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ON THE MOMENTS OF THE 3-PARAMETER GOMPERTZ DISTRIBUTION

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ABSTRACT

Gompertz distribution is a classical probability distribution extensively used in actuarial science, reliability, and survival analysis. Gompertz distribution also plays a role in various fields, such as biology, economics, and marketing analysis. Some extensions of this distribution have been studied and applied to various problems. In this article, we are concerned with some statistical properties of a 3-parameter Gompertz distribution. This extension of the Gompertz distribution introduced has been used in studying competing risk survival analysis. Our main results are the derivation of moments of this distribution and other statistical properties related to moments, such as moment generating function, mean residual life function, mean inactivity time and Lorenz curve. These results will serve as a complement to the theoretical aspect of the analysis of the distribution.

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1. INTRODUCTION

In statistics, probability distributions provide a tool to analyze real-world problems. Regarding human mortality modeling, the Gompertz distribution is a classical yet popular distribution that has been proven to model mortality adequately. Gompertz distribution first appeared as the Gompertz mortality model, introduced by Benjamin Gompertz in 1825, who stated that human mortality increases exponentially with age [1]. Because of its significant use in mortality modeling, Gompertz distribution is extensively used in actuarial science and demographic analysis. Besides mortality analysis, Gompertz distribution also appears helpful in various fields. For example, Bemmaor and Glady [2] use Gompertz distribution in modeling customer lifetime, Ohishi [3] analyzes software reliability using Gompertz distribution, and in biology studied by Economos [4].

Gompertz distribution is a continuous distribution with nonnegative support and two positive parameters; one is the scale parameter, and the other one is the shape parameter. To know more about the behavior of a distribution, one can study several mathematical and statistical properties of the distribution. The mathematical and statistical properties of Gompertz distribution have been thoroughly studied. Moments, variance, skewness, and kurtosis of Gompertz distribution were provided by Lenart [5]. Castellares et al. [6] provide a closed-form expression for Gompertz mean residual life. Dey et al.[6] derived several properties of the Gompertz distribution, such as its moment-generating function, entropies, Bonferroni and Lorenz curve, and order statistics. In addition, Dey et al. did parameter estimation of the Gompertz model using various estimation methods.

While the Gompertz distribution proved helpful in some areas of applied mathematics, there are some cases where classical distribution needs to provide adequate models for data. Therefore, there is a need to develop a new class of distribution. There are many extensions of Gompertz distribution. The most popular extension is the Gompertz-Makeham distribution [1], introduced by Makeham. Gompertz-Makeham distribution is obtained by adding a constant parameter to hazard rate of Gompertz distribution. The Gompertz-Makeham distribution is another classical distribution in mortality modeling, where it can fit better to mortality data than the Gompertz distribution. More recently, Ieren et al., in [7] used power transform $X^{1/\theta}$, where X is a random variable from the Gompertz distribution, and introduced a Power Gompertz distribution. Adubisi et al. [8] extended the power Gompertz distribution by exponentiating the cumulative distribution function of Power Gompertz distribution, that is $[G(y)]^{\lambda}$, where $G(y)$ is a cumulative distribution function of Power Gompertz distribution. For other extensions, see [9], [10], [11], [12] and reference therein.

In the case of medical studies, the hazard rate function of a distribution plays an important role in investigating survival data. The hazard rate of Gompertz distribution is monotone increasing if its shape parameter is positive, and vice versa. However, in clinical settings, some observed hazard rates have unimodal shape rather than monotone shape [13]. This motivates Haile et al. [14] to extend the Gompertz distribution by modifying its hazard rate. With this modification, they obtained a more flexible hazard rate that can capture the unimodal shape seen in clinical settings. The new distribution is called a '3-parameters Gompertz distribution', which is used to analyze survival data of competing risk, specifically in breast cancer data. Shama et al. [10] also used this new distribution to fit several real-world data.

In their article, Haile et al. also provide basic properties of the new distribution. However, no article studies several mathematical and statistical properties of the 3-parameters Gompertz distribution. Therefore, our aim in this paper is to develop some of the properties of the 3-parameters Gompertz. The primary purpose of this paper is to derive the moments of the distribution and some other properties related to moments, such as moment-generating function, incomplete moments, and life expectancy. These properties will help investigate some behaviors of the newly developed 3-parameter Gompertz distribution. For example, the distribution shape is indicated by the value of skewness and kurtosis, where both statistics use moments. The analysis of the expected value of the time-to-death of electronic devices is measured by its mean residual life, which can be computed using mean and first incomplete moment. The result of this research complements the theoretical standpoint of a 3-parameter Gompertz distribution.

2. RESEARCH METHODS

This research deals with moments and some other properties related to moments of 3-parameter Gompertz distribution, defined in [14]. Our main results are structured as follows: First, we derive the formula for moments and incomplete moments of the distribution. We then provide the formula for mean, variance, skewness, and kurtosis. Moreover, we also show a numerical illustration of the value of those four statistics. The following properties will be formulated: mean residual life, mean inactivity time, and Bonferroni and Lorenz curve. These three formulas can be considered a corollary of moment and incomplete moment formulas, as they can be calculated from the first and incomplete moment. We end the discussion by deriving the formula for the moment generating function (mgf), and we will see that the derivation of mgf is quite similar to the derivation of moments. This research is done using a literature study and thoroughly reviewing previous research on probability distribution. In the next subsections, we present the definition of a 3-parameter Gompertz distribution. Then, we briefly explain the required definitions for some statistical properties of distributions.

2.1 3-Parameter Gompertz Distribution

The 3-parameter Gompertz distribution introduced by Haile et al. [14] extends the Gompertz distribution. As its name suggests, this distribution has three shape parameters: one parameter defined on positive real numbers and the other two parameters defined on real numbers. However, in this article, we restrict all three parameters to positive real numbers since only in this setting is the 3-parameter Gompertz distribution proper. Moreover, the computation is easier in this setting. In this case we write $X \sim \mathrm{ThGo}(\alpha, \beta, \eta)$ if random variable X follows the 3-parameter Gompertz distribution where $\alpha, \beta\, \gamma$ are all positive parameters. The cumulative distribution function (cdf) and probability distribution function (pdf) of X, respectively, is as follows:

$$F(x) = 1 - \exp\left(-\frac{\alpha}{\beta\eta}\left(e^{\eta e^{\beta x}} - e^{\eta}\right)\right) \tag{1}$$

and

$$f(x) = \alpha \exp\left(\beta x + \eta e^{\beta x} - \frac{\alpha}{\beta\eta}\left(e^{\eta e^{\beta x}} - e^{\eta}\right)\right) \tag{2}$$

where $\alpha, \beta, \eta > 0$ and $x \geq 0$. When we let $\eta \to 0^+$ we obtain the classical Gompertz distribution. That is, if we let $\eta \to 0^+$ in Equation (1) and (2), then we get the cdf and pdf of Gompertz distribution. The cdf and pdf of Gompertz distribution, respectively, is given by

$$G(x) = 1 - \exp\left(-\frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right),$$
$$g(x) = \alpha \exp\left(\beta x - \frac{\alpha}{\beta}\left(e^{\beta x} - 1\right)\right),$$

where $\alpha > 0$ is shape parameter, $\beta > 0$ is scale parameter and $x \geq 0$ [1].

2.2 Moments and Incomplete Moments

For a continuous, nonnegative, random variable X, the r-th moment of X is defined as

$$\mu_r = E[X^r] = \int_0^\infty x f(x)\, \mathrm{d}x$$

and the r-th incomplete moment of X is defined as

$$m_r(x) = \int_0^x y^r f(y)\mathrm{d}y$$

where $f(x)$ is the probability density function (pdf) of X [15]. It is clear that $m_r(x) \to E[X^r]$ if $x \to \infty$. Some properties of a distribution can be described using moments and incomplete moments. The mean, variance, skewness, and kurtosis of a distribution rely on the first four moments. More precisely, we have the following relation for variance, skewness, and kurtosis, respectively:

$$\mathrm{Var}(X) = \mu_2 - \mu_1^2 \tag{3a}$$

$$\mathrm{Skew}(X) = \frac{\mu_3 - 3\mu_1\sigma^2 + 2\mu_1^3}{\sigma^3} \tag{3b}$$

$$\mathrm{Kurt}(X) = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu^2\sigma^2 + 3\mu_1^4}{\sigma^4} \tag{3c}$$

2.3 Mean Residual Life and Mean Inactivity Time Function

Suppose that X is a nonnegative random variable. For a fixed real number $x \geq 0$, define a random variable $R_x = X - x|X > x$. Then, $E[R_x]$ is the mean residual life (MRL) of X **[16]** and

$$e_x = E[X - x|X > x] = \frac{1}{S(x)}\int_x^\infty (t - x)f(t)\mathrm{d}t. \tag{4}$$

Mean residual life also known as life expectancy at age x, the measure of the expected or average time to failure of a system, given that it is still active at age x.

There is 'dual' of random variable R_x and its expectation. If we let of $I_x = x - X|X < x$, the expectation of I_x is called mean incativity time (MIT) of X **[17]** and we have

$$\zeta(x) = E[x - X|X < x] = \frac{1}{F(x)}\int_0^x (x - t)f(t)\mathrm{d}t. \tag{5}$$

Mean inactivity time measures the average time of a system has been inactive when measured at time x, given that it is already inactive before time x. Mean residual life and mean inactivity time are important role in reliability analysis and actuarial science. For example, the random variable R_x is important in survival analysis of human mortality and the calculation of insurance loss when there is a deductible **[18]**.

2.4 Lorenz and Bonferroni Curve

For continuous random variable X, the Lorenz curve is defined as

$$L(p) = \frac{1}{\mu_1}\int_0^p F^{-1}(t)\mathrm{d}t, \qquad 0 \leq p \leq 1 \tag{6}$$

where $F^{-1}(p)$ is the quantile function of X associated with probability p. Bonferroni curve is defined as

$$B(p) = \frac{1}{p\mu_1}\int_0^p F^{-1}(t)\mathrm{d}t, \qquad 0 \leq p \leq 1. \tag{7}$$

Lorenz curve and Bonferroni curve are two important quantities to measure inequality. This concept is useful in economics, for example, in measuring inequality of income distribution **[19]**.

2.5 Moment Generating Function

The moment generating function (mgf) of nonnegative random variable X is defined as

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x) \mathrm{d}x$$

where the domain for t must contain 0. That is, the integral exists (converge) in some neighborhood of 0.

2.6 Special Functions

Next, we present some special functions that is needed in our results. For complex numbers s and z and for nonnegative integer n, the generalized integro-exponential function $E_s^n(z)$, introduced by Milgram [20], is defined as

$$E_s^n(z) = \frac{1}{\Gamma(n+1)} \int_1^\infty (\ln w)^n w^{-s} e^{-zw} \mathrm{d}w \tag{8}$$

where

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} \mathrm{d}t, \tag{9}$$

for complex number u with positive real part, is a gamma function. If u is a nonnegative integer, we have $\Gamma(u) = (u-1)!$. For further details about the generalized integro-exponential function, see Milgram [20] and Pogany et al., [21].

Another special integral is the incomplete generalized integro-exponential function. This function appears in Reyes et al., [12]. For $s, z \in \mathbb{C}$, $x \geq 1$, and nonnegative integer n, the incomplete generalized integro-exponential function is defined as

$$E_s^n(z; x) = \frac{1}{\Gamma(n+1)} \int_1^x (\ln w)^n w^{-s} e^{-zw} \mathrm{d}w \tag{10}$$

It is clear that $\lim\limits_{x\to\infty} E_s^n(z; x) = E_s^n(z)$.

For any real number x and nonnegative integer n, the Pochhammer Symbol $(x)_n$, also known as shifted factorial or rising factorial, is defined as [22]

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1). \tag{11}$$

3. RESULTS AND DISCUSSION

3.1 Moments

For the rest of this paper, we define a constant $\xi = \alpha e^{\eta}/\beta\eta$ to simplify the presentation. Our first result is the expression of moments of ThGO distribution, given in the following theorem.

Theorem 1. *If $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta, \eta > 0$, the r-th moments of X is*

$$\mu_r = E[X^r] = \frac{\eta\xi e^{\xi}}{\beta^r} \sum_{k,m\geq 0} \sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m r!}{k!\, n!\, (m-n)!} \xi^k e^{(n+1)\eta}\, E_0^r\big((n+1)\eta\big) \tag{12}$$

Proof. First, we rewrite the pdf of X as

$$f(x) = \alpha e^{\eta} e^{\beta x}\, e^{\eta(e^{\beta x}-1)} \exp\left(-\frac{\alpha e^{\eta}}{\beta\eta}\left(e^{\eta(e^{\beta x}-1)} - 1\right)\right)$$

Now, the moment of X is

$$\mu_r = \int_0^{\infty} x^r f(x)\mathrm{d}x = \alpha e^{\eta} \int_0^{\infty} x^r e^{\beta x}\, e^{\eta(e^{\beta x}-1)} \exp\left(-\frac{\alpha e^{\eta}}{\beta\eta}\left(e^{\eta(e^{\beta x}-1)} - 1\right)\right) \mathrm{d}x.$$

First, we make a substitution $u = e^{\eta(e^{\beta x}-1)}$, then $x = \frac{1}{\beta}\ln\left(1+\frac{\ln u}{\eta}\right)$ and we have

$$\begin{aligned}\mu_r &= \frac{\alpha e^{\eta}}{\beta^{r+1}\eta} \exp\left(\frac{\alpha e^{r}}{\beta\eta}\right) \int_1^{\infty} \left(\ln\left(1+\frac{u}{\eta}\right)\right)^r \exp\left(-\frac{\alpha e^{\eta}}{\beta\eta}u\right) \mathrm{d}u \\ &= \frac{\xi e^{\xi}}{\beta^r} \int_1^{\infty} \left(\ln\left(1+\frac{\ln u}{\eta}\right)\right)^r e^{-\xi x}\mathrm{d}u.\end{aligned} \tag{13}$$

For the integral in **Equation (13)**, we substitute $u = \frac{1}{1-v}$, then $v = 1 - \frac{1}{u}$ and $\mathrm{d}v = (1-u)^{-2}\mathrm{d}u$. After this substitution, we expand the exponential and binomial term to obtain

$$\begin{aligned}\int_1^{\infty} \left(\ln\left(1+\frac{\ln x}{\eta}\right)\right)^r \exp(-\xi x)\,\mathrm{d}x &= \int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r \exp\left(-\xi\frac{1}{1-u}\right)(1-u)^{-2}\mathrm{d}u \\ &= \int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r \sum_{k=0}^{\infty}\frac{(-1)^k\xi^k}{k!}(1-u)^{-k-2}\mathrm{d}u \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\xi^k}{k!}\int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r (1-u)^{-k-2}\mathrm{d}u \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\xi^k}{k!}\int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r \sum_{m=0}^{\infty}\binom{-k-2}{m}(-u)^m\,\mathrm{d}u \\ &= \sum_{k,m=0}^{\infty}\frac{(-1)^k(k+2)_m\xi^k}{k!\,m!}\int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r u^m\mathrm{d}u.\end{aligned}$$

Where we expand the binomial coefficient $\binom{-k-2}{m}$ and write it out as

$$\binom{-k-2}{m} = \frac{(-k-2)(-k-3)\ldots(-k-2-m+1)}{m!} = \frac{(-1)^m}{m!}(k+2)_m$$

using the Pochhammer symbol in **Equation (11)**. Keep in mind that binomial series $(1-u)^{\alpha}$ for real α converge only for $|u| < 1$. Hence, we have

$$\begin{aligned}&\int_1^{\infty} \left(\ln\left(1+\frac{\ln x}{\eta}\right)\right)^r \exp(-\xi x)\,\mathrm{d}x \\ &\qquad = \sum_{k,m=0}^{\infty}\frac{(-1)^k(k+2)_m\xi^k}{k!\,m!}\int_0^1 \left[\ln\left(1-\frac{\ln(1-u)}{\eta}\right)\right]^r u^m\mathrm{d}u\end{aligned} \tag{14}$$

Next, we set $u = 1 - e^{-v}$, then $v = -\ln(1-u)$ and the integral on the right-hand side of **Equation (14)** expanded into

$$\begin{aligned}\int_0^1 \left[\ln\left(1 - \frac{\ln(1-u)}{\eta}\right)\right]^r u^m \, \mathrm{d}u &= \int_0^\infty \left[\ln\left(1 + \frac{v}{\eta}\right)\right]^r (1-e^{-v})^m e^{-v} \mathrm{d}v \\ &= \int_0^\infty \left[\ln\left(1 + \frac{v}{\eta}\right)\right]^r \sum_{n=0}^{m} \binom{m}{n} (-1)^n e^{-(n+1)v} \, \mathrm{d}v \\ &= \sum_{n=0}^{m} (-1)^n \binom{m}{n} \int_0^\infty \left[\ln\left(1 + \frac{v}{\eta}\right)\right]^r e^{-(n+1)v} \mathrm{d}v \\ &= \sum_{n=0}^{m} (-1)^n \binom{m}{n} \int_1^\infty (\ln w)^r e^{-(n+1)(w-1)\eta} \eta \, \mathrm{d}w \\ &= \sum_{n=0}^{m} \frac{(-1)^n m!}{n!\,(m-n)!} \eta e^{(n+1)\eta} r! \, E_0^r\big((n+1)\eta\big)\end{aligned}$$

where the last integral is written as a generalized integro-exponential function using **Equation 8**. Hence, we conclude that

$$\int_0^1 \left[\ln\left(1 - \frac{\ln(1-u)}{\eta}\right)\right]^r u^m \, du = \eta \sum_{n=0}^{m} \frac{(-1)^n m!\, r!}{n!\,(m-n)!} e^{(n+1)\eta} E_0^r\big((n+1)\eta\big). \tag{15}$$

Combining **Equation (15)** and **Equation (14)** into **Equation (13)**, we immediately have **Equation (12)**. The proof is completed.

Knowing the moments of distribution allows us to calculate the mean, variance, skewness, and kurtosis of the ThGo distribution from **Equation (3)** (a) through **Equation (3)** (d). To illustrate the value of these four statistics, we use some sets of values of parameters α, β, and η. We use small values for each parameter since the estimated value of each parameter is usually small in mortality analysis. This situation also happened in Haile et al. **[14]**. Of course, we can also consider other ranges of values for each parameter. The value of these parameters and the resulting statistics are presented in **Table 1**. The computation is done in Wolfram Mathematica.

Table 1. Values of Mean, Variance, Skewness, and Kurtosis of the ThGo Distribution for Different Sets of Parameter Values

Parameter			Statistics			
α	β	η	**Mean**	**Variance**	**Skewness**	**Kurtosis**
0.1	1	0.001	2.00929	0.82281	32.45271	2.33258
		0.05	1.81123	0.59051	38.92445	2.34637
		1	0.77474	0.10919	38.27561	2.34794
	2	0.001	1.29208	0.25646	49.45923	2.54893
		0.05	1.13072	0.16172	66.00985	2.81908
		1	0.48330	0.02953	66.03239	2.87946
	5	0.001	0.67960	0.04905	86.05756	3.00466
		0.05	0.56558	0.02467	138.96207	3.99010
		1	0.24178	0.00452	138.14382	4.04802
2	1	0.01	0.35747	0.07839	7.24917	3.73433
		0.5	0.22100	0.02981	7.26389	3.59028

Parameter			Statistics			
α	β	η	**Mean**	**Variance**	**Skewness**	**Kurtosis**
		2	0.05730	0.00247	5.92632	4.94704
		0.01	0.29471	0.04287	9.35945	2.95564
	2	0.5	0.18116	0.01575	9.65628	2.73635
		2	0.05065	0.00162	6.98782	3.79193
		0.01	0.20655	0.01501	14.70462	2.37370
	5	0.5	0.12432	0.00501	16.40731	2.16541
		2	0.03903	0.00071	10.09051	2.68525

***Data source**: Own computation.*

From **Table 1**, we can see that the variance is very small for each set of parameters. As η increases, the mean and variance of ThGO distribution both decreases. However, the skewness is quite large and increases with η. Positive skewness means that the pdf is leaning to the left. Meanwhile, the kurtosis is fairly small and varies from all parameters.

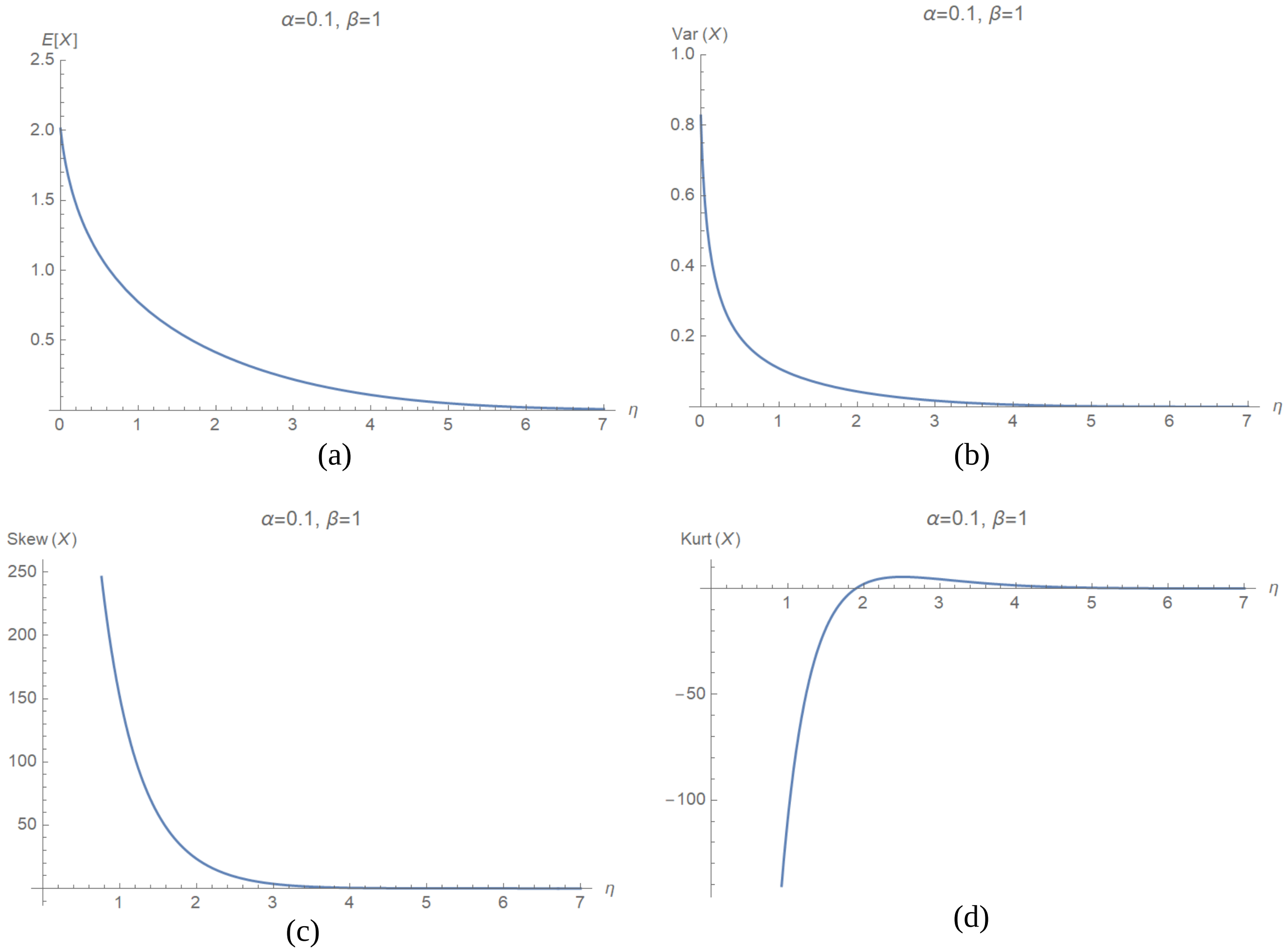

Figure 1. The graph of: (a) mean, (b) variance, (c) skewness, and (d) kurtosis of the ThGo distribution with $\alpha = 0.1$, $\beta = 1$, and η varies.

Source: Own computation

Furthermore, we present the graph of the mean, variance, skewness, and kurtosis of ThGo distribution with respect to η and fixed α and β are given in **Figure 1**. These graphs will illustrate the change of mean, variance, skewness, and kurtosis as $\eta > 0$ increases. That means we shall see how this distribution differ from Gompertz distribution ($\eta = 0$) in those statistics.

3.2 Incomplete Moments

The r-th incomplete moments of ThGO distribution is given in the following theorem.

Theorem 2. *If $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta, \eta > 0$, the r-th incomplete moments of X is*

$$m_r(x) = \frac{\xi e^{\xi}}{\beta^r} \sum_{k,m\geq 0} \sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m r!}{k!\, n!\, (m-n)!} \xi^k e^{(n+1)\eta} E_0^{r-1}\left((n+1)\eta; e^{\beta x}\right) \tag{16}$$

Proof. The proof using the same process used in the proof of **Theorem 1**. First, by substituting $u = e^{\eta(e^{\beta y}-1)}$, we obtain

$$m_r(x) = \int_0^x y^r f(y) \mathrm{d}y = \frac{\xi e^{\xi}}{\beta^r} \int_1^{\rho} \ln^r\left(1 + \frac{\ln u}{\eta}\right) e^{-\xi u} \mathrm{d}u \tag{17}$$

where $\rho = e^{\eta(e^{\beta x}-1)}$. Next, substitute $u = \frac{1}{1-v}$, we have

$$\begin{aligned} \int_1^{\rho} \left[\ln\left(1 + \frac{\ln u}{\eta}\right)\right]^k \mathrm{e}^{-\xi u} \mathrm{d}u &= \int_0^{1-\frac{1}{\rho}} \ln^r\left(1 - \frac{\ln(1-v)}{\eta}\right) e^{-\frac{\xi}{1-v}} (1-v)^{-2} \mathrm{d}v \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \xi^k \int_0^{1-\frac{1}{\rho}} \ln^r\left(1 - \frac{\ln(1-v)}{\eta}\right) (1-v)^{k-2} \mathrm{d}v \\ &= \sum_{k,m\geq 0} \frac{(-1)^k (k+2)_m}{k!\, m!} \xi^k \int_0^{1-\frac{1}{\rho}} \ln^r\left(1 - \frac{\ln(1-v)}{\eta}\right) v^m \mathrm{d}v. \end{aligned} \tag{18}$$

Setting $u = 1 - e^{-v}$ gives

$$\begin{aligned} \int_0^{1-\frac{1}{\rho}} \ln^r\left(1 - \frac{\ln(1-v)}{\eta}\right) v^m \mathrm{d}v &= \int_0^{\ln\rho} \ln^r\left(1 + \frac{w}{\eta}\right) (1-e^{-w})^m \mathrm{e}^{-w} \mathrm{d}w \\ &= \sum_{n=0}^{m} (-1)^n \binom{m}{n} \int_0^{\ln\rho} \ln^r\left(1 + \frac{w}{\eta}\right) e^{-(n+1)w} \mathrm{d}w \\ &= \sum_{n=0}^{m} (-1)^n \binom{m}{n} \int_1^{e^{\beta x}} \ln^r w\, e^{-(n+1)\eta(w-1)} \mathrm{d}w \\ &= \sum_{n=0}^{m} \frac{(-1)^n m!\, r!}{n!\, (m-n)!} \eta e^{(n+1)\eta} E_0^r\left((n+1)\eta; e^{\beta x}\right). \end{aligned} \tag{19}$$

Combining **Equation (19)** and **Equation (18)** into **Equation (17)**, we have **Equation (16)**.

The first incomplete moment has several corollaries. In the following subsections, we present some structural properties of the ThGo distribution that can be derived from the first incomplete moments. These are mean residual life, mean inactivity time, and Lorenz and Bonferroni curve. Although these are corollaries of incomplete moments, these properties are of interest.

3.3 Mean Residual Life and Mean Inactivity Time

The mean residual life and mean inactivity time are defined in **Equation (4)** and **Equation (5)**, respectively. After some algebra, we can write the formula for MRL and MIT in terms of the first moment and first incomplete moment, specifically,

$$e_x = \frac{\mu_1 - m_1(x)}{S(x)} - x. \tag{20}$$

and

$$\zeta(x) = x - \frac{m_1(x)}{F(x)}. \tag{21}$$

where μ_1 is the first moment (mean) of X, $m_1(x)$ is the first incomplete moment of X, $F(x)$ is the cdf of X, and $S(x) = 1 - F(x)$ is the survival function of X. In the case of $X \sim \mathrm{ThGo}(\alpha, \beta, \eta)$, we have the following result for MRL of X.

Theorem 4. *If $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta, \eta > 0$, and $x \geq 0$, we have*

$$\begin{aligned} e_x = \frac{\xi\eta}{\beta}\kappa_{\alpha,\beta,\eta}(x) \sum_{k,m\geq 0}\sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k \eta e^{(n+1)\eta}\Big[E_0^1\big((n+1)\eta\big) \\ - E_0^1\left((n+1)\eta; e^{\beta x}\right)\Big] - x. \end{aligned} \tag{22}$$

where $\kappa_{\alpha,\beta,\eta}(x) = \exp\left(\frac{\alpha \exp(\eta e^{\beta x})}{\beta\eta}\right)$.

Proof. It immediately follows from **Equation (20)** and the formula for the moment, which needed to be completed in the previous discussion.

$$\begin{aligned} e_x &= \frac{1}{\exp\left(-\frac{\alpha}{\beta\eta}\left(e^{\eta e^{\beta x}} - e^{\eta}\right)\right)}\Bigg(\frac{\xi e^{\xi}}{\beta}\sum_{k,m\geq 0}\sum_{n=0}^{m}\frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k\eta e^{(n+1)\eta}[E_0^1\big((n+1)\eta\big) \\ &\quad - E_0^1\left((n+1)\eta; e^{\beta x}\right)\Bigg) - x \\ e_x &= \frac{\xi\eta}{\beta}\kappa_{\alpha,\beta,\eta}(x)\sum_{k,m\geq 0}\sum_{n=0}^{m}\frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k\eta e^{(n+1)\eta}\left[E_0^1\big((n+1)\eta\big) - E_0^1\left((n+1)\eta; e^{\beta x}\right)\right] - x. \end{aligned}$$

This concludes the proof.

As for MIT of X, the result is as follows.

Theorem 5. *If $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta, \eta > 0$, and $x \geq 0$, we have*

$$\zeta(x) = x - \frac{\kappa_{\alpha,\beta,\eta}(x)}{\kappa_{\alpha,\beta,\eta}(x) - \xi}\frac{\xi e^{\xi}}{\beta}\sum_{k,m\geq 0}\sum_{n=0}^{m}\frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k e^{(n+1)\eta}E_0^1\left((n+1)\eta; e^{\beta x}\right). \tag{23}$$

Proof. It follows immediately from **Equation (21)** and the formula for an incomplete moment.

3.4 Lorenz and Bonferroni Curve

We can represent the Lorenz and Bonferroni curves in **Equation (6)** and **Equation (7)**, respectively, in terms of the first moment and the first incomplete moment of X. Since $F^{-1}(p) = x$ equivalent to $F(x) = p$, where $0 \leq p \leq 1$, we can calculate the Lorenz and Bonferroni curve using the following formula, that follows from substitution $F(x) = p$ in **Equation** (6) and **Equation (7)**:

$$L\big(F(x)\big) = \frac{m_1(x)}{\mu_1}, \qquad B\big(F(x)\big) = \frac{L\big(F(x)\big)}{F(x)}. \tag{24}$$

For the ThGo distribution, we have the following.

Theorem 6. *If $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta, \eta > 0$, and $x \geq 0$, we have*

$$L\big(F(x)\big) = \frac{\sum_{k,m\geq 0}\sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k e^{(n+1)\eta}\left(\frac{e^{-1} - e^{-\exp(\beta x)}}{n+1}\right)}{\eta^2 \sum_{k,m\geq 0}\sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m}{k!\,n!\,(m-n)!}\xi^k e^{(n+1)\eta}\, E_0^1\big((n+1)\eta\big)}. \tag{25}$$

Proof. The result follows by substituting first moment and incomplete moment into **Equation (23)**.

3.5 Moment Generating Function

The mgf of the ThGo distribution is given in the following theorem.

Theorem 7. *The moment generating function of $X \sim ThGo(\alpha, \beta, \eta)$ where $\alpha, \beta\,, \eta > 0$ is*

$$M(t) = \frac{\xi e^{\xi}}{\eta^{t/\beta}} \sum_{k,m\geq 0}\sum_{n=0}^{m} \frac{(-1)^{k+n}(k+2)_m}{k!\,(n+1)!\,(m-n)!}\frac{\xi^k e^{(n+1)\eta}}{(n+1)^{t/\beta}}\Gamma\left(\frac{t}{\beta}+1, (n+1)\eta\right) \tag{26}$$

where for complex number z with positive real part and $x \geq 0$

$$\Gamma(z, x) = \int_x^{\infty} t^{z-1} e^{-t} \mathrm{d}t$$

denotes the upper incomplete gamma function.

Proof. The proof is highly similar to the derivation of incomplete moments. First, we have

$$M(t) = \int_0^{\infty} e^{tx} f(x)\mathrm{d}x = \alpha e^{\eta}\int_0^{\infty} e^{tx} e^{\beta x}\, e^{\eta(e^{\beta x}-1)} \exp\left(-\frac{\alpha e^{\eta}}{\beta\eta}\left(e^{\eta(e^{\beta x}-1)} - 1\right)\right)\mathrm{d}x. \tag{27}$$

Substitute $u = e^{\eta(e^{\beta y}-1)}$ into the right-hand side of **Equation (16)**, we have

$$M(t) = \xi e^{\xi}\int_1^{\infty}\left(1 + \frac{\ln u}{\eta}\right)^{t/\beta} \exp(-\xi u)\,\mathrm{d}u. \tag{28}$$

Next, substitute $v = 1 - 1/u$ into the integral on **Equation (17)** and expand the exponential and binomial term to obtain

$$M(t)=\xi e^{\xi}\sum_{k,m\geq 0}\frac{(-1)^k(k+2)_m}{k!\,m!}\xi^k\int_0^1\left(1-\frac{\ln(1-v)}{\eta}\right)^{t/\beta}v^m\mathrm{d}v. \tag{29}$$

In the integral on **Equation (18)**, let $v=1-e^{-w}$ and then simplifying the integral so that

$$\begin{aligned}\int_0^1\left(1-\frac{\ln(1-v)}{\eta}\right)^{t/\beta}v^m\mathrm{d}v&=\int_0^\infty\left(1+\frac{w}{\eta}\right)^{t/\beta}(1-e^{-w})^m\mathrm{e}^{-w}\mathrm{d}w\\&=\sum_{n=0}^{m}(-1)^n\binom{m}{n}\int_0^\infty\left(1+\frac{w}{\eta}\right)^{t/\beta}e^{-(n+1)w}\mathrm{d}w\\&=\sum_{n=0}^{m}(-1)^n\binom{m}{n}\eta e^{(n+1)\eta}\int_1^\infty w^{t/\beta}e^{-(n+1)\eta w}\mathrm{d}w\\&=\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{\eta e^{(n+1)\eta}}{\big((n+1)\eta\big)^{t/\beta+1}}\int_{(n+1)\eta}^\infty w^{t/\beta}e^{-w}\mathrm{d}w\\&=\sum_{n=0}^{m}\frac{(-1)^n m!}{(n+1)!\,(m-n)!}\frac{e^{(n+1)\eta}}{\big((n+1)\eta\big)^{t/\beta}}\Gamma\left(\frac{t}{\beta}+1,(n+1)\eta\right).\end{aligned} \tag{30}$$

Combining **Equation (29)** and **Equation (30)** into **Equation (27)**, we obtain **Equation (26)** and the proof is complete.

4. CONCLUSIONS

In this paper, we have established the formula for moments and incomplete moments, as well as several properties related to them, of the 3-parameter Gompertz distribution, such as mean residual life and mean inactivity time function, which is a useful concept in reliability analysis, Lorenz and Bonferroni Curve, which is a useful notion in economics. We illustrate and visualize the mean, variance, skewness, and kurtosis using some sets of parameter values. This will provide a glimpse into the distribution behavior. The results should add to the understanding of the distribution structure from a theoretical point of view.

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