BAREKENG: Journal of Mathematics and Its Applications
March 2024 Volume 18 Issue 1 Page 0565–0578
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss1pp0565-0578

SOME CONSTRUCTION OF 8N-DIMENSIONAL PERFECT MAGIC CUBE WITH ARITHMETIC SEQUENCE

Ulil Albab Mu'min [1*], Bib Paruhum Silalahi[2], Sugi Guritman[3]

[1,2,3]*Departement of Mathematics, Faculty of Mathematics and Natural Science, IPB University*
Dramaga Street, Campus IPB Dramaga, Bogor, 16680, Indonesia

*Corresponding author's e-mail: * **ulilalbabmumin@apps.ipb.ac.id***

Article History:

Received: 1st December 2023
Revised: 2nd February 2024
Accepted: 6th February 2024

Keywords:

Arithmetic sequence;
Magic cube;
Magic square;
Perfect magic cube.

ABSTRACT

A magic square whose dimensions are expanded is called a magic cube. A magic cube whose properties are expanded is called a perfect magic cube. The perfect magic cube problem is how to arrange m^3 numbers in an $m \times m \times m$ cube (matrix) such that the sum of rows, columns, pillars, diagonals (planes and spaces) produce a magic constant of the cube. In this paper, it will be studied how to construct a perfect magic cube of order $8n$ for $n \geq 1$ whose entries contain an arithmetic sequence with the difference (β) which is set to find specific patterns, and the algorithm for constructing a perfect magic cube is then implemented into programming language to solve large orders.

How to cite this article:

U. A. Mu'min, B. P. Silalahi and S. Guritman., "SOME CONSTRUCTION OF 8N-DIMENSIONAL PERFECT MAGIC CUBE WITH ARITHMETIC SEQUENCE," *BAREKENG: J. Math. & App.*, vol. 18, iss. 1, pp. 0565-0578, March, 2024.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

In prehistoric times there have been studies and the earliest known examples of magic square is the Lo-Shu turtle from ancient China [1]. The magic square problem is how to arrange a number of m^2 numbers in an $m \times m$ square box (matrix) such that it satisfies the magic square property, namely the sum of the numbers in each row, column, and main diagonal are the same [2]. Research on the magic square construction has been conducted by [3], [4], [5]. The magic cube is a dimensional extension of the magic square. A magic cube is a number of magic squares arranged in such a way that it satisfies a magic cube property, namely the sum of the numbers in each row, column, pillar, and diagonal of the space is the same or in other words it is called the magic constant of cube [6]. The magic cube problem is how to arrange a number of m^3 numbers in a cube (matrix) $m \times m \times m$ such that it produces a magic constant of cube [7]. The magic cube is divided into several types including; simple, diagonal, pantriagonal, diagonal pantriagonal, pandiagonal, and perfect magic cube [8]. Some previous studies on the completion of magic cube construction have been carried out by [9], [10], [11]. Recently, several types of research have been published related to the magic square and cube that applied their properties into some scopes such as cryptography, information security, public key, secret sharing, remote access control, applied mathematics, and number theory [12].

A perfect magic cube is an expansion of the properties of a magic cube that has the properties that the sum of the numbers in each row, column, pillar, diagonal space, and diagonal plane is the same. Research on perfect magic cubes with natural number sequence entries has been done per case of each order in previous years [13]. Therefore, this paper discusses how to construct a generalized perfect magic cube of order $8n$ for $n \geq 1$ whose entries contain an arithmetic sequence with initial number (α) and difference (β) which is set to find a certain pattern, and the algorithm for constructing the perfect magic cube is then implemented into a programming language to solve larger orders.

2. RESEARCH METHODS

This research will discuss the construction of a perfect magic cube of order $8n$ where n is the index for $n \geq 1$ with the following research stages:

1. Literature study and formulate the problem of determining the settings for manually forming a perfect magic cube of order $8n$.
2. Prove the existence of the perfect magic cube of order $8n$ for $n = 1,2$ with arithmetic sequence.
3. Construct the perfect magic cube of order $8n$ algorithm for $n = 1,2$ with arithmetic sequence.
4. Recapitulate and test the feasibility of the algorithm up to large order and then construct a perfect magic cube of order $8n$ for $n \geq 1$ with arithmetic sequence using Python software.
5. Prove the existence of the perfect magic cube of order $8n$ for $n \geq 1$ with arithmetic sequence.

2.1 Magic Square, Magic Cube and Perfect Magic Cube with Arithmetic Sequence

In this section, the definitions and properties of magic square, magic cube, and perfect magic cube with arithmetic sequence entries are presented. It also presents illustrations and theorems related to magic square, magic cube and perfect magic cube.

Definition 1. [14]. An arithmetic sequence is a sequence of numbers $\{a_m\}$ with difference (β) if for a fixed natural number m and for all $k = 1,2, \dots, m$,

$$a_k - a_{k-1} = \beta,$$

where a_0 = initial number = α.

Based on **Definition 1**, an arithmetic sequence can be written in **Equation (1)** as follows:

$$\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, \dots, \alpha+(m-2)\beta, \alpha+(m-1)\beta, \quad (1)$$

$$\alpha+\alpha+\beta+\alpha+2\beta+\cdots+\alpha+(m-2)\beta+\alpha+(m-1)\beta = \frac{m}{2}(\alpha+\alpha+(m-1)\beta. \quad (2)$$

An arithmetic series is the sum of the arithmetic sequence written in **Equation (2) [15]**.

Definition 2. [16], [17]. The magic square can be denoted as follows

$$C = \left[c_{i,j}: 1 \leq i, j \leq m\right]$$

is an $m \times m$ matrix (square) containing an arithmetic sequence whose entries appear exactly once then arranged such that the sum of rows, columns, and main diagonals yields the magic constant of square $\frac{m}{2}(2\alpha + (m^2 - 1)\beta)$.

Example 1. [5]. The magic square with arithmetic sequence entries is illustrated in **Figure 1**(a).

Definition 3. [18]. A three-dimensional matrix $m \times m \times m$ consisting of m rows, m columns, and m layer is called a cubic array of order m or can be written $C = c_{i,j,k}$, $i, j, k \in \{1,2,3, \ldots, m\}$ where $c_{i,j,k}$ is the i th row element, j th column element, and k th file of the array (layer). The set of m elements of

i. $\{c_{i+l,j,k} \ : \ l = 1,2, \ldots, m\}$ is a column,

ii. $\{c_{i,j+l,k} \ : \ l = 1,2, \ldots, m\}$ is a row,

iii. $\{c_{i,j,k+l} \ : \ l = 1,2, \ldots, m\}$ is a pillar,

iv. $\{c_{i+l,j+l,k} \ : \ l = 1,2, \ldots, m\}$, $\{c_{i+l,j,k+l} \ : \ l = 1,2, \ldots, m\}$, and $\{c_{i,j+l,k+l} \ : \ l = 1,2, \ldots, m\}$ are the plane diagonals,

v. $\{c_{i+l,j+l,k+l} \ : \ l = 1,2, \ldots, m\}$ and $\{c_{i+l,t+1-(j+l),k+l} \ : \ l = 1,2, \ldots, m\}$ are the space diagonals, where $t = m$.

Definition 4. [19], [20]. The magic cube is a generalization of the magic square which can be denoted as follows

$$C = \left[c_{i,j,k}: 1 \leq i, j, k \leq m\right],$$

is an $m \times m \times m$ matrix (cubic array) containing an arithmetic sequence whose entries appear exactly once then arranged such that the sum of rows, columns, pillars, and space diagonals yields the magic constant of the cube.

Example 2. [21]. The magic cube with arithmetic sequence entries is illustrated in **Figure 1**(b).

Lemma 1. [18]. *The magic cube containing an arithmetic sequence has the magic constant of cube* $\frac{m}{2}(2\alpha + (m^3 - 1)\beta)$.

Proof. Suppose that a number of arithmetic rows summed in **Equation (2)** are then placed in m rows, m columns, and m layers of a cube matrix

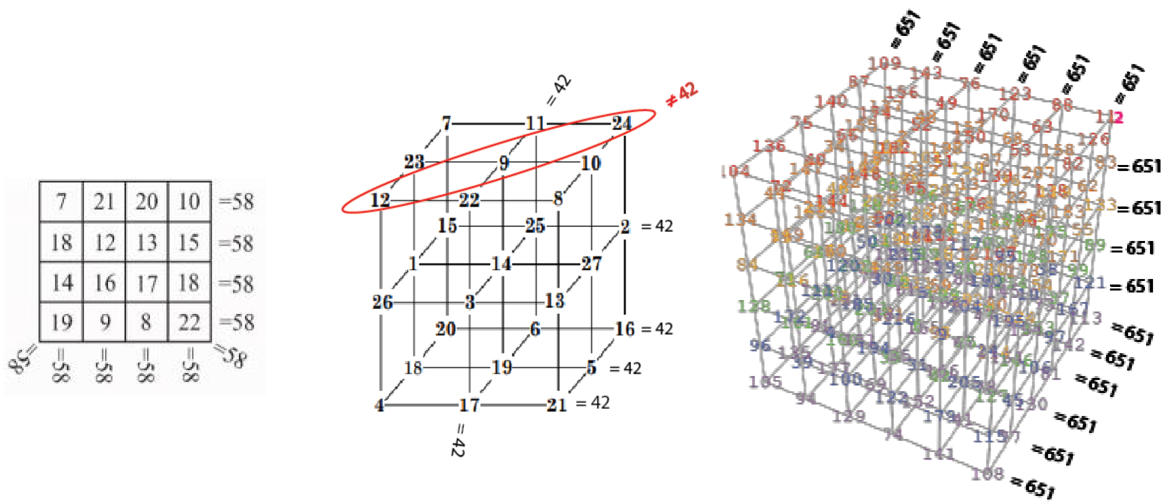
$$\alpha + (\alpha + \beta) + (\alpha + 2\beta) + \cdots + \alpha + (m^3 - 1)\beta = \frac{m^3}{2}(\alpha + \alpha + (m^3 - 1)\beta), \tag{3}$$

Since only one row, column, pillar, or diagonal space of the magic cube is needed, then

$$S_c = \frac{\frac{m^3}{2}(2\alpha + (m^3 - 1)\beta)}{m^2} = \frac{m}{2}(2\alpha + (m^3 - 1)\beta). \blacksquare \tag{4}$$

Definition 5. [18]. The perfect magic cube is a cubic array containing an arithmetic sequence whose entries appear exactly once arranged so that the sum of rows, columns, pillars, plane diagonals, and space diagonals produces the magic constant of the cube based on **Equation (4)**.

Example 3. [22]. The perfect magic cube with arithmetic sequence entries is illustrated in **Figure 1**(c).



(a) (b) (c)

Figure 1. (a) magic square of order four with $\alpha = 7, \beta = 1$, and the magic constant of squares sums to 58, (b) magic cube of order three with $\alpha = 1, \beta = 1$, and the magic constant of cubes sums to 42, (c) perfect magic cube of order six with $\alpha = 1, \beta = 1$, and the magic constant of cubes sums to 651

The magic cube that has arithmetic sequence entries can be determined using **Equation (4)** with the following information:

$Sc =$ the sum of a rows, columns, pillars, and space diagonals,

$m =$ order of the magic cube,

$\alpha =$ initial number,

$\beta =$ the difference between each number,

$\alpha, \beta \in \mathbb{Z}$ [23].

A perfect magic cube has different properties from a magic cube. The difference in properties between the magic cube and the perfect magic cube of order m causes the construction of the perfect magic cube to exist for $m \geq 1$, $m \neq 2,3,4$. The case of $m = 1$ has a trivial solution [24].

Theorem 1 [25]. *There is no perfect magic cube of order two.*
Proof. Let $C_2 = (c_{i,j,k})$, $i, j, k \in 1,2$ perfect magic cube of order two whose entries are arithmetic sequence with each element i rows, j columns, and k layers. Based on **Equation (4)** in **Lemma 1**, the magic constant of cube S_c. Take any square matrix in a layer e.g.:

$$C_2 = \begin{pmatrix} c_{1,1,1} & c_{1,2,1} \\ c_{2,1,1} & c_{2,2,1} \end{pmatrix},$$

based on the properties of magic cube $c_{1,1,1} + c_{1,2,1} = c_{1,1,1} + c_{2,1,1} = Sc$, $c_{1,2,1} = c_{2,1,1}$, since each entry must be different, it is a contradiction with **Definition 5**.■

Theorem 2 [18]. *There is no perfect magic cube of order three.*

Proof. Let $C_3 = (c_{i,j,k})$, $i, j, k \in 1,2,3$ perfect magic cube of order three whose entries are arithmetic sequence with each element i rows, j columns, and k layers. Based on **Equation (4)** in **Lemma 1**, the magic constant of cube S_c. Then it holds

$$c_{1,1,k} + c_{2,2,k} + c_{3,3,k} = Sc,\ c_{1,2,k} + c_{2,2,k} + c_{3,2,k} = Sc,$$

$$c_{1,3,k} + c_{2,2,k} + c_{3,1,k} = Sc,\ c_{2,1,k} + c_{2,2,k} + c_{2,3,k} = Sc,$$

these imply that $c_{2,2,k} = Sc/3$ for $k = 1,2$, and 3, a contradiction with **Definition 5**.■

Theorem 3 [18]. *There is no perfect magic cube of order four.*

Proof. Let $C_4 = (c_{i,j,k}), i, j, k \in 1,2,3,4$ perfect magic cube of order four whose entries are arithmetic sequence with each element i rows, j columns, and k layers. Based on **Equation (4)** in **Lemma 1**, the magic constant of cube S_c. First, it will be proved that each layer of the cube has the sum of its four corners elements S_c. Then it holds

$$c_{1,1,k} + c_{1,2,k} + c_{1,3,k} + c_{1,4,k} = c_{1,1,k} + c_{2,2,k} + c_{3,3,k} + c_{4,4,k} = c_{1,1,k} + c_{2,1,k} + c_{3,1,k} + c_{4,1,k} = Sc,$$

$$c_{1,4,k} + c_{2,3,k} + c_{3,2,k} + c_{4,1,k} = c_{1,4,k} + c_{2,4,k} + c_{3,4,k} + c_{4,4,k} = c_{4,1,k} + c_{4,2,k} + c_{4,3,k} + c_{4,4,k} = Sc,$$

these imply that

$$2(c_{1,1,k} + c_{1,4,k} + c_{4,1,k} + c_{4,4,k}) + \sum_{i=1}^{4}\sum_{j=1}^{4} c_{i,j,k} = 6(Sc),$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} c_{i,j,k} = 4(Sc),$$

$$c_{1,1,k} + c_{1,4,k} + c_{4,1,k} + c_{4,4,k} = Sc.$$

So, the sum of the corners is Sc for $k = 1,2,3,$ and 4. Consider only the corners perfect magic cube. Thus, we have

$$c_{1,1,1} + c_{1,1,4} + c_{4,1,1} + c_{4,1,4} = c_{4,1,1} + c_{4,1,4} + c_{4,4,1} + c_{4,4,4} = c_{1,1,1} + c_{1,1,4} + c_{4,4,1} + c_{4,4,4} = Sc,$$

hence

$$\begin{aligned} c_{1,1,1} + c_{1,1,4} &= \frac{1}{2}\big((c_{1,1,1} + c_{1,1,4} + c_{4,1,1} + c_{4,1,4}) + (c_{4,1,1} + c_{4,1,4} + c_{4,4,1} + c_{4,4,4}) \\ &\quad -(c_{1,1,1} + c_{1,1,4} + c_{4,4,1} + c_{4,4,4})\big), \\ &= \frac{1}{2}(Sc + Sc - Sc) = \frac{Sc}{2}. \end{aligned}$$

Therefore, any two corners sum up to $Sc/2$. Similarly, it can be shown that $c_{1,1,1} + c_{1,4,1} = Sc/2$, thus $c_{1,1,4} = c_{1,4,1}$, a contradiction with **Definition 5**.■

3. RESULTS AND DISCUSSION

The construction of a perfect magic cube is inspired by the construction of a magic square. The idea of constructing a magic square is to change/swap entries. The method of determining the pattern by assigning points can also construct a magic square. The combination of the two methods is used to construct the perfect magic cube of order $8n$ for $n \geq 1$. Visualizing a three-dimensional image (cube) can be simplified with a two-dimensional image. The following is the pattern of the perfect magic cube of order 8.

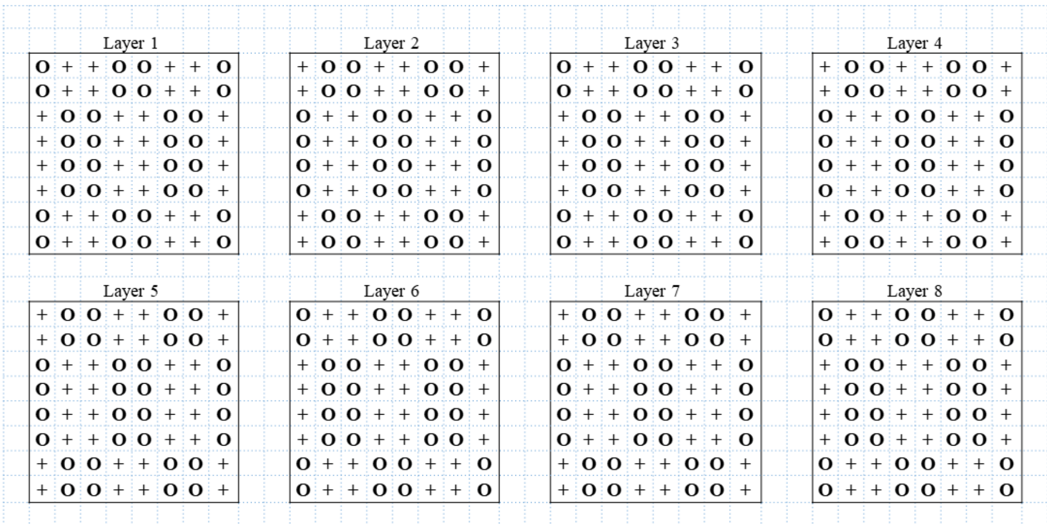


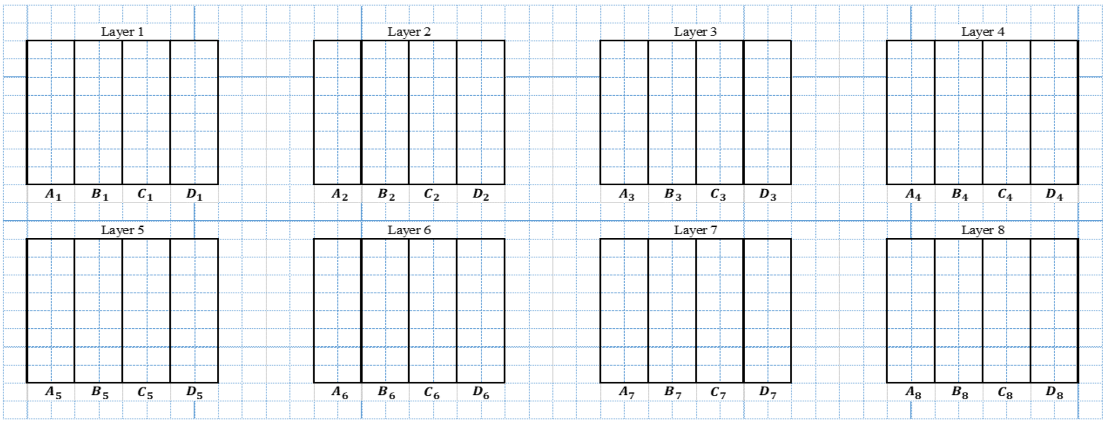
Figure 2. The pattern of Constructing the Perfect Magic cube of order 8

Each layer in the **Figure 2** is a partition of the perfect magic cube of order 8. Each layer can be assumed to be a matrix. Note that, each matrix has the the sum of the numbers of **O** and + elements in each row, column, main diagonal, and pillar are the same. This pattern will result in the number of rows, columns, pillars, and diagonals having the same value (magic constant of the cube). We shall now proceed to the construction of a perfect magic cube of order $8n$ for $n = 1,2$ with illustrations in each step and for $n > 2$ it will be proved by a theorem.

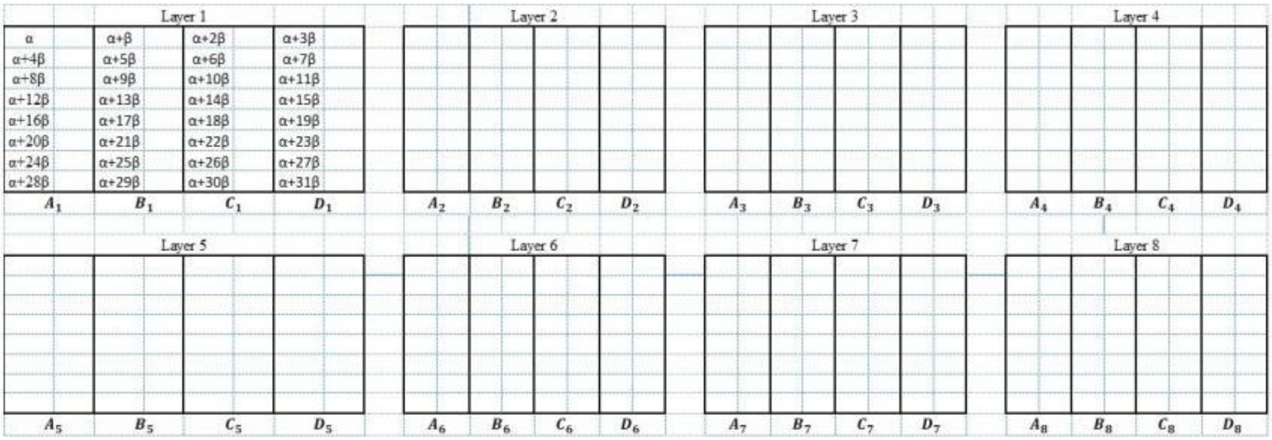
Theorem 4. *There exists a perfect magic cube of order* 8 *with arithmetic sequence.*

Proof. Define $Q_u =$ a u-th layer or matrix with $u = 1,2,3,\dots,t$ and $t = 8n$ partitioned from the cubic array matrix $C = [c_{i,j,k}: 1 \leq i,j,k \leq 8]$. Given an arithmetic sequence with initial number (α) and difference (β). Here is the construction of perfect magic cube of order $8n$ for $n = 1$.

1. Column partition the created even-order $(8n) \times (8n)$ matrix Q_u into four column matrices, e.g. $Q_u = [A_u, B_u, C_u, D_u]$.



2. Place the initial number in Q_1 in the first row and column of matrix A_1, continue to matrix B_1, C_1, and D_1 then continue to the second row of the first column repeatedly until the first column in matrix A_1, B_1, C_1, and D_1 is filled.



3. Place the next number (the difference $\left(\left(\frac{8n^2}{2}\right)\beta\right)$ from the previous column) to fill the next column in the matrix columns A_1, B_1, C_1, and D_1 following **Step 2**. It means, place $(\alpha + 32\beta)$ in the next column in the first row following **Step 2** until layer Q_1 is filled. Place the next number (the difference $((8n)^2\beta)$ from the previous layer) following **Step 2** and **Step 3** on $Q_2, Q_3, \dots, Q_{8n}$ until all matrices are filled. It means, place $(\alpha + 64\beta)$ starting from the first row of the first column of the second layer and then fill the next entry following **Step 2** and **Step 3** until layer $Q_2, Q_3, \dots, Q_{8n}$ are filled.

Layer 1							
α	α+32β	α+β	α+33β	α+2β	α+34β	α+3β	α+35β
α+4β	α+36β	α+5β	α+37β	α+6β	α+38β	α+7β	α+39β
α+8β	α+40β	α+9β	α+41β	α+10β	α+42β	α+11β	α+43β
α+12β	α+44β	α+13β	α+45β	α+14β	α+46β	α+15β	α+47β
α+16β	α+48β	α+17β	α+49β	α+18β	α+50β	α+19β	α+51β
α+20β	α+52β	α+21β	α+53β	α+22β	α+54β	α+23β	α+55β
α+24β	α+56β	α+25β	α+57β	α+26β	α+58β	α+27β	α+59β
α+28β	α+60β	α+29β	α+61β	α+30β	α+62β	α+31β	α+63β
A_1		B_1		C_1		D_1	

Layer 2							
α+64β	α+96β	α+65β	α+97β	α+66β	α+98β	α+67β	α+99β
α+68β	α+100β	α+69β	α+101β	α+70β	α+102β	α+71β	α+103β
α+72β	α+104β	α+73β	α+105β	α+74β	α+106β	α+75β	α+107β
α+76β	α+108β	α+77β	α+109β	α+78β	α+110β	α+79β	α+111β
α+80β	α+112β	α+81β	α+113β	α+82β	α+114β	α+83β	α+115β
α+84β	α+116β	α+85β	α+117β	α+86β	α+118β	α+87β	α+119β
α+88β	α+120β	α+89β	α+121β	α+90β	α+122β	α+91β	α+123β
α+92β	α+124β	α+93β	α+125β	α+94β	α+126β	α+95β	α+127β
A_2		B_2		C_2		D_2	

Layer 3							
α+128β	α+160β	α+129β	α+161β	α+130β	α+162β	α+131β	α+163β
α+132β	α+164β	α+133β	α+165β	α+134β	α+166β	α+135β	α+167β
α+136β	α+168β	α+137β	α+169β	α+138β	α+170β	α+139β	α+171β
α+140β	α+172β	α+141β	α+173β	α+142β	α+174β	α+143β	α+175β
α+144β	α+176β	α+145β	α+177β	α+146β	α+178β	α+147β	α+179β
α+148β	α+180β	α+149β	α+181β	α+150β	α+182β	α+151β	α+183β
α+152β	α+184β	α+153β	α+185β	α+154β	α+186β	α+155β	α+187β
α+156β	α+188β	α+157β	α+189β	α+158β	α+190β	α+159β	α+191β
A_3		B_3		C_3		D_3	

Layer 4							
α+192β	α+224β	α+193β	α+225β	α+194β	α+226β	α+195β	α+227β
α+196β	α+228β	α+197β	α+229β	α+198β	α+230β	α+199β	α+231β
α+200β	α+232β	α+201β	α+233β	α+202β	α+234β	α+203β	α+235β
α+204β	α+236β	α+205β	α+237β	α+206β	α+238β	α+207β	α+239β
α+208β	α+240β	α+209β	α+241β	α+210β	α+242β	α+211β	α+243β
α+212β	α+244β	α+213β	α+245β	α+214β	α+246β	α+215β	α+247β
α+216β	α+248β	α+217β	α+249β	α+218β	α+250β	α+219β	α+251β
α+220β	α+252β	α+221β	α+253β	α+222β	α+254β	α+223β	α+255β
A_4		B_4		C_4		D_4	

Layer 5							
α+256β	α+288β	α+257β	α+289β	α+258β	α+290β	α+259β	α+291β
α+260β	α+292β	α+261β	α+293β	α+262β	α+294β	α+263β	α+295β
α+264β	α+296β	α+265β	α+297β	α+266β	α+298β	α+267β	α+299β
α+268β	α+300β	α+269β	α+301β	α+270β	α+302β	α+271β	α+303β
α+272β	α+304β	α+273β	α+305β	α+274β	α+306β	α+275β	α+307β
α+276β	α+308β	α+277β	α+309β	α+278β	α+310β	α+279β	α+311β
α+280β	α+312β	α+281β	α+313β	α+282β	α+314β	α+283β	α+315β
α+284β	α+316β	α+285β	α+317β	α+286β	α+318β	α+287β	α+319β
A_5		B_5		C_5		D_5	

Layer 6							
α+320β	α+352β	α+321β	α+353β	α+322β	α+354β	α+323β	α+355β
α+324β	α+356β	α+325β	α+357β	α+326β	α+358β	α+327β	α+359β
α+328β	α+360β	α+329β	α+361β	α+330β	α+362β	α+331β	α+363β
α+332β	α+364β	α+333β	α+365β	α+334β	α+366β	α+335β	α+367β
α+336β	α+368β	α+337β	α+369β	α+338β	α+370β	α+339β	α+371β
α+340β	α+372β	α+341β	α+373β	α+342β	α+374β	α+343β	α+375β
α+344β	α+376β	α+345β	α+377β	α+346β	α+378β	α+347β	α+379β
α+348β	α+380β	α+349β	α+381β	α+350β	α+382β	α+351β	α+383β
A_6		B_6		C_6		D_6	

Layer 7							
α+384β	α+416β	α+385β	α+417β	α+386β	α+418β	α+387β	α+419β
α+388β	α+420β	α+389β	α+421β	α+390β	α+422β	α+391β	α+423β
α+392β	α+424β	α+393β	α+425β	α+394β	α+426β	α+395β	α+427β
α+396β	α+428β	α+397β	α+429β	α+398β	α+430β	α+399β	α+431β
α+400β	α+432β	α+401β	α+433β	α+402β	α+434β	α+403β	α+435β
α+404β	α+436β	α+405β	α+437β	α+406β	α+438β	α+407β	α+439β
α+408β	α+440β	α+409β	α+441β	α+410β	α+442β	α+411β	α+443β
α+412β	α+444β	α+413β	α+445β	α+414β	α+446β	α+415β	α+447β
A_7		B_7		C_7		D_7	

Layer 8							
α+448β	α+480β	α+449β	α+481β	α+450β	α+482β	α+451β	α+483β
α+452β	α+484β	α+453β	α+485β	α+454β	α+486β	α+455β	α+487β
α+456β	α+488β	α+457β	α+489β	α+458β	α+490β	α+459β	α+491β
α+460β	α+492β	α+461β	α+493β	α+462β	α+494β	α+463β	α+495β
α+464β	α+496β	α+465β	α+497β	α+466β	α+498β	α+467β	α+499β
α+468β	α+500β	α+469β	α+501β	α+470β	α+502β	α+471β	α+503β
α+472β	α+504β	α+473β	α+505β	α+474β	α+506β	α+475β	α+507β
α+476β	α+508β	α+477β	α+509β	α+478β	α+510β	α+479β	α+511β
A_8		B_8		C_8		D_8	

4. Suppose $X_u = [A_u, B_u]$ and $Y_u = [C_u, D_u]$ then $Q_u = [X_u, Y_u]$.

Layer 1							
α	α+32β	α+β	α+33β	α+2β	α+34β	α+3β	α+35β
α+4β	α+36β	α+5β	α+37β	α+6β	α+38β	α+7β	α+39β
α+8β	α+40β	α+9β	α+41β	α+10β	α+42β	α+11β	α+43β
α+12β	α+44β	α+13β	α+45β	α+14β	α+46β	α+15β	α+47β
α+16β	α+48β	α+17β	α+49β	α+18β	α+50β	α+19β	α+51β
α+20β	α+52β	α+21β	α+53β	α+22β	α+54β	α+23β	α+55β
α+24β	α+56β	α+25β	α+57β	α+26β	α+58β	α+27β	α+59β
α+28β	α+60β	α+29β	α+61β	α+30β	α+62β	α+31β	α+63β
X_1				Y_1			

Layer 2							
α+64β	α+96β	α+65β	α+97β	α+66β	α+98β	α+67β	α+99β
α+68β	α+100β	α+69β	α+101β	α+70β	α+102β	α+71β	α+103β
α+72β	α+104β	α+73β	α+105β	α+74β	α+106β	α+75β	α+107β
α+76β	α+108β	α+77β	α+109β	α+78β	α+110β	α+79β	α+111β
α+80β	α+112β	α+81β	α+113β	α+82β	α+114β	α+83β	α+115β
α+84β	α+116β	α+85β	α+117β	α+86β	α+118β	α+87β	α+119β
α+88β	α+120β	α+89β	α+121β	α+90β	α+122β	α+91β	α+123β
α+92β	α+124β	α+93β	α+125β	α+94β	α+126β	α+95β	α+127β
X_2				Y_2			

Layer 3							
α+128β	α+160β	α+129β	α+161β	α+130β	α+162β	α+131β	α+163β
α+132β	α+164β	α+133β	α+165β	α+134β	α+166β	α+135β	α+167β
α+136β	α+168β	α+137β	α+169β	α+138β	α+170β	α+139β	α+171β
α+140β	α+172β	α+141β	α+173β	α+142β	α+174β	α+143β	α+175β
α+144β	α+176β	α+145β	α+177β	α+146β	α+178β	α+147β	α+179β
α+148β	α+180β	α+149β	α+181β	α+150β	α+182β	α+151β	α+183β
α+152β	α+184β	α+153β	α+185β	α+154β	α+186β	α+155β	α+187β
α+156β	α+188β	α+157β	α+189β	α+158β	α+190β	α+159β	α+191β
X_3				Y_3			

Layer 4							
α+192β	α+224β	α+193β	α+225β	α+194β	α+226β	α+195β	α+227β
α+196β	α+228β	α+197β	α+229β	α+198β	α+230β	α+199β	α+231β
α+200β	α+232β	α+201β	α+233β	α+202β	α+234β	α+203β	α+235β
α+204β	α+236β	α+205β	α+237β	α+206β	α+238β	α+207β	α+239β
α+208β	α+240β	α+209β	α+241β	α+210β	α+242β	α+211β	α+243β
α+212β	α+244β	α+213β	α+245β	α+214β	α+246β	α+215β	α+247β
α+216β	α+248β	α+217β	α+249β	α+218β	α+250β	α+219β	α+251β
α+220β	α+252β	α+221β	α+253β	α+222β	α+254β	α+223β	α+255β
X_4				Y_4			

Layer 5							
α+256β	α+288β	α+257β	α+289β	α+258β	α+290β	α+259β	α+291β
α+260β	α+292β	α+261β	α+293β	α+262β	α+294β	α+263β	α+295β
α+264β	α+296β	α+265β	α+297β	α+266β	α+298β	α+267β	α+299β
α+268β	α+300β	α+269β	α+301β	α+270β	α+302β	α+271β	α+303β
α+272β	α+304β	α+273β	α+305β	α+274β	α+306β	α+275β	α+307β
α+276β	α+308β	α+277β	α+309β	α+278β	α+310β	α+279β	α+311β
α+280β	α+312β	α+281β	α+313β	α+282β	α+314β	α+283β	α+315β
α+284β	α+316β	α+285β	α+317β	α+286β	α+318β	α+287β	α+319β
X_5				Y_5			

Layer 6							
α+320β	α+352β	α+321β	α+353β	α+322β	α+354β	α+323β	α+355β
α+324β	α+356β	α+325β	α+357β	α+326β	α+358β	α+327β	α+359β
α+328β	α+360β	α+329β	α+361β	α+330β	α+362β	α+331β	α+363β
α+332β	α+364β	α+333β	α+365β	α+334β	α+366β	α+335β	α+367β
α+336β	α+368β	α+337β	α+369β	α+338β	α+370β	α+339β	α+371β
α+340β	α+372β	α+341β	α+373β	α+342β	α+374β	α+343β	α+375β
α+344β	α+376β	α+345β	α+377β	α+346β	α+378β	α+347β	α+379β
α+348β	α+380β	α+349β	α+381β	α+350β	α+382β	α+351β	α+383β
X_6				Y_6			

Layer 7							
α+384β	α+416β	α+385β	α+417β	α+386β	α+418β	α+387β	α+419β
α+388β	α+420β	α+389β	α+421β	α+390β	α+422β	α+391β	α+423β
α+392β	α+424β	α+393β	α+425β	α+394β	α+426β	α+395β	α+427β
α+396β	α+428β	α+397β	α+429β	α+398β	α+430β	α+399β	α+431β
α+400β	α+432β	α+401β	α+433β	α+402β	α+434β	α+403β	α+435β
α+404β	α+436β	α+405β	α+437β	α+406β	α+438β	α+407β	α+439β
α+408β	α+440β	α+409β	α+441β	α+410β	α+442β	α+411β	α+443β
α+412β	α+444β	α+413β	α+445β	α+414β	α+446β	α+415β	α+447β
X_7				Y_7			

Layer 8							
α+448β	α+480β	α+449β	α+481β	α+450β	α+482β	α+451β	α+483β
α+452β	α+484β	α+453β	α+485β	α+454β	α+486β	α+455β	α+487β
α+456β	α+488β	α+457β	α+489β	α+458β	α+490β	α+459β	α+491β
α+460β	α+492β	α+461β	α+493β	α+462β	α+494β	α+463β	α+495β
α+464β	α+496β	α+465β	α+497β	α+466β	α+498β	α+467β	α+499β
α+468β	α+500β	α+469β	α+501β	α+470β	α+502β	α+471β	α+503β
α+472β	α+504β	α+473β	α+505β	α+474β	α+506β	α+475β	α+507β
α+476β	α+508β	α+477β	α+509β	α+478β	α+510β	α+479β	α+511β
X_8				Y_8			

5. Mark the entries in each corner area of the matrix with sizes $\frac{1}{4}(8n) \times \frac{1}{8}(8n)$ and $\frac{1}{2}(8n) \times \frac{1}{4}(8n)$ of the center areas of matrices X_u and Y_u in the $Q_{1:2:(4n-1)}$ and $Q_{t-(4n-2):2:t}$ layers.

Layer 1							
α	α+32β	α+β	α+33β	α+2β	α+34β	α+3β	α+35β
α+4β	α+36β	α+5β	α+37β	α+6β	α+38β	α+7β	α+39β
α+8β	α+40β	α+9β	α+41β	α+10β	α+42β	α+11β	α+43β
α+12β	α+44β	α+13β	α+45β	α+14β	α+46β	α+15β	α+47β
α+16β	α+48β	α+17β	α+49β	α+18β	α+50β	α+19β	α+51β
α+20β	α+52β	α+21β	α+53β	α+22β	α+54β	α+23β	α+55β
α+24β	α+56β	α+25β	α+57β	α+26β	α+58β	α+27β	α+59β
α+28β	α+60β	α+29β	α+61β	α+30β	α+62β	α+31β	α+63β
X_1				Y_1			

Layer 2							
α+64β	α+96β	α+65β	α+97β	α+66β	α+98β	α+67β	α+99β
α+68β	α+100β	α+69β	α+101β	α+70β	α+102β	α+71β	α+103β
α+72β	α+104β	α+73β	α+105β	α+74β	α+106β	α+75β	α+107β
α+76β	α+108β	α+77β	α+109β	α+78β	α+110β	α+79β	α+111β
α+80β	α+112β	α+81β	α+113β	α+82β	α+114β	α+83β	α+115β
α+84β	α+116β	α+85β	α+117β	α+86β	α+118β	α+87β	α+119β
α+88β	α+120β	α+89β	α+121β	α+90β	α+122β	α+91β	α+123β
α+92β	α+124β	α+93β	α+125β	α+94β	α+126β	α+95β	α+127β
X_2				Y_2			

Layer 3							
α+128β	α+160β	α+129β	α+161β	α+130β	α+162β	α+131β	α+163β
α+132β	α+164β	α+133β	α+165β	α+134β	α+166β	α+135β	α+167β
α+136β	α+168β	α+137β	α+169β	α+138β	α+170β	α+139β	α+171β
α+140β	α+172β	α+141β	α+173β	α+142β	α+174β	α+143β	α+175β
α+144β	α+176β	α+145β	α+177β	α+146β	α+178β	α+147β	α+179β
α+148β	α+180β	α+149β	α+181β	α+150β	α+182β	α+151β	α+183β
α+152β	α+184β	α+153β	α+185β	α+154β	α+186β	α+155β	α+187β
α+156β	α+188β	α+157β	α+189β	α+158β	α+190β	α+159β	α+191β
X_3				Y_3			

Layer 4							
α+192β	α+224β	α+193β	α+225β	α+194β	α+226β	α+195β	α+227β
α+196β	α+228β	α+197β	α+229β	α+198β	α+230β	α+199β	α+231β
α+200β	α+232β	α+201β	α+233β	α+202β	α+234β	α+203β	α+235β
α+204β	α+236β	α+205β	α+237β	α+206β	α+238β	α+207β	α+239β
α+208β	α+240β	α+209β	α+241β	α+210β	α+242β	α+211β	α+243β
α+212β	α+244β	α+213β	α+245β	α+214β	α+246β	α+215β	α+247β
α+216β	α+248β	α+217β	α+249β	α+218β	α+250β	α+219β	α+251β
α+220β	α+252β	α+221β	α+253β	α+222β	α+254β	α+223β	α+255β
X_4				Y_4			

Layer 5							
α+256β	α+288β	α+257β	α+289β	α+258β	α+290β	α+259β	α+291β
α+260β	α+292β	α+261β	α+293β	α+262β	α+294β	α+263β	α+295β
α+264β	α+296β	α+265β	α+297β	α+266β	α+298β	α+267β	α+299β
α+268β	α+300β	α+269β	α+301β	α+270β	α+302β	α+271β	α+303β
α+272β	α+304β	α+273β	α+305β	α+274β	α+306β	α+275β	α+307β
α+276β	α+308β	α+277β	α+309β	α+278β	α+310β	α+279β	α+311β
α+280β	α+312β	α+281β	α+313β	α+282β	α+314β	α+283β	α+315β
α+284β	α+316β	α+285β	α+317β	α+286β	α+318β	α+287β	α+319β
X_5				Y_5			

Layer 6							
α+320β	α+352β	α+321β	α+353β	α+322β	α+354β	α+323β	α+355β
α+324β	α+356β	α+325β	α+357β	α+326β	α+358β	α+327β	α+359β
α+328β	α+360β	α+329β	α+361β	α+330β	α+362β	α+331β	α+363β
α+332β	α+364β	α+333β	α+365β	α+334β	α+366β	α+335β	α+367β
α+336β	α+368β	α+337β	α+369β	α+338β	α+370β	α+339β	α+371β
α+340β	α+372β	α+341β	α+373β	α+342β	α+374β	α+343β	α+375β
α+344β	α+376β	α+345β	α+377β	α+346β	α+378β	α+347β	α+379β
α+348β	α+380β	α+349β	α+381β	α+350β	α+382β	α+351β	α+383β
X_6				Y_6			

Layer 7							
α+384β	α+416β	α+385β	α+417β	α+386β	α+418β	α+387β	α+419β
α+388β	α+420β	α+389β	α+421β	α+390β	α+422β	α+391β	α+423β
α+392β	α+424β	α+393β	α+425β	α+394β	α+426β	α+395β	α+427β
α+396β	α+428β	α+397β	α+429β	α+398β	α+430β	α+399β	α+431β
α+400β	α+432β	α+401β	α+433β	α+402β	α+434β	α+403β	α+435β
α+404β	α+436β	α+405β	α+437β	α+406β	α+438β	α+407β	α+439β
α+408β	α+440β	α+409β	α+441β	α+410β	α+442β	α+411β	α+443β
α+412β	α+444β	α+413β	α+445β	α+414β	α+446β	α+415β	α+447β
X_7				Y_7			

Layer 8							
α+448β	α+480β	α+449β	α+481β	α+450β	α+482β	α+451β	α+483β
α+452β	α+484β	α+453β	α+485β	α+454β	α+486β	α+455β	α+487β
α+456β	α+488β	α+457β	α+489β	α+458β	α+490β	α+459β	α+491β
α+460β	α+492β	α+461β	α+493β	α+462β	α+494β	α+463β	α+495β
α+464β	α+496β	α+465β	α+497β	α+466β	α+498β	α+467β	α+499β
α+468β	α+500β	α+469β	α+501β	α+470β	α+502β	α+471β	α+503β
α+472β	α+504β	α+473β	α+505β	α+474β	α+506β	α+475β	α+507β
α+476β	α+508β	α+477β	α+509β	α+478β	α+510β	α+479β	α+511β
X_8				Y_8			

6. Mark the entries in each corner area and the center of the matrix other than in **Step 5** in the $Q_{2:2:(4n)}$ and $Q_{t-(4n-1):2:(t-1)}$ layers.

Layer 1							
α	α+32β	α+β	α+33β	α+2β	α+34β	α+3β	α+35β
α+4β	α+36β	α+5β	α+37β	α+6β	α+38β	α+7β	α+39β
α+8β	α+40β	α+9β	α+41β	α+10β	α+42β	α+11β	α+43β
α+12β	α+44β	α+13β	α+45β	α+14β	α+46β	α+15β	α+47β
α+16β	α+48β	α+17β	α+49β	α+18β	α+50β	α+19β	α+51β
α+20β	α+52β	α+21β	α+53β	α+22β	α+54β	α+23β	α+55β
α+24β	α+56β	α+25β	α+57β	α+26β	α+58β	α+27β	α+59β
α+28β	α+60β	α+29β	α+61β	α+30β	α+62β	α+31β	α+63β
X_1				Y_1			

Layer 2							
α+64β	α+96β	α+65β	α+97β	α+66β	α+98β	α+67β	α+99β
α+68β	α+100β	α+69β	α+101β	α+70β	α+102β	α+71β	α+103β
α+72β	α+104β	α+73β	α+105β	α+74β	α+106β	α+75β	α+107β
α+76β	α+108β	α+77β	α+109β	α+78β	α+110β	α+79β	α+111β
α+80β	α+112β	α+81β	α+113β	α+82β	α+114β	α+83β	α+115β
α+84β	α+116β	α+85β	α+117β	α+86β	α+118β	α+87β	α+119β
α+88β	α+120β	α+89β	α+121β	α+90β	α+122β	α+91β	α+123β
α+92β	α+124β	α+93β	α+125β	α+94β	α+126β	α+95β	α+127β
X_2				Y_2			

Layer 3							
α+128β	α+160β	α+129β	α+161β	α+130β	α+162β	α+131β	α+163β
α+132β	α+164β	α+133β	α+165β	α+134β	α+166β	α+135β	α+167β
α+136β	α+168β	α+137β	α+169β	α+138β	α+170β	α+139β	α+171β
α+140β	α+172β	α+141β	α+173β	α+142β	α+174β	α+143β	α+175β
α+144β	α+176β	α+145β	α+177β	α+146β	α+178β	α+147β	α+179β
α+148β	α+180β	α+149β	α+181β	α+150β	α+182β	α+151β	α+183β
α+152β	α+184β	α+153β	α+185β	α+154β	α+186β	α+155β	α+187β
α+156β	α+188β	α+157β	α+189β	α+158β	α+190β	α+159β	α+191β
X_3				Y_3			

Layer 4							
α+192β	α+224β	α+193β	α+225β	α+194β	α+226β	α+195β	α+227β
α+196β	α+228β	α+197β	α+229β	α+198β	α+230β	α+199β	α+231β
α+200β	α+232β	α+201β	α+233β	α+202β	α+234β	α+203β	α+235β
α+204β	α+236β	α+205β	α+237β	α+206β	α+238β	α+207β	α+239β
α+208β	α+240β	α+209β	α+241β	α+210β	α+242β	α+211β	α+243β
α+212β	α+244β	α+213β	α+245β	α+214β	α+246β	α+215β	α+247β
α+216β	α+248β	α+217β	α+249β	α+218β	α+250β	α+219β	α+251β
α+220β	α+252β	α+221β	α+253β	α+222β	α+254β	α+223β	α+255β
X_4				Y_4			

Layer 5							
α+256β	α+288β	α+257β	α+289β	α+258β	α+290β	α+259β	α+291β
α+260β	α+292β	α+261β	α+293β	α+262β	α+294β	α+263β	α+295β
α+264β	α+296β	α+265β	α+297β	α+266β	α+298β	α+267β	α+299β
α+268β	α+300β	α+269β	α+301β	α+270β	α+302β	α+271β	α+303β
α+272β	α+304β	α+273β	α+305β	α+274β	α+306β	α+275β	α+307β
α+276β	α+308β	α+277β	α+309β	α+278β	α+310β	α+279β	α+311β
α+280β	α+312β	α+281β	α+313β	α+282β	α+314β	α+283β	α+315β
α+284β	α+316β	α+285β	α+317β	α+286β	α+318β	α+287β	α+319β
X_5				Y_5			

Layer 6							
α+320β	α+352β	α+321β	α+353β	α+322β	α+354β	α+323β	α+355β
α+324β	α+356β	α+325β	α+357β	α+326β	α+358β	α+327β	α+359β
α+328β	α+360β	α+329β	α+361β	α+330β	α+362β	α+331β	α+363β
α+332β	α+364β	α+333β	α+365β	α+334β	α+366β	α+335β	α+367β
α+336β	α+368β	α+337β	α+369β	α+338β	α+370β	α+339β	α+371β
α+340β	α+372β	α+341β	α+373β	α+342β	α+374β	α+343β	α+375β
α+344β	α+376β	α+345β	α+377β	α+346β	α+378β	α+347β	α+379β
α+348β	α+380β	α+349β	α+381β	α+350β	α+382β	α+351β	α+383β
X_6				Y_6			

Layer 7							
α+384β	α+416β	α+385β	α+417β	α+386β	α+418β	α+387β	α+419β
α+388β	α+420β	α+389β	α+421β	α+390β	α+422β	α+391β	α+423β
α+392β	α+424β	α+393β	α+425β	α+394β	α+426β	α+395β	α+427β
α+396β	α+428β	α+397β	α+429β	α+398β	α+430β	α+399β	α+431β
α+400β	α+432β	α+401β	α+433β	α+402β	α+434β	α+403β	α+435β
α+404β	α+436β	α+405β	α+437β	α+406β	α+438β	α+407β	α+439β
α+408β	α+440β	α+409β	α+441β	α+410β	α+442β	α+411β	α+443β
α+412β	α+444β	α+413β	α+445β	α+414β	α+446β	α+415β	α+447β
X_7				Y_7			

Layer 8							
α+448β	α+480β	α+449β	α+481β	α+450β	α+482β	α+451β	α+483β
α+452β	α+484β	α+453β	α+485β	α+454β	α+486β	α+455β	α+487β
α+456β	α+488β	α+457β	α+489β	α+458β	α+490β	α+459β	α+491β
α+460β	α+492β	α+461β	α+493β	α+462β	α+494β	α+463β	α+495β
α+464β	α+496β	α+465β	α+497β	α+466β	α+498β	α+467β	α+499β
α+468β	α+500β	α+469β	α+501β	α+470β	α+502β	α+471β	α+503β
α+472β	α+504β	α+473β	α+505β	α+474β	α+506β	α+475β	α+507β
α+476β	α+508β	α+477β	α+509β	α+478β	α+510β	α+479β	α+511β
X_8				Y_8			

7. Exchange the matrix entries marked in **Step 5** and **Step 6** by crossing and reflection respectively to form a perfect magic cube (PMC).

Layer 1

α+511β	α+32β	α+β	α+478β	α+509β	α+34β	α+3β	α+476β
α+507β	α+36β	α+5β	α+474β	α+505β	α+38β	α+7β	α+472β
α+8β	α+471β	α+502β	α+41β	α+10β	α+469β	α+500β	α+43β
α+12β	α+467β	α+498β	α+45β	α+14β	α+465β	α+496β	α+47β
α+16β	α+463β	α+494β	α+49β	α+18β	α+461β	α+492β	α+51β
α+20β	α+459β	α+490β	α+53β	α+22β	α+457β	α+488β	α+55β
α+487β	α+56β	α+25β	α+454β	α+485β	α+58β	α+27β	α+452β
α+483β	α+60β	α+29β	α+450β	α+481β	α+62β	α+31β	α+448β
X_1				Y_1			

Layer 2

α+64β	α+415β	α+446β	α+97β	α+66β	α+413β	α+444β	α+99β
α+68β	α+411β	α+442β	α+101β	α+70β	α+409β	α+440β	α+103β
α+439β	α+104β	α+73β	α+406β	α+437β	α+106β	α+75β	α+404β
α+435β	α+108β	α+77β	α+402β	α+433β	α+110β	α+79β	α+400β
α+431β	α+112β	α+81β	α+398β	α+429β	α+114β	α+83β	α+396β
α+427β	α+116β	α+85β	α+394β	α+425β	α+118β	α+87β	α+392β
α+88β	α+391β	α+422β	α+121β	α+90β	α+389β	α+420β	α+123β
α+92β	α+387β	α+418β	α+125β	α+94β	α+385β	α+416β	α+127β
X_2				Y_2			

Layer 3

α+383β	α+160β	α+129β	α+350β	α+381β	α+162β	α+131β	α+348β
α+379β	α+164β	α+133β	α+346β	α+377β	α+166β	α+135β	α+344β
α+136β	α+343β	α+374β	α+169β	α+138β	α+341β	α+372β	α+171β
α+140β	α+339β	α+370β	α+173β	α+142β	α+337β	α+368β	α+175β
α+144β	α+335β	α+366β	α+177β	α+146β	α+333β	α+364β	α+179β
α+148β	α+331β	α+362β	α+181β	α+150β	α+329β	α+360β	α+183β
α+359β	α+184β	α+153β	α+326β	α+357β	α+186β	α+155β	α+324β
α+355β	α+188β	α+157β	α+322β	α+353β	α+190β	α+159β	α+320β
X_3				Y_3			

Layer 4

α+192β	α+287β	α+318β	α+225β	α+194β	α+286β	α+316β	α+227β
α+196β	α+283β	α+314β	α+229β	α+198β	α+282β	α+312β	α+231β
α+311β	α+232β	α+201β	α+278β	α+309β	α+234β	α+203β	α+276β
α+307β	α+236β	α+205β	α+274β	α+305β	α+238β	α+207β	α+272β
α+303β	α+240β	α+209β	α+270β	α+301β	α+242β	α+211β	α+268β
α+299β	α+244β	α+213β	α+236β	α+297β	α+246β	α+215β	α+264β
α+216β	α+264β	α+294β	α+249β	α+218β	α+261β	α+292β	α+251β
α+220β	α+260β	α+290β	α+253β	α+222β	α+257β	α+288β	α+255β
X_4				Y_4			

Layer 5

α+256β	α+223β	α+254β	α+289β	α+258β	α+221β	α+252β	α+291β
α+260β	α+219β	α+250β	α+293β	α+262β	α+217β	α+248β	α+295β
α+247β	α+296β	α+265β	α+214β	α+245β	α+298β	α+267β	α+212β
α+243β	α+300β	α+269β	α+210β	α+241β	α+302β	α+271β	α+208β
α+239β	α+304β	α+273β	α+206β	α+237β	α+306β	α+275β	α+204β
α+235β	α+308β	α+277β	α+202β	α+233β	α+310β	α+279β	α+200β
α+280β	α+199β	α+230β	α+313β	α+282β	α+197β	α+228β	α+315β
α+284β	α+195β	α+226β	α+317β	α+286β	α+193β	α+224β	α+319β
X_5				Y_5			

Layer 6

α+191β	α+352β	α+321β	α+158β	α+189β	α+354β	α+323β	α+156β
α+187β	α+356β	α+325β	α+154β	α+185β	α+358β	α+327β	α+152β
α+328β	α+151β	α+182β	α+361β	α+330β	α+149β	α+180β	α+363β
α+332β	α+147β	α+178β	α+365β	α+334β	α+145β	α+176β	α+367β
α+336β	α+143β	α+174β	α+369β	α+338β	α+141β	α+172β	α+371β
α+340β	α+139β	α+170β	α+373β	α+342β	α+137β	α+168β	α+375β
α+167β	α+376β	α+345β	α+134β	α+165β	α+378β	α+347β	α+132β
α+163β	α+380β	α+349β	α+130β	α+161β	α+382β	α+351β	α+128β
X_6				Y_6			

Layer 7

α+384β	α+95β	α+126β	α+417β	α+386β	α+93β	α+124β	α+419β
α+388β	α+91β	α+122β	α+421β	α+390β	α+89β	α+120β	α+423β
α+119β	α+424β	α+393β	α+86β	α+117β	α+426β	α+395β	α+84β
α+115β	α+428β	α+397β	α+82β	α+113β	α+430β	α+399β	α+80β
α+111β	α+432β	α+401β	α+78β	α+109β	α+434β	α+403β	α+76β
α+107β	α+436β	α+405β	α+74β	α+105β	α+438β	α+407β	α+72β
α+408β	α+71β	α+102β	α+441β	α+410β	α+69β	α+100β	α+443β
α+412β	α+67β	α+98β	α+445β	α+414β	α+65β	α+96β	α+447β
X_7				Y_7			

Layer 8

α+63β	α+480β	α+449β	α+30β	α+61β	α+482β	α+451β	α+28β
α+59β	α+484β	α+453β	α+26β	α+57β	α+486β	α+455β	α+24β
α+456β	α+23β	α+54β	α+489β	α+458β	α+490β	α+52β	α+491β
α+460β	α+19β	α+50β	α+493β	α+462β	α+494β	α+48β	α+495β
α+464β	α+15β	α+46β	α+497β	α+466β	α+498β	α+44β	α+499β
α+468β	α+11β	α+42β	α+501β	α+470β	α+502β	α+37β	α+503β
α+39β	α+504β	α+473β	α+6β	α+37β	α+506β	α+475β	α+4β
α+35β	α+508β	α+477β	α+2β	α+33β	α+510β	α+479β	α
X_8				Y_8			

The yellow background represents the **O** element and the white background represents the $+$ element in **Figure 2**. In the **Step 7** illustration, the sum of the entries of each row, column, pillar, diagonal plane, and diagonal space is the same. This can be proven using **Equation (4)** in **Lemma 1** or can be calculated manually in a row, column, pillar, diagonal space and diagonal plane as follows:

i. Choose any rows, e.g:

$$\sum_{j=1}^{8} c_{1,j,8} = c_{1,1,8} + c_{1,2,8} + \cdots + c_{1,8,8} = (\alpha + 63\beta) + (\alpha + 480\beta) + \cdots + (\alpha + 28\beta) = 8\alpha + 2044\beta.$$

ii. Choose any columns, e.g:

$$\sum_{i=1}^{8} c_{i,4,5} = c_{1,4,5} + c_{2,4,5} + \cdots + c_{8,4,5} = (\alpha + 289\beta) + (\alpha + 293\beta) + \cdots + (\alpha + 317\beta) = 8\alpha + 2044\beta.$$

iii. Choose any pillars, e.g:

$$\sum_{k=1}^{8} c_{1,1,k} = c_{1,1,1} + c_{1,1,2} + \cdots + c_{1,1,8} = (\alpha + 511\beta) + (\alpha + 64\beta) + \cdots + (\alpha + 63\beta) = 8\alpha + 2044\beta.$$

iv. Choose any plane diagonals, e.g:

$$\sum_{i=j=1}^{8} c_{i,j,3} = c_{1,1,3} + c_{2,2,3} + \cdots + c_{8,8,3} = (\alpha + 383\beta) + (\alpha + 164\beta) + \cdots + (\alpha + 320\beta) = 8\alpha + 2044\beta.$$

v. Choose any space diagonals, e.g:

$$\sum_{i=j=k=1}^{8} c_{i,j,k} = c_{1,1,1} + c_{2,2,2} + \cdots + c_{8,8,8} = (\alpha + 511\beta) + (\alpha + 411\beta) + \cdots + \alpha = 8\alpha + 2044\beta. \blacksquare$$

Theorem 5. *There exists a perfect magic cube of order* 16 *with arithmetic sequence.*

Proof. Define Q_u = a u-th layer or matrix with $u = 1,2,3,\ldots,t$ and $t = 8n$ partitioned from the cubic array matrix $C = [c_{i,j,k}: 1 \le i,j,k \le 16]$. Given an arithmetic sequence with initial number (α) and difference

(β). The following results of constructing a perfect magic cube of order $8n$ for $n = 2$ using **Steps 1-7** in **Theorem 4** are illustrated in **Figure 3**.

Layer 1

α+4095β	α+4031β	α+128β	α+192β	α+β	α+65β	α+3966β	α+3902β	α+4093β	α+4029β	α+130β	α+194β	α+3β	α+67β	α+3964β	α+3900β
α+4091β	α+4027β	α+132β	α+196β	α+5β	α+69β	α+3962β	α+3898β	α+4089β	α+4025β	α+134β	α+198β	α+7β	α+71β	α+3960β	α+3896β
α+4000β	α+4023β	α+136β	α+200β	α+9β	α+73β	α+3958β	α+3894β	α+4085β	α+4021β	α+138β	α+202β	α+11β	α+75β	α+3956β	α+3892β
α+4083β	α+4019β	α+140β	α+204β	α+13β	α+77β	α+3954β	α+3890β	α+4081β	α+4017β	α+142β	α+206β	α+15β	α+79β	α+3952β	α+3888β
α+16β	α+80β	α+3951β	α+3887β	α+4078β	α+4014β	α+145β	α+209β	α+18β	α+82β	α+3949β	α+3885β	α+4076β	α+4012β	α+147β	α+211β
α+20β	α+84β	α+3947β	α+3883β	α+4074β	α+4010β	α+149β	α+213β	α+22β	α+86β	α+3945β	α+3881β	α+4072β	α+4008β	α+151β	α+215β
α+24β	α+88β	α+3943β	α+3879β	α+4070β	α+4006β	α+153β	α+217β	α+26β	α+90β	α+3941β	α+3877β	α+4068β	α+4004β	α+155β	α+219β
α+28β	α+92β	α+3939β	α+3875β	α+4066β	α+40002	α+157β	α+221β	α+30β	α+94β	α+3937β	α+3873β	α+4064β	α+4000β	α+159β	α+223β
α+32β	α+96β	α+3935β	α+3871β	α+4062β	α+3998β	α+161β	α+225β	α+34β	α+98β	α+3933β	α+3869β	α+4060β	α+3996β	α+163β	α+227β
α+36β	α+100β	α+3931β	α+3867β	α+4058β	α+3994β	α+165β	α+229β	α+38β	α+102β	α+3929β	α+3865β	α+4056β	α+3992β	α+167β	α+231β
α+40β	α+104β	α+3927β	α+3863β	α+4054β	α+3990β	α+169β	α+233β	α+42β	α+106β	α+3925β	α+3861β	α+4052β	α+3988β	α+171β	α+235β
α+44β	α+108β	α+3923β	α+3859β	α+4050β	α+3986β	α+173β	α+237β	α+46β	α+110β	α+3921β	α+3857β	α+4048β	α+3984β	α+175β	α+239β
α+4047β	α+3983β	α+176β	α+240β	α+49β	α+113β	α+3918β	α+3854β	α+4045β	α+3981β	α+178β	α+242β	α+51β	α+115β	α+3916β	α+3852β
α+4043β	α+3979β	α+180β	α+244β	α+53β	α+117β	α+3914β	α+3850β	α+4041β	α+3977β	α+182β	α+246β	α+55β	α+119β	α+3912β	α+3848β
α+4039β	α+3975β	α+184β	α+248β	α+57β	α+121β	α+3910β	α+3846β	α+4037β	α+3973β	α+186β	α+250β	α+59β	α+123β	α+3908β	α+3844β
α+4035β	α+3971β	α+188β	α+252β	α+61β	α+125β	α+3906β	α+3842β	α+4033β	α+3969β	α+190β	α+254β	α+63β	α+127β	α+3904β	α+3840β

Layer 2

α+256β	α+320β	α+3711β	α+3647β	α+3838β	α+3774β	α+385β	α+449β	α+258β	α+322β	α+3709β	α+3645β	α+3836β	α+3772β	α+387β	α+451β
α+260β	α+324β	α+3707β	α+3643β	α+3834β	α+3770β	α+389β	α+453β	α+262β	α+326β	α+3705β	α+3641β	α+3832β	α+3768β	α+391β	α+455β
α+264β	α+328β	α+3703β	α+3639β	α+3830β	α+3766β	α+393β	α+457β	α+266β	α+330β	α+3701β	α+3637β	α+3828β	α+3764β	α+395β	α+459β
α+268β	α+332β	α+3699β	α+3635β	α+3826β	α+3762β	α+397β	α+461β	α+270β	α+334β	α+3697β	α+3633β	α+3824β	α+3760β	α+399β	α+463β
α+3823β	α+3759β	α+400β	α+464β	α+273β	α+337β	α+3694β	α+3630β	α+3821β	α+3757β	α+402β	α+466β	α+275β	α+339β	α+3692β	α+3628β
α+3819β	α+3755β	α+404β	α+468β	α+277β	α+341β	α+3690β	α+3626β	α+3817β	α+3753β	α+406β	α+470β	α+279β	α+343β	α+3688β	α+3624β
α+3815β	α+3751β	α+408β	α+472β	α+281β	α+345β	α+3686β	α+3622β	α+3813β	α+3749β	α+410β	α+474β	α+283β	α+347β	α+3684β	α+3620β
α+3811β	α+3747β	α+412β	α+476β	α+285β	α+349β	α+3682β	α+3618β	α+3809β	α+3745β	α+414β	α+478β	α+287β	α+351β	α+3680β	α+3616β
α+3807β	α+3743β	α+416β	α+480β	α+289β	α+353β	α+3678β	α+3614β	α+3805β	α+3741β	α+418β	α+482β	α+291β	α+355β	α+3676β	α+3612β
α+3803β	α+3739β	α+420β	α+484β	α+293β	α+357β	α+3674β	α+3610β	α+3801β	α+3737β	α+422β	α+486β	α+295β	α+359β	α+3672β	α+3608β
α+3799β	α+3735β	α+424β	α+488β	α+297β	α+361β	α+3670β	α+3606β	α+3797β	α+3733β	α+426β	α+490β	α+299β	α+363β	α+3668β	α+3604β
α+3795β	α+3731β	α+428β	α+492β	α+301β	α+365β	α+3666β	α+3602β	α+3793β	α+3729β	α+430β	α+494β	α+303β	α+367β	α+3664β	α+3600β
α+304β	α+368β	α+3663β	α+3599β	α+3790β	α+3726β	α+433β	α+497β	α+306β	α+370β	α+3661β	α+3597β	α+3788β	α+3724β	α+435β	α+499β
α+308β	α+372β	α+3659β	α+3595β	α+3786β	α+3722β	α+437β	α+501β	α+310β	α+374β	α+3657β	α+3593β	α+3784β	α+3720β	α+439β	α+503β
α+312β	α+376β	α+3655β	α+3591β	α+3782β	α+3718β	α+441β	α+505β	α+314β	α+378β	α+3653β	α+3589β	α+3780β	α+3716β	α+443β	α+507β
α+316β	α+380β	α+3651β	α+3587β	α+3778β	α+3714β	α+445β	α+509β	α+318β	α+382β	α+3649β	α+3585β	α+3776β	α+3712β	α+447β	α+511β

⋮

Layer 16

α+255β	α+191β	α+3968β	α+4032β	α+3841β	α+3905β	α+126β	α+62β	α+253β	α+189β	α+3970β	α+4034β	α+3843β	α+3907β	α+124β	α+60β
α+251β	α+187β	α+3972β	α+4036β	α+3845β	α+3909β	α+122β	α+58β	α+249β	α+185β	α+3974β	α+4038β	α+3847β	α+3911β	α+120β	α+56β
α+247β	α+183β	α+3976β	α+4040β	α+3849β	α+3913β	α+118β	α+54β	α+245β	α+181β	α+3978β	α+4042β	α+3851β	α+3915β	α+116β	α+52β
α+243β	α+179β	α+3980β	α+4044β	α+3853β	α+3917β	α+114β	α+50β	α+241β	α+177β	α+3982β	α+4046β	α+3855β	α+3919β	α+112β	α+48β
α+3856β	α+3920β	α+111β	α+47β	α+238β	α+174β	α+3985β	α+4049β	α+3858β	α+3922β	α+109β	α+45β	α+236β	α+172β	α+3987β	α+4051β
α+3860β	α+3924β	α+107β	α+43β	α+234β	α+170β	α+3989β	α+4053β	α+3862β	α+3926β	α+105β	α+41β	α+232β	α+168β	α+3991β	α+4055β
α+3864β	α+3928β	α+103β	α+39β	α+230β	α+166β	α+3993β	α+4057β	α+3866β	α+3930β	α+101β	α+37β	α+228β	α+164β	α+3995β	α+4059β
α+3868β	α+3932β	α+99β	α+35β	α+226β	α+162β	α+3997β	α+4061β	α+3870β	α+3934β	α+97β	α+33β	α+224β	α+160β	α+3999β	α+4063β
α+3872β	α+3936β	α+95β	α+31β	α+222β	α+158β	α+4001β	α+4065β	α+3874β	α+3938β	α+93β	α+29β	α+220β	α+156β	α+4003β	α+4067β
α+3876β	α+3940β	α+91β	α+27β	α+218β	α+154β	α+4005β	α+4069β	α+3878β	α+3942β	α+89β	α+25β	α+216β	α+152β	α+4007β	α+4071β
α+3880β	α+3944β	α+87β	α+23β	α+214β	α+150β	α+4009β	α+4073β	α+3882β	α+3946β	α+85β	α+21β	α+212β	α+148β	α+4011β	α+4075β
α+3884β	α+3948β	α+83β	α+19β	α+210β	α+146β	α+4013β	α+4077β	α+3886β	α+3950β	α+81β	α+17β	α+208β	α+144β	α+4015β	α+4079β
α+207β	α+143β	α+4016β	α+4080β	α+3889β	α+3953β	α+78β	α+14β	α+205β	α+141β	α+4018β	α+4082β	α+3891β	α+3955β	α+76β	α+12β
α+203β	α+139β	α+4020β	α+4084β	α+3893β	α+3957β	α+74β	α+10β	α+201β	α+137β	α+4022β	α+4086β	α+3895β	α+3959β	α+72β	α+8β
α+199β	α+135β	α+4024β	α+4088β	α+3897β	α+3961β	α+70β	α+6β	α+197β	α+133β	α+4026β	α+4090β	α+3899β	α+3963β	α+68β	α+4β
α+195β	α+131β	α+4028β	α+4092β	α+3901β	α+3965β	α+66β	α+2β	α+193β	α+129β	α+4030β	α+4094β	α+3903β	α+3967β	α+64β	α

Figure 3. Perfect magic cube of order 16 with arithmetic sequence

As the order of perfect magic cube increases, the construction pattern (yellow background) of perfect magic cube also expands as well as for perfect magic cube of order $8n$ for $n \geq 2$. In **Figure 3**, the sum of the entries of each row, column, pillar, diagonal plane, and diagonal space is the same. This can be proven using **Equation (4)** in **Lemma 1** or can be calculated manually in a row, column, pillar, diagonal space and diagonal plane as follows:

i. Choose any rows, e.g.:

$$\sum_{j=1}^{16} c_{2,j,4} = c_{2,1,4} + c_{2,2,4} + c_{2,3,4} + c_{2,4,4} + \cdots + c_{2,13,4} + c_{2,14,4} + c_{2,15,4} + c_{2,16,4}$$

$$= (\alpha + 772\beta) + (\alpha + 836\beta) + \cdots + (\alpha + 903\beta) + (\alpha + 967\beta) = 16\alpha + 32760\beta.$$

ii. Choose any columns, e.g.:

$$\sum_{i=1}^{16} c_{i,10,14} = c_{1,10,14} + c_{2,10,14} + c_{3,10,14} + c_{4,10,14} \ldots + c_{14,10,14} + c_{15,10,14} + c_{16,10,14}$$

$$= (\alpha + 701\beta) + (\alpha + 697\beta) + \cdots + (\alpha + 645\beta) + (\alpha + 641\beta) = 16\alpha + 32760\beta.$$

iii. Choose any pillars, e.g.:

$$\sum_{k=1}^{16} c_{3,8,k} = c_{3,8,1} + c_{3,8,2} + c_{3,8,3} + c_{3,8,4} \dots + c_{3,8,13} + c_{3,8,14} + c_{3,8,15} + c_{3,8,16}$$

$$= (\alpha + 3894\beta) + (\alpha + 457\beta) + \cdots + (\alpha + 3785\beta) + (\alpha + 54\beta) = 16\alpha + 32760\beta.$$

iv. Choose any plane diagonals, e.g.:

$$\sum_{j=k=1}^{16} c_{1,j,k} = c_{1,1,1} + c_{1,2,2} + c_{1,3,3} + c_{1,4,4} + \cdots + c_{1,14,14} + c_{1,15,15} + c_{1,16,16}$$

$$= (\alpha + 4095\beta) + (\alpha + 320\beta) + \cdots + (\alpha + 3715\beta) + (\alpha + 60\beta) = 16\alpha + 32760\beta.$$

v. Choose any space diagonals, e.g.:

$$\sum_{i=j=k=1}^{16} c_{i,t+1-j,k} = c_{1,16,1} + c_{2,15,2} + c_{3,14,3} + c_{4,13,4} + \cdots + c_{14,3,14} + c_{15,2,15} + c_{16,1,16}$$

$$= (\alpha + 3900\beta) + (\alpha + 391\beta) + \cdots + (\alpha + 3704\beta) + (\alpha + 195\beta)$$

$$= 16\alpha + 32760\beta. \blacksquare$$

Theorem 6. *There exists a perfect magic cube of order $8n$ for $n \geq 1$ with arithmetic sequence.*

Proof. By **Definition 5**, given an arithmetic sequence of $(8n)^3$ numbers to satisfy the entries of perfect magic cube with initial number α and difference β for each $\alpha, \beta \in \mathbb{Z}$. Since $n \geq 1$, we take $n = 1$ as the basis.

i. We will prove the existence of perfect magic cube of order $8n$ for $n = 1$.

By **Theorem 4**, there exists the perfect magic cube of order 8 with magic constant of cube as follows

$$\frac{8}{2}(2\alpha + (8^3 - 1)\beta).$$

ii. We will prove the existence of perfect magic cube of order $8n$ for $n = 2$.

Since the algorithm for constructing the perfect magic cube of order 8 is the basis of construction pattern, the algorithm for constructing the perfect magic cube of order 16 follows the algorithm for constructing the perfect magic cube of order 8 (**Steps 1-7)** in **Theorem 4** or can be denoted $8 < 16$. Consequently, there exists a perfect magic cube of order 16 in **Theorem 5** with magic constant of cube as follows

$$\frac{8(2)}{2}\left(2\alpha + \left(\left(8(2)\right)^3 - 1\right)\beta\right).$$

iii. Furthermore, it is assumed that for the case $n = k$, there exists a perfect magic cube of order $8k$ for k natural numbers. Based on this assumption, it will be proved that perfect magic cube of order $8(k + 1)$ also exists. That means it will be proved that the algorithm for constructing the perfect magic cube of order $8(k + 1)$ follows the algorithm for constructing the perfect magic cube of order 8 (**Steps 1-7)** in **Theorem 4** or can be denoted $8 < 8(k + 1)$.

Assuming that there exists the perfect magic cube of order $8k$, it is known that

a. The algorithm for constructing the perfect magic cube of order $8k$ follows the algorithm for constructing the perfect magic cube of order 8 (**Steps 1-7)** in **Theorem 4** or can be denoted $8 < 8k$. Since there exists perfect magic cube of order $8, 16, \dots, 8k$.

b. the algorithm for constructing the perfect magic cube of order $8(k + 1)$ follows the algorithm for constructing the perfect magic cube of order $8k$ or can be denoted $8k < 8(k + 1)$. Since there is the perfect magic cube of order $8, 16, \dots, 8k, 8(k + 1)$.

Since the algorithm for constructing the perfect magic cube of order $8k$ follows the algorithm for constructing the perfect magic cube of order 8 (**Steps 1-7)** in **Theorem 4** and the algorithm for constructing the perfect magic cube of order $8(k + 1)$ follows the algorithm for constructing the perfect magic cube of order $8k$, then based on the transitive property, it can be concluded that the algorithm for constructing the perfect magic cube of order $8(k+1)$ follows the algorithm for constructing the perfect magic cube of order 8 (**Steps 1-7)** in **Theorem 4** or can be denoted $8 < 8(k+1)$. By doing inductive proof, it has been shown that **Theorem 6** there exists a perfect magic cube of order $8k$, then there exists the perfect magic cube of order $8(k+1)$ with magic constant of cube as follows

$$\frac{8(k+1)}{2}\left(2\alpha+\left(\left(8(k+1)\right)^{3}-1\right)\beta\right).$$

So, it has been proved that **Theorem 6** holds for all $n \geq 1$.■

4. CONCLUSIONS

In this paper, the construction of a perfect magic cube of order $8n$ is based on pattern formation and algorithm. Constructing the perfect magic cube of order $8n$ with arithmetic sequence entries with initial number (α) and difference (β) can be adjusted for $\alpha, \beta \in \mathbb{Z}$. The pattern and algorithm of the perfect magic cube of order $8n$ for $n = 1$ serve as the basis for constructing the perfect magic cube of order $8n$ for $n \geq 1$.

5. REFERENCES

[1] P. B. Dhandapani, V. Leiva, and C. Martin-Barreiro, "Construction of a repetitive magic square with Ramanujan's number as its product," *Heliyon*, vol. 9, no. 1, p. e12046, 2023, doi: 10.1016/j.heliyon.2022.e12046.

[2] Z. Duan, J. Liu, J. Li, and C. Tian, "Improved even order magic square construction algorithms and their applications in multi-user shared electronic accounts," *Theor. Comput. Sci.*, vol. 607, pp. 391–410, 2015, doi: 10.1016/j.tcs.2015.07.053.

[3] O. M. Baksalary, D. Trenkler, and G. Trenkler, "On most perfect magic square of order four," *Linear Multilinear Algebr.*, pp. 1–13, Jul. 2018.

[4] B. Gupta, "Algorithm for doubly even magic square," *Int. Adv. Res. J. Sci. Eng. Technol.*, vol. 6, no. 3, pp. 112–117, 2019.

[5] U. A. Mu'min and B. P. Silalahi, "Konstruksi persegi ajaib dengan entri bilangan bulat (magic square construction with integer entry)," *J. Mat. dan Pendidik. Mat.*, vol. 6, no. 2, pp. 166–176, 2021.

[6] S. Duangmoon, P. Kanokladarom, S. Pongprasert, T. Rungratgasame, and N. Srithaisong, "Affine magic cubes of order 4: concepts and applications," *Int. J. Math. Math. Sci.*, vol. 2022, 2022, doi: 10.1155/2022/2578562.

[7] B. J. Freyberg, "Constructing a tournament with a magic cube," *Am. Math. Mon.*, vol. 128, no. 8, pp. 719–725, 2021.

[8] C. Ashbacher and L. Widmer, "Magic cube classes," *J. Recreat. Math.*, pp. 120–121, 2010, doi: 10.5951/mt.43.6.0290.

[9] S. H. Jamel, T. Herawan, and M. M. Deris, "Steps for constructing magic cube using two orthogonal Latin squares and a magic square," *Malaysian Tech. Univ. Conf. Eng. Technol.*, pp. 1–4, 2009.

[10] L. U. Uko and T. L. Barron, "A generalization of Trenkler's magic cubes formula," *Recreat. Math. Mag.*, vol. 4, no. 8, pp. 39–45, 2017, doi: 10.1515/rmm-2017-0019.

[11] F. Liao and H. Xie, "On the construction of pandiagonal magic cubes," *Mathematics*, vol. 11, no. 5, pp. 1–23, 2023, doi: 10.3390/math11051185.

[12] O. A. Dawood, A. M. S. Rahma, and A. M. J. Abdul Hossen, "Generalized method for constructing magic cube by folded magic squares," *Int. J. Intell. Syst. Appl.*, vol. 8, no. 1, pp. 1–8, 2016, doi: 10.5815/ijisa.2016.01.01.

[13] C. Boyer, "Perfect magic cubes," *Multimagie*, 2020. http://www.multimagie.com/English/Perfectcubes.htm (accessed Nov. 01, 2022).

[14] G. Guardia and N. B. Do Nascimento, "Arithmetic sequence with multiple reasons," *Am. J. Eng. Res. (AJER*, vol. 7, no. 5, pp. 370–376, 2018.

[15] J. F. T. Rabago, "On sequences of numbers in generalized arithmetic and geometric progressions," *Palest. J. Math.*, vol. 4, no. 1, pp. 170–176, 2015.

[16] J. Arroyo, "A proof of the " Magicness " of the siam construction of a magic square," *Rose-hulman Undergrad. Math. J.*, vol. 19, no. 1, 2018.

[17] M. C. Tastan and H. Ilhan, "Magic Square Based SM for UAV Communication," *TechRxiv*, pp. 1–11, 2023.

[18] B. Alspach and K. Heinrich, "Perfect magic cubes of order 4m," *Fibonacci Q.*, vol. 19, no. 2, pp. 97–106, 1981.

[19] S. H. Jamel, T. Herawan, and M. M. Deris, "A cryptographic algorithm based in hybrid cubes," *Int. Conf. Comput. Sci. its Appl.*, pp. 175–187, 2010.

[20] M. Trenkler, "An algorithm for making magic cubes," *Pi ME J.*, vol. 12, no. 2, pp. 105–106, 2005.

[21] H. . Heinz and J. R. Hendricks, *Magic Square Lexicon: Illustrated*. HDH, 2000.

[22] E. W. Weisstein, "Perfect Magic Cube," *Wolfram Mathworld*, 2003. https://mathworld.wolfram.com/PerfectMagicCube.html (accessed May 01, 2022).

[23] W. S. Andrews, *Magic Squares and Cubes*. USA: Open Court Publishing, 1917.

[24] M. Trenkler, "Magic cubes," *Math. Gaz.*, vol. 82, no. 493, pp. 56–61, 1998, doi: 10.5840/monist190616312.

[25] M. Trenkler, "A construction of magic cubes," *Math. Gaz.*, vol. 84, no. 499, pp. 36–41, 2000, doi: 10.2307/3621472.