based on the result of the parameter estimation from Table 7 and the conclusion result given in Figure 11, we concluded that the best method to estimate the parameter θ of the standard fluorescence life data is using LLF for underestimation with $\hat{\theta}_L = 0.480173$ and MELF for overestimation with $\hat{\theta}_M = 0.410872$.

4. CONCLUSIONS

Parameter estimation of the exponentially distributed survival data using the Bayesian method under symmetric and asymmetric loss functions has been derived, and the closed form of the estimates has been obtained. The result showed that for the symmetric loss function, the best estimation method is to use the Minimum Expected Loss Function (MELF) for overestimation and the Squared Error Loss Function (SELF) for underestimation. While for asymmetric loss function, the best estimation method is General Entropy Loss Function (GELF) with $\alpha_1 > 0$ for overestimation and Linex Loss Function (LLF) with $\alpha_2 < 0$ for underestimation. Overall, the best estimation method from both symmetric and asymmetric loss function is MELF for overestimation and LLF with $\alpha_2 < 0$ for underestimation. Further study can be done to determine how to select whether the estimated parameter is underestimation or overestimation and apply the result of this research to choose the best estimation method.

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