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CRAMER'S RULE IN MIN-PLUS ALGEBRA

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ABSTRACT

Cramer's rule is a method for solving a system of linear equations in conventional algebra. The system of linear equations $Ax = b$ *can be solved using Cramer's rule if* $\det(A) \neq 0$. *Max-plus algebra is a set* $\mathbb{R}_{max} = \{-\infty\} \cup \mathbb{R}$ *where* $\mathbb{R}$ *is a set of real numbers, equipped with binary operations* $\oplus$ *and* $\otimes$ *where* $a \oplus b = \max(a, b)$ *and* $a \otimes b = a + b$. *Min-plus Algebra is a set* $\mathbb{R}_{min} = \mathbb{R} \cup \{+\infty\}$ *where* $\mathbb{R}$ *is a set of real numbers, equipped with binary operations* $\oplus'$ *and* $\otimes$ *where* $a \oplus' b = \min(a, b)$ *and* $a \otimes b = a + b$. *In max-plus algebra, Cramer's rule has been formulated to solve a system of linear equations. Because max-plus algebra is isomorphic to min-plus algebra, Cramer's rule can be formulated into min-plus algebra. The purpose of this research is to determine the sufficient conditions for a system of linear equations can be solved using Cramer's rule. The method used in this research is a literature study that reviews previous research related to min-plus algebra, max-plus algebra, and Cramer's rule in max-plus algebra. By using the appropriate analogy in max-plus algebra, we can determine the sufficient conditions so that a system of linear equations in min-plus algebra can be solved using Cramer's rule. Based on the research, the sufficient conditions for a system of linear equations can be solved using Cramer's rule are* $sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$ *for* $1 \le i \le n$ *and* $dom(A) < \varepsilon'$ *with the Cramer's rule is* $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$. *For an invertible matrix A, Cramer's rule can be written as* $x_i \otimes perm(A) = perm(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$.

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1. INTRODUCTION

Cramer's rule is one of the rules that can be used to solve a system of linear equations $Ax = b$ where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^{n\times 1}$, and $b \in \mathbb{R}^{n\times 1}$ in conventional algebra. A system of linear equations $Ax = b$ can be solved using Cramer's rule when A is a non-singular matrix, or we can say $\det(A) \neq 0$ [1], [2]. Max-plus algebra is a set $\mathbb{R}_{max} = \{-\infty\} \cup \mathbb{R}$ where $\mathbb{R}$ is a set of real numbers, equipped with two binary operations $\oplus$ and $\otimes$ where $x \oplus y = max(x, y)$ and $x \otimes y = x + y$ [3], [4]. The structure $(\mathbb{R}_{max}, \oplus, \otimes)$ is a semifield with a multiplication identity element $e = 0$ and an addition identity element $\varepsilon = -\infty$ [5], [6], [7]. In max-plus algebra, the determinant of a matrix does not have a direct analogy to conventional algebra because of the absence of additive inverse. However, there are two approaches called dominant and permanent which defined as a determinant of a matrix over max-plus algebra [8]. In 1988, Cramer's rule has been developed by Olsder and Ross [8], [9] that is $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $1 \leq i \leq n$ where a_j is the j^{th} column of matrix A. As in conventional algebra, the solution of a system of linear equations $A \otimes x = b$ in max-plus algebra where $A \in \mathbb{R}_{max}^{n\times n}$, $x \in \mathbb{R}_{max}^{n\times 1}$, and $b \in \mathbb{R}_{max}^{n\times 1}$ may not exist, even when $dom(A) > \varepsilon$. An additional condition is required so that the system of linear equations can be solved using Cramer's rule, that is $sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$ for $1 \leq i \leq n$ where a_j is the j^{th} column of matrix A [8]. When the solution exists, it is not necessarily unique.

There is another structure that has a similar structure as max-plus algebra called min-plus algebra. Min-plus algebra is a set $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$, where $\mathbb{R}$ is a set of real numbers, equipped with two binary operations $\oplus'$ and $\otimes$, where $x \oplus' y = min(x, y)$ and $x \otimes y = x + y$ [3], [10]. The structure $(\mathbb{R}_{min}, \oplus', \otimes)$ is a semifield with a multiplication identity element $e = 0$ and an addition identity element $\varepsilon' = +\infty$ [5], [10]. In 2011 and 2013, Musthofa [11] and Rudito [12] researched a system of linear equations. Also, in 2021, Diena [13] discussed about solving a system of linear equation in min-plus algebra using a reduction and discrepancy matrix. In the same year, Siswanto et al.[14] have researched about the determinant of a matrix over min-plus algebra. Similarly, in max-plus algebra, the determinant of a matrix over min-plus algebra has no direct analogy to conventional algebra. The determinant is defined in two approaches called dominant and permanent over min-plus algebra. Siswanto et al. also discussed the relation between those two approaches. The dominant and permanent have the same value if the corresponding matrix is invertible. Due to the structural similarity of max-plus and min-plus algebra, by using the appropriate analogy in max-plus algebra, Cramer's rule can be formulated into min-plus algebra. The purpose of this research is to determine the sufficient conditions for a system of linear equations in min-plus algebra to be solved using Cramer's rule.

2. RESEARCH METHODS

The method used in this research is a literature study that reviews previous research such as books, journals, or articles related to min-plus algebra, matrix over min-plus algebra, and system of linear equations in min-plus algebra. It also uses references related to max-plus algebra and Cramer's rule in max-plus algebra. The steps carried out in this study are studying material related to Cramer's rule in max-plus algebra. Then, determine Cramer's rule in min-plus algebra with dominant and permanent matrix using the appropriate analogy in max-plus algebra. After that, determine the definition of the sign of a matrix over min-plus algebra. Then, determine the sufficient condition so that a system of linear equations $\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}$ in min-plus algebra can be solved using Cramer's rule. Then, make a conclusion.

3. RESULTS AND DISCUSSION

In conventional algebra when a matrix $A \in \mathbb{R}^{n\times n}$ is a non-singular matrix that is $\det(A) \neq 0$, Cramer's rule yields the solution of a system of linear equations $Ax = b$. The solution given according to [1] and [2] is

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \dots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is a matrix obtained by replacing entries in the j^{th} column of matrix A with entries in the matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Cramer's rule also has been developed in max-plus algebra by Olsder and Roos [9], that is

$$x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$$

for $1 \leq i \leq n$ where a_j is the j^{th} column of matrix A. Unlike in conventional algebra, $dom(A) > \varepsilon$ is not sufficient for Cramer's rule to yield the solution. It requires an additional condition that is

$$sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$$

for $1 \leq i \leq n$ where a_j is the j^{th} column of matrix A.

In min-plus algebra, the determinant of a matrix over min-plus algebra is defined in two approaches called dominant and permanent [14]. Dominant of $A \in \mathbb{R}_{min}^{n\times n}$ according to Siswanto[14] is

$$dom(A) = \begin{cases} lowest\ exponent\ in\ \det(z^A), & if\ \det(z^A) \neq 0 \\ \varepsilon', & if\ \det(z^A) = 0 \end{cases}$$

where z^A is the $n \times n$ matrix with the entries $z^{a_{ij}}$, where a_{ij} is entries of A and z is variable as follow.

$$z^A = \begin{bmatrix} z^{a_{11}} & z^{a_{12}} & \cdots & z^{a_{1n}} \\ z^{a_{21}} & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ z^{a_{n1}} & \cdots & \cdots & z^{a_{nn}} \end{bmatrix},$$

we can calculate the $\det(z^A)$ as in conventional algebra.

For $A \in \mathbb{R}_{min}^{n\times n}$, permanent of A is defined by

$$perm(A) = \bigoplus_{\sigma\in P_n}{}' \bigotimes_{i=1}^{n} (a_{i\sigma(i)})$$

where P_n is set of all permutation $\{1,2,\dots,n\}$ [14].

There are some relations between dominant and permanent. Before discussing it, we first discuss the invertible matrix over min-plus algebra. A matrix $A \in \mathbb{R}_{min}^{n\times n}$ is said to be invertible if and only if there exists a permutation σ and $\lambda_i \in \mathbb{R}_{min}$ where $\lambda_i \neq \varepsilon'$ for $i \in \{1,2,\dots,n\}$ such that $A = P_\sigma \otimes D(\lambda_i)$ where $D(\lambda_i)$ is a diagonal matrix

$$D(\lambda_i) = \begin{bmatrix} \lambda_1 & \varepsilon' & \varepsilon' & \cdots & \varepsilon' \\ \varepsilon' & \lambda_2 & \varepsilon' & \cdots & \varepsilon' \\ \varepsilon' & \varepsilon' & \lambda_3 & \cdots & \varepsilon' \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \varepsilon' & \varepsilon' & \varepsilon' & \cdots & \lambda_n \end{bmatrix}$$

[15]. Here are the relations between dominant and permanent [14].

(1) If $A \in \mathbb{R}_{min}^{n\times n}$ invertible then $dom(A) \neq \varepsilon'$ and $perm(A) \neq \varepsilon'$.
(2) If $A \in \mathbb{R}_{min}^{n\times n}$ invertible then $dom(A) = perm(A)$.

In the same analogy as max-plus algebra, we can write Cramer's rule in min-plus algebra as follows.

$$x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n),$$

for $i = 1,2,\dots,n$, where a_j is the j^{th} column of matrix A and $i \leq j \leq n$. Unlike in conventional algebra, in min-plus algebra $dom(A) < \varepsilon'$ is not sufficient for the Cramer's rule to yield a solution, an additional condition is required so that the Cramer's rule yields the solution of a system of linear equations $A \otimes x = b$ where $A \in \mathbb{R}_{min}^{n\times n}$, $x \in \mathbb{R}_{min}^{n\times 1}$, and $b \in \mathbb{R}_{min}^{n\times 1}$. The additional condition is

$$sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A) \tag{1}$$

for $i = 1,2,\dots,n$, where a_j is the j^{th} column of matrix A and $i \leq j \leq n$. The definition of sign matrix in min-plus algebra is given in **Definition 1** below.

Definition 1. Let P_n a set of all permutations $\{1,2,\dots,n\}$ and $t_1, t_2, \dots, t_L$ are all possible values such that $t_j = \bigoplus_{i=1}^{n}(a_{i\sigma(i)})$ for a permutation $\sigma \in P_n$. Let

$$S_j = \left\{\sigma \in P_n \middle| t_j = \otimes_{i=1}^{n}\left(a_{i\sigma(i)}\right)\right\},$$

$$S_{je} = \{\sigma \in S_j | \sigma \in P\_n\},$$
$$S_{jo} = \{\sigma \in S_j | \sigma \in P_n^o\},$$
$$k_{je} = |S_{je}|,$$
$$k_{jo} = |S_{jo}|,$$

where P_n^e is set of even permutations of P_n and P_n^o is set of odd permutations of P_n. For $t_j = dom(A)$ and $1 \leq j \leq L$, defined

$$sign(A) = \begin{cases} 1, & if\ k_{je} - k_{jo} > 0 \\ -1, & if\ k_{je} - k_{jo} < 0, \end{cases}$$

if $dom(A) = \varepsilon'$ then $sign(A) = \varepsilon'$.

The following is an example of determining the sign of a matrix.

Example 1. Given a matrix $A \in \mathbb{R}_{min}^{3\times 3}$

$$A = \begin{pmatrix} 6 & 1 & 3 \\ 3 & 7 & 2 \\ 1 & 2 & 4 \end{pmatrix},$$

since $n = 3$, then there are $3! = 6$ permutations that shown in Table 1.

Table 1. All Permutations of $n = 3$ and Values of t_j

Permutation σ_j	$t_j = \otimes_{i=1}^{3} (a_{i\sigma_j(i)})$
$\sigma_1 = 1,2,3$	$t_1 = 6 \otimes 7 \otimes 4 = 17$
$\sigma_2 = 1,3,2$	$t_2 = 6 \otimes 2 \otimes 2 = 10$
$\sigma_3 = 2,1,3$	$t_3 = 1 \otimes 3 \otimes 4 = 8$
$\sigma_4 = 2,3,1$	$t_4 = 1 \otimes 2 \otimes 1 = 4$
$\sigma_5 = 3,1,2$	$t_5 = 3 \otimes 3 \otimes 2 = 8$
$\sigma_6 = 3,2,1$	$t_6 = 3 \otimes 7 \otimes 1 = 11$

Observing the dominant of matrix A,

$$\det(z^A) = \det\begin{pmatrix} z^6 & z^1 & z^3 \\ z^3 & z^7 & z^2 \\ z^1 & z^2 & z^4 \end{pmatrix} = z^{17} + z^4 + z^8 - (z^{11} + z^8 + z^{10}),$$

we obtained $\det(z^A) = z^4 - z^{10} - z^{11} + z^{17}$ and $dom(A) = 4$. From Table 1, we got that $dom(A) = 4 = t_4$, then obtained $S_4, S_{4e}, S_{4o}, k_{4e}$, and k_{4o} as follows.

$$S_4 = \left\{\sigma_j \in P_3 \middle| t_4 = \otimes_{i=1}^{3} \left(a_{i\sigma_j(i)}\right)\right\} = \{\sigma_4\},$$

$$S_{4e} = \{\sigma_j \in S_4 | \sigma_j \in P_3^e\} = \{\sigma_4\},$$

$$S_{4o} = \{\sigma_j \in S_4 | \sigma_j \in P_3^o\} = \{\},$$

$$k_{4e} = |S_{4e}| = 1,$$

$$k_{4o} = |S_{4o}| = 0,$$

then we obtained $k_{4e} - k_{4o} = 1 - 0 = 1 > 0$ so that $sign(A) = 1$.

The Theorem 1 below is about the sufficient conditions so that the system of linear equations $A \otimes x = b$ in min-plus algebra, where $A \in \mathbb{R}_{min}^{n\times n}$, $x \in \mathbb{R}_{min}^{n\times 1}$, and $b \in \mathbb{R}_{min}^{n\times 1}$, can be solved using Cramer's rule.

Theorem 1. *If $sign(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) = sign(A)$ for $1 \leq i \leq n$ and $dom(A) < \varepsilon'$ then the solution of system of linear equations $A \otimes x = b$ can be obtained from $x_i \otimes dom(A) = dom(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$.*

Proof. Let $Ax = b$ be a system of linear equations. Express the system in the form of z^A and z^b, we get the following equation.

$$z^A \xi = z^b \tag{2}$$

Since $dom(A) < \varepsilon'$ then $\det(z^A) \neq 0$ and (2) can be solved using Cramer's rule which the solution is

$$\xi_i = \frac{\det\left(z^{a_1}, \dots, z^{a_{i-1}}, z^b, z^{a_{i+1}}, \dots, z^{a_n}\right)}{\det(z^A)} \tag{3}$$

for $1 \leq i \leq n$.

If $z \to \infty$ then the value of ξ_i determined by the dominants of right-hand side matrices of (3). The value of $\det\left(z^{a_1}, \dots, z^{a_{i-1}}, z^b, z^{a_{i+1}}, \dots, z^{a_n}\right)$ will leads to the value of $dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ in the form of $z^{dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)}$, likewise for $\det(z^A)$.

If we write

$$d_i = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$$

for $1 \leq i \leq n$ then obtained

$$\xi_i \approx c_i \frac{z^{d_i}}{z^{dom(A)}} \approx c_i z^{d_i - dom(A)} \tag{4}$$

for $1 \leq i \leq n$ and some constants c_i. According to the assumption that **Equation (1)** holds, the constant c_i is positive.

Substituting **Equation (4)** into **Equation (2)** yields

$$\sum_{j=1}^{n} c_j z^{a_{ij} + d_j - dom(A)} \approx z^{b_i}$$

for $1 \leq i \leq n$, which means

$$\bigoplus_{j=1}^{n}{}' \left(a_{ij} + d_j - dom(A)\right) = b_i$$

for $1 \leq i \leq n$. So, if $x_i = d_i - dom(A)$ for $1 \leq i \leq n$ then x_i is the solution of $A \otimes x = b$ or we can express it as $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$. Hence, the solution of a system of linear equations $A \otimes x = b$ can be obtained from $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ is proven. □

It should be noted that even if the condition (1) is not met, Cramer's rule sometimes still can yield a solution. Here is an example.

Example 2. Given a system of linear equations $A \otimes x = b$

$$\begin{pmatrix} 1 & 3 & 5 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix}. \tag{5}$$

First, we observe that $\det(z^A) = -z^3 + 2z^4 - z^6 - z^8 + z^9$ which gives us that 3 is the lowest exponent in $\det(z^A)$. Therefore, $dom(A) = 3$ and $sign(A) = -1$. Similarly, observe the $dom(A_i)$ and $sign(A_i)$ where $A_i = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $i \in \{1,2,3\}$ then, we get $dom(A_1) = 8$, $dom(A_2) = 9$, and $dom(A_3) = 8$. Also, we get $sign(A_1) = -1$, $sign(A_2) = 1$, and $sign(A_3) = -1$. We can see that the sign values are not all the same, $sign(A) = sign(A_1) = sign(A_3) = -1$ and $sign(A_2) = 1$. We try using Cramer's rule it still gives us the solution of the system (5), that is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}.$$

The solution generated by Cramer's rule is not always unique, as in **Example 3** below.

Example 3. Given a system of linear equations $A \otimes x = b$

$$\begin{pmatrix} -6 & 0 & 5 \\ 9 & 2 & 7 \\ 1 & -4 & 10 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix}. \tag{6}$$

Observe the $\det(z^A)$ and $\det(z^{A_i})$ where $A_i = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $i = \{1,2,3\}$. Then we get $dom(A) = -3$, $dom(A_1) = 0$, $dom(A_2) = 2$, and $dom(A_3) = 2$, also we get $sign(A) = sign(A_1) = sign(A_2) = sign(A_3) = -1$. Therefore, according to **Theorem 1**, system of linear equations (6) can be solved using Cramer's rule. The Cramer's rule yields the solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}.$$

Moreover, there is also another solution that is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}.$$

Here we can conclude that the solution generated by Cramer's rule is not always unique.

Instead of expressing in dominant, if the matrix $A \in \mathbb{R}_{min}^{n\times n}$ is invertible then the solution of system of linear equation $A \otimes x = b$ can be expressed in permanent as follow.

$$x_i \otimes perm(A) = perm(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n). \tag{7}$$

Corollary 1 below shows the sufficient conditions so that the system of linear equations $A \otimes x = b$ in min-plus algebra, where $A \in \mathbb{R}_{min}^{n\times n}$, $x \in \mathbb{R}_{min}^{n\times 1}$, $b \in \mathbb{R}_{min}^{n\times 1}$ and A is invertible, can be solved using Cramer's rule (7).

Corollary 1. *If a matrix* $A \in \mathbb{R}_{min}^{n\times n}$ *invertible and* $sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$ *for* $1 \le i \le n$ *then the solution of system of linear equations* $A \otimes x = b$ *can be obtained from* $x_i \otimes perm(A) = perm(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$.

Proof. We have A is invertible so that $dom(A) = perm(A)$, as well $dom(A) = perm(A) \neq \varepsilon'$. According to **Theorem 1**, $x_i \otimes perm(A) = perm(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ yields the solution of $A \otimes x = b$. □

Here is an example of using **Equation (7)** for solving a system of linear equation.

Example 4. Given a system of linear equations $A \otimes x = b$ where A is an invertible matrix

$$\begin{pmatrix} 1 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 9 & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 5 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 16 \\ 1 \\ 5 \end{pmatrix}. \tag{8}$$

First, we show that A is invertible. Matrix $A = \begin{pmatrix} 1 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 9 & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 5 \end{pmatrix}$ can be expressed as the product of permutation matrix P_σ and diagonal matrix $D(\lambda_i)$ as follow.

$$\begin{aligned} A &= P_\sigma \otimes D(\lambda_i) \\ &= \begin{pmatrix} e & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & e & \varepsilon' \\ \varepsilon' & e & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & e \end{pmatrix} \otimes \begin{pmatrix} 1 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 9 & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \varepsilon' & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & 9 & \varepsilon' \\ \varepsilon' & 3 & \varepsilon' & \varepsilon' \\ \varepsilon' & \varepsilon' & \varepsilon' & 5 \end{pmatrix}. \end{aligned} \tag{9}$$

From **Equation (7)**, we can conclude that A is invertible. Now, observe the $\det(z^A)$ and $\det(z^{A_i})$ where $A_i = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $i = \{1,2,3,4\}$.

$$z^A = \begin{pmatrix} z^1 & z^{\varepsilon'} & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{\varepsilon'} & z^{\varepsilon'} & z^9 & z^{\varepsilon'} \\ z^{\varepsilon'} & z^3 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{\varepsilon'} & z^{\varepsilon'} & z^{\varepsilon'} & z^5 \end{pmatrix},$$

then we get $\det(z^A) = -z^{17}, dom(A) = 17$ and $sign(A) = -1$.

$$A_1 = \begin{pmatrix} 2 & \varepsilon' & \varepsilon' & \varepsilon' \\ 16 & \varepsilon' & 9 & \varepsilon' \\ 1 & 3 & \varepsilon' & \varepsilon' \\ 5 & \varepsilon' & \varepsilon' & 5 \end{pmatrix}, z^{A_1} = \begin{pmatrix} z^2 & z^{\varepsilon'} & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{16} & z^{\varepsilon'} & z^9 & z^{\varepsilon'} \\ z^1 & z^3 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^5 & z^{\varepsilon'} & z^{\varepsilon'} & z^5 \end{pmatrix},$$

then we get $det(z^{A_1}) = -z^{19}$, $dom(A_1) = 19$ and $sign(A_1) = -1$.

$$A_2 = \begin{pmatrix} 1 & 2 & \varepsilon' & \varepsilon' \\ \varepsilon' & 16 & 9 & \varepsilon' \\ \varepsilon' & 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 5 & \varepsilon' & 5 \end{pmatrix}, z^{A_2} = \begin{pmatrix} z^1 & z^2 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{\varepsilon'} & z^{16} & z^9 & z^{\varepsilon'} \\ z^{\varepsilon'} & z^1 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{\varepsilon'} & z^5 & z^{\varepsilon'} & z^5 \end{pmatrix},$$

then we get $\det(z^{A_2}) = -z^{16}, dom(A_2) = 16$ and $sign(A_2) = -1$.
By the same step for z^{A_3} and z^{A_4}, we get $sign(A) = sign(A_1) = sign(A_2) = sign(A_3) = sign(A_4) = -1$.

Based on **Corollary 1**, the solution of **Equation (8)** can be obtained from **Equation (7)**.

Next, find $perm(A)$ and $perm(A_i)$ where $A_i = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $i \in \{1,2,3,4\}$. We obtain $perm(A) = \min(1 \otimes 17, \varepsilon', \varepsilon', \varepsilon') = 18$, $perm(A_1) = \min(2 \otimes 17, \varepsilon', \varepsilon', \varepsilon') = 19$, $perm(A_2) = \min(1 \otimes 15, \varepsilon', \varepsilon', \varepsilon') = 16$, $perm(A_3) = \min(1 \otimes 24, \varepsilon', \varepsilon', \varepsilon') = 25$, $perm(A_4) = \min(1 \otimes 17, \varepsilon', \varepsilon', \varepsilon') = 18$. By using **Equation (7)**, we obtain the solution of **Equation (8)** is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 7 \\ 0 \end{pmatrix}.$$

It should be noted that even if the system of linear equations does not fulfill the sufficient condition in **Corollary 1**, Cramer's rule sometimes still can yield a solution. Here is an example.

Example 5. Given a system of linear equations $A \otimes x = b$ where A is an invertible matrix

$$\begin{pmatrix} \varepsilon' & \varepsilon' & 4 \\ 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \varepsilon' \\ 1 \\ 0 \end{pmatrix}. \tag{10}$$

First, we show that A is invertible. Matrix $A = \begin{pmatrix} \varepsilon' & \varepsilon' & 4 \\ 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' \end{pmatrix}$ can be expressed as the product of permutation matrix P_σ and diagonal matrix $D(\lambda_i)$ as follow.

$$\begin{aligned} A &= P_\sigma \otimes D(\lambda_i) \\ &= \begin{pmatrix} \varepsilon' & \varepsilon' & e \\ e & \varepsilon' & \varepsilon' \\ \varepsilon' & e & \varepsilon' \end{pmatrix} \otimes \begin{pmatrix} 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' \\ \varepsilon' & \varepsilon' & 4 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon' & \varepsilon' & 4 \\ 1 & \varepsilon' & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' \end{pmatrix}. \end{aligned} \tag{11}$$

From **Equation (11)**, we can conclude that A in invertible. Now, by the same method as **Example 4** we observe the det (z^A) and det (z^{A_i}) where $A_i = (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$ for $i \in \{1,2,3\}$.

$$z^A = \begin{pmatrix} z^{\varepsilon'} & z^{\varepsilon'} & z^4 \\ z^1 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^{\varepsilon'} & z^0 & z^{\varepsilon'} \end{pmatrix},$$

then we get $\det(z^A) = z^5, dom(A) = 5$ and $sign(A) = +1$.

$$A_1 = \begin{pmatrix} \varepsilon' & \varepsilon' & 4 \\ 1 & \varepsilon' & \varepsilon' \\ 0 & 0 & \varepsilon' \end{pmatrix}, z^{A_1} = \begin{pmatrix} z^{\varepsilon'} & z^{\varepsilon'} & z^4 \\ z^1 & z^{\varepsilon'} & z^{\varepsilon'} \\ z^0 & z^0 & z^{\varepsilon'} \end{pmatrix},$$

then we get $det(z^{A_1}) = z^5$, $dom(A_1) = 5$ and $sign(A_1) = +1$.

$$A_2 = \begin{pmatrix} \varepsilon' & \varepsilon' & 4 \\ 1 & 1 & \varepsilon' \\ \varepsilon' & 0 & \varepsilon' \end{pmatrix}, z^{A_2} = \begin{pmatrix} z^{\varepsilon'} & z^{\varepsilon'} & z^4 \\ z^1 & z^1 & z^{\varepsilon'} \\ z^0 & z^0 & z^{\varepsilon'} \end{pmatrix},$$

then we get $\det(z^{A_2}) = z^5, dom(A_2) = 5$ and $sign(A_2) = +1$.

$$A_3 = \begin{pmatrix} \varepsilon' & \varepsilon' & \varepsilon' \\ 1 & \varepsilon' & 1 \\ \varepsilon' & 0 & 0 \end{pmatrix}, z^{A_3} = \begin{pmatrix} z^{\varepsilon'} & z^{\varepsilon'} & z^{\varepsilon'} \\ z^1 & z^{\varepsilon'} & z^1 \\ z^{\varepsilon'} & z^0 & z^0 \end{pmatrix},$$

then we get $\det(z^{A_3}) = 0, dom(A_3) = \varepsilon'$ and $sign(A_3) = \varepsilon'$. We can see that the $sign(A) = sign(A_1) = sign(A_2) = +1$ but $sign(A_3) = \varepsilon'$. We try using **Equation (7)** it still gives us the solution of the system (10). Calculate the permanents, we get $perm(A) = \min(\varepsilon', \varepsilon', 5, \varepsilon', \varepsilon', \varepsilon') = 5$, $perm(A_1) = \min(\varepsilon', \varepsilon', 5, \varepsilon', \varepsilon', \varepsilon') = 5$, $perm(A_2) = \min(\varepsilon', \varepsilon', 5, \varepsilon', \varepsilon', \varepsilon') = 5$ and $perm(A_3) = \min(\varepsilon', \varepsilon', \varepsilon', \varepsilon', \varepsilon', \varepsilon') = \varepsilon'$. By using **Equation (7)**, we obtain the solution of **Equation (10)** is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \varepsilon' \\ 1 \\ 0 \end{pmatrix}.$$

4. CONCLUSIONS

Based on the result and discussion, the sufficient conditions so that the system of linear equations $A \otimes x = b$ has a solution in min-plus algebra, where $A \in \mathbb{R}_{min}^{n\times n}$, $x \in \mathbb{R}_{min}^{n\times 1}$ and $b \in \mathbb{R}_{min}^{n\times 1}$, is $sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$ for $1 \le i \le n$ and $dom(A) < \varepsilon'$, with the solution using Cramer's rule is $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$. For an invertible matrix A, the solution can be expressed as $x_i \otimes perm(A) = perm(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$.

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