

## ALGORITHM FOR CONSTRUCTING THE TRIPLE UNIT GRAPH OF TYPE II OF RING $\mathbb{Z}_n$ USING PYTHON

Ika Nur Fitriani<sup>1</sup>, Vika Yugi Kurniawan<sup>2\*</sup>

<sup>1,2</sup>Departement of Mathematics, Faculty of Mathematics and Natural Science, Universitas Sebelas Maret  
Jln. Ir. Sutami No 36A, Kentingan, Surakarta, 57126, Indonesia

Corresponding author's e-mail: \*[vikayugi@staff.uns.ac.id](mailto:vikayugi@staff.uns.ac.id)

### ABSTRACT

#### Article History:

Received: 1<sup>st</sup> January 2024

Revised: 7<sup>th</sup> April 2024

Accepted: 28<sup>th</sup> June 2024

Published: 1<sup>st</sup> September 2024

#### Keywords:

The Triple Unit Graph of type II;

A ring of integers Modulo  $n$ ;

Construct of  $TU2(\mathbb{Z}_n)$ ;

Python Programming;

Characteristics of  $TU2(\mathbb{Z}_n)$

Let  $R$  be a commutative ring with  $U(R)$  as the set of all unit elements in  $R$ . This paper introduces a new graph associated with the ring  $R$ , called the triple unit graph of type II, denoted by  $TU2(R)$  with the vertex set is  $R - \{0,1\}$ . In  $TU2(R)$ , two distinct vertices,  $u$  and  $v$ , are adjacent if there exists  $w \in R - \{0,1\}$  with  $w \neq u$  and  $w \neq v$  such that  $uvw \in U(R)$ . This paper focuses on the algorithm for constructing  $TU2(\mathbb{Z}_n)$  using Python. This research uses the literature study research method. The Python programming language can be used to observe the characteristic result of the graph. From the patterns generated by the algorithm, some characteristics of  $TU2(\mathbb{Z}_n)$  are obtained. For example, if  $n$  is a prime and  $n \geq 5$ , then  $TU2(\mathbb{Z}_n)$  is a connected graph, a complete graph, a regular graph, and a Hamiltonian graph



This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-ShareAlike 4.0 International License.

#### How to cite this article:

I. N. Fitriani and V. Y. Kurniawan, "ALGORITHM FOR CONSTRUCTING THE TRIPLE UNIT GRAPH OF TYPE II OF RING  $\mathbb{Z}_n$  USING PYTHON," *BAREKENG: J. Math. & App.*, vol. 18, iss. 3, pp. 1639-1648, September, 2024.

Copyright © 2024 Author(s)

Journal homepage: <https://ojs3.unpatti.ac.id/index.php/barekeng/>

Journal e-mail: [barekeng.math@yahoo.com](mailto:barekeng.math@yahoo.com); [barekeng.journal@mail.unpatti.ac.id](mailto:barekeng.journal@mail.unpatti.ac.id)

**Research Article · Open Access**

## 1. INTRODUCTION

The graph theory approach to algebraic structures is a method that utilizes graph concepts to investigate the properties of algebraic structures. In 1988, Beck [1] introduced a connection between graph theory and commutative ring theory by studying the notion of coloring a commutative ring  $R$ . Subsequently, Anderson and Naseer [2] continued this investigation into the colorings of commutative rings in 1993. Later in 1999, Anderson and Livingston [3] developed research on the subgraph  $\Gamma(R)$  of the zero divisor graph of  $R$ . The concept of the zero divisor graph has been extensively studied by several researchers, including [4], [5], [6], and [7].

In 1990, Grimaldi [8] defined a unit graph  $G(\mathbb{Z}_n)$  based on the unit elements of  $\mathbb{Z}_n$ , where the vertices of  $G(\mathbb{Z}_n)$  are the elements of  $\mathbb{Z}_n$ . Subsequently, in 2010, Ashrafi et al. [9] generalized the unit graph  $G(\mathbb{Z}_n)$  to  $G(R)$  for any ring  $R$ , where  $R$  is an arbitrary associative ring with nonzero identity and they obtained various characterization results for finite commutative rings related to connectivity, diameter, girth, and the planarity of  $G(R)$ . Then in 2019, Satyanarayana et al. [10] introduced a unit graph of type-1 of a ring, denoted by  $UG1(R)$  with  $V(UG1(R)) = U(R)$ , where two distinct vertices,  $x$  and  $y$ , are adjacent if  $xy = 1$  and  $x \neq 1 \neq y$ . Then in 2022, Satyanarayana et al. [11] extended the previous research to define the unit graph of type-2 of ring  $R$ , denoted by  $UG2(R)$ , which is a subgraph of the unit graph of type-1. In  $UG2(R)$ , two distinct vertices,  $x$  and  $y$ , are adjacent if  $xy = 1$  and  $x \neq y$ . Furthermore, Celikel [12] introduced the triple zero graph of  $R$ , denoted by  $TZ\Gamma(R)$ , which is an undirected graph with vertices  $TZ(R)$ . In  $TZ\Gamma(R)$ , two distinct vertices,  $a$  and  $b$ , are adjacent if only if  $ab \neq 0$  and there exists a non-zero element  $c$  of  $R$  such that  $ac \neq 0$ ,  $bc \neq 0$ , and  $abc = 0$ .

According to Suharto [13], Python is a high-level programming language capable of directly converting a set of instruction codes into machine code when the program is executed. Python is known for its ease of learning and use, with a readable and understandable syntax. It is widely employed in solving problems in algebraic structures and graph theory. Several researchers have utilized Python for these purposes, as evidenced by [14] and [15]. Additionally, [16], [17], [18], [19] have utilized Python to construct graphs.

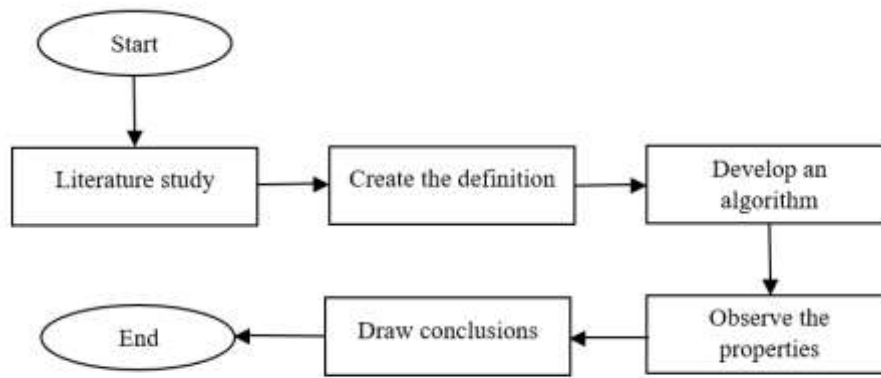
Building upon previous studies, this research introduces a new graph combining the concepts of the triple zero graph in Celikel [12] and the unit graph of type-2 in Satyanarayana [11]. This new graph is termed the triple unit graph of type II of the commutative ring  $R$  and denoted by  $TU2(R)$ . The vertex set in  $TU2(R)$  is  $R - \{0,1\}$ . In  $TU2(R)$ , two distinct vertices,  $u$  and  $v$ , are adjacent if there exists  $w \in R - \{0,1\}$  with  $w \neq u$  and  $w \neq v$  such that  $uvw \in U(R)$ , where  $U(R)$  represents the set of all unit elements in  $R$ . This research also develops the construction of  $TU2(R)$  from various variants of the ring  $R$  is necessary to investigate the properties of  $TU2(R)$ . It will take a long time if done manually. Then, if it is constructed using a computer program, such as Python, it will be faster. Therefore, the researcher developed an algorithm using Python programming to construct  $TU2(R)$  of the ring of integers modulo  $n$ . By analyzing the output generated through Python programming, we can observe the properties of  $TU2(R)$ .

## 2. RESEARCH METHODS

The research uses the literature study method by collecting data and information as well as studying references from various sources such as books, and journals, focusing on algebraic structure, graph theory, Python programming, and especially the triple unit graph of type II of a commutative ring  $R$ . The steps involved in this research are as follows.

- a. Learning basic definitions and theorems related to graph theory and algebraic structure.
- b. Create the definition of the triple unit graph of type II of a commutative ring  $R$ .
- c. Developing an algorithm for the triple unit graph of type II of ring  $\mathbb{Z}_n$  using Python.
- d. Observing the properties of the triple unit graph of type II of ring  $\mathbb{Z}_n$ .
- e. Concluding.

In general, the research procedure is depicted in the flowchart shown in **Figure 1**.



**Figure 1.** Research procedure chart

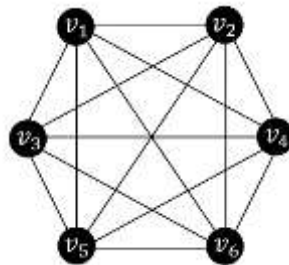
The following explains several definitions that underlie this research, including the basic concepts of graphs, rings, and the triple unit graph of type II.

## 2.1 Basic Concepts of Graphs

The following are some basic definitions of graphs taken from Marsudi [20] and Chartrand and Zhang [21].

**Definition 1.** A graph  $G$  is a pair  $(V, E)$  where  $V(G)$  is a finite nonempty set of vertices in a graph  $G$  called vertices and  $E$  is a finite set of pairs of vertices in a graph  $G$  called edges. The number of vertices in a graph  $G$  is often called the order and is denoted by  $|V(G)|$ , whereas the number of edges in a graph  $G$  is called the size and is denoted by  $|E(G)|$ .

Given an example of the graph  $G$  shown in **Figure 2**.



**Figure 2.** Graph  $G$

In **Figure 2**, the graph  $G$  has  $|V(G)| = 6$  with the vertex set  $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and has  $|E(G)| = 15$  with the edge set  $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_6, v_5v_6\}$ .

**Definition 2.** Suppose  $G = (V, E)$  is a graph. If two vertices,  $u$  and  $v$ , in  $G$  are connected by an edge  $uv$ , then  $u$  and  $v$  are said to be adjacent. Meanwhile, if edge  $uv$  exists, then the vertex  $u$  and edge  $uv$  are said to be incident.

Based on **Figure 2**, the vertex  $v_1$  and  $v_2$  are connected by an edge  $v_1v_2$ , so  $v_1$  and  $v_2$  are adjacent vertices, and vertex  $v_1$  are incident to the edge  $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ , and  $v_1v_6$ .

**Definition 3.** The degree of a vertex  $v$  in graph  $G$  is the number of edges incident to the vertex  $v$ . The degree of vertex  $v$  is denoted by  $\deg_G(v)$ .

In **Figure 2**, the graph  $G$  has a degree of 5 at each vertex.

**Definition 4.** A regular graph in graph  $G$  is a graph in which every vertex has the same degree. If  $\deg(v) = r$  for every vertex  $v$  of  $G$ , where  $0 \leq r \leq n - 1$ , then  $G$  is a  $r$ -regular graph or regular of degree  $r$ . The number

of edges in an  $r$ -regular graph with  $n$  vertex is  $\frac{nr}{2}$ .

In **Figure 2**, every vertex of graph  $G$  has the same degree of 5. Therefore, the graph  $G$  is a 5-regular graph or regular of degree 5.

**Definition 5.** A  $u - v$  walk in graph  $G$  is an alternating sequence of vertices and edges in  $G$  that starts from the vertex  $u$  and ends at the vertex  $v$ . A  $u - v$  trail in graph  $G$  is an alternating sequence of  $u - v$  walk that does not repeat any edge. A  $u - v$  path in graph  $G$  is an alternating sequence of  $u - v$  walk or  $u - v$  trail that does not repeat any vertex.

Based on **Figure 2**,  $u - v$  walk,  $u - v$  trail, and  $u - v$  path can be determined as follows:  $v_3 - v_4$  walk:  $v_3, v_3v_5, v_5, v_5v_3, v_3, v_3v_4, v_4$ .  $v_2 - v_5$  trail:  $v_2, v_2v_3, v_3, v_3v_6, v_6, v_6v_2, v_2, v_2v_5, v_5$ .  $v_1 - v_6$  path:  $v_1, v_1v_2, v_2, v_2v_4, v_4, v_4v_6, v_6$ .

**Definition 6.** A graph  $G$  is a connected graph if there exists a  $u - v$  path between two distinct vertices  $u$  and  $v$  in graph  $G$ . A graph  $G$  that is not connected is called a disconnected graph.

**Figure 2** is an example of a connected graph because it has a  $u - v$  path between two distinct vertices.

**Definition 7.** A graph  $G$  is an empty graph if its edge set is empty (contains no edges) but has at least one vertex.

**Definition 8.** A graph  $G$  is a complete graph if every pair of distinct vertices is adjacent. A complete graph with  $n$  vertices is denoted by  $K_n$ .

**Figure 2** is an example of a complete graph because every pair of distinct vertices is adjacent. A graph  $G$  is a complete graph  $K_6$  because it has 6 vertices.

**Definition 9.** A graph  $G$  is called a Hamiltonian graph if it contains a Hamiltonian cycle. A Hamiltonian cycle is a closed cycle that contains all vertices of  $G$ .

The graph  $G$  of **Figure 2** contains a Hamiltonian cycle. A Hamiltonian cycle in the graph  $G$  is  $v_1, v_1v_6, v_6, v_6v_4, v_4, v_4v_2, v_2, v_2v_5, v_5, v_5v_3, v_3, v_3v_1, v_1$ .

**Definition 10.** The diameter of a graph  $G$  is the greatest distance between two vertices of a connected graph  $G$ , denoted by  $diam(G)$ .

In **Figure 2**, the diameter of graph  $G$  is 1 because graph  $G$  is a complete graph  $K_6$ , where every vertex pair in this graph is connected by exactly one edge.

**Definition 11.** The girth of a graph  $G$  is the length of the shortest cycle in the graph  $G$ , denoted by  $g(G)$ .

In **Figure 2**, the girth of graph  $G$  is 3 in the cycle  $v_1, v_1v_5, v_5, v_5v_4, v_4, v_4v_1, v_1$ .

## 2.2 The Triple Unit Graph of Type II of Commutative Ring

Before we define the triple unit graph of type II, we first recall the definition of the unit graph of type II taken from Satyanarayana et al. [11] and the triple zero graph taken from Celikel [12].

**Definition 12.** Let  $R$  be a finite commutative ring with 1. A graph  $G(V, E)$  is said to be a unit graph of type-2 if  $V = U(R)$  and  $E = \{\overline{xy} | x, y \in U(R) \text{ such that } xy = 1 \text{ and } x \neq y\}$ . The unit graph of type-2 of a ring  $R$  is denoted by  $UG2(R)$ .

**Definition 13.** The triple zero graph of  $R$  is an undirected graph is denoted by  $TZ\Gamma(R)$  with vertices  $TZ(R)$ . If two distinct elements  $a$  and  $b$  are adjacent, then  $(a, b)$  is an edge and is denoted by  $a \sim b$ . Two distinct vertices  $a$  and  $b$  are adjacent if and only if  $ab \neq 0$  and there exists an element  $c \in R \setminus \{0\}$  such that  $ac \neq 0, bc \neq 0$ , and  $abc = 0$ .

Based on the unit graph of type-2 and the triple zero graph, a new graph definition is given, namely the triple unit graph of type II of a commutative ring  $R$ .

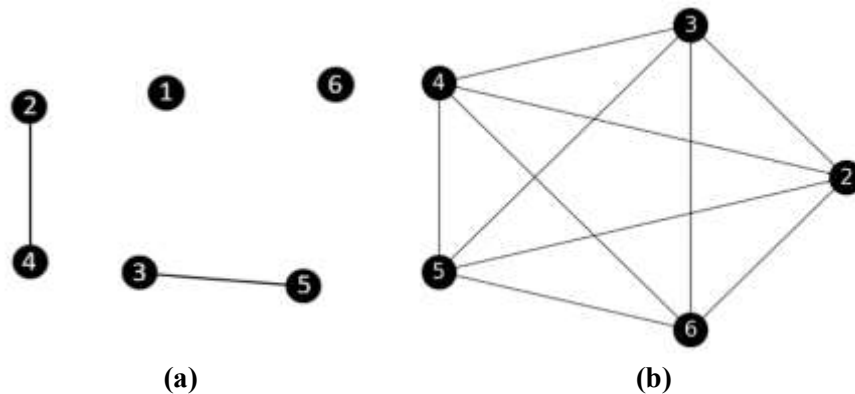
**Definition 14.** The triple unit graph of type II on a commutative ring  $R$ , denoted by  $TU2(R)$ , is a simple graph with a vertex set  $R^* = R - \{0, 1\}$ . In  $TU2(R)$ , two distinct vertices,  $u$  and  $v$ , are adjacent if there exists  $w \in R - \{0, 1\}$  with  $w \neq u$  and  $w \neq v$  such that  $uvw \in U(R)$ , where  $U(R)$  is the set of all unit elements in  $R$ .

**Example 1.** Given an example of the unit graph of type-2 and the triple unit graph of type II formed by the ring  $R = \mathbb{Z}_7$ . We knew that  $\mathbb{Z}_7 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$ . To determine the unit element of the ring  $\mathbb{Z}_7$ , the Cayley Table is used as in **Table 1**.

**Table 1.** Cayley Table of  $\mathbb{Z}_7$

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{1}$	$\bar{3}$	$\bar{5}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{6}$	$\bar{2}$	$\bar{5}$	$\bar{1}$	$\bar{4}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{1}$	$\bar{5}$	$\bar{2}$	$\bar{6}$	$\bar{3}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{6}$	$\bar{4}$	$\bar{2}$
$\bar{6}$	$\bar{0}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

Based on **Table 1**, the unit element in  $\mathbb{Z}_7$  is found to be  $U(\mathbb{Z}_7) = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$ . According to the **Definition 12**, we obtain the vertex set  $V(UG2(\mathbb{Z}_7)) = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$  and the edge set  $E(UG2(\mathbb{Z}_7)) = \{(\bar{2}, \bar{4}), (\bar{3}, \bar{5})\}$ . According to the **Definition 14**, we obtain the vertex set  $V(TU2(\mathbb{Z}_7)) = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$ . In  $TU2(\mathbb{Z}_7)$ , two distinct vertices,  $u$  and  $v$ , are adjacent if there exists  $w \in \mathbb{Z}_7 - \{0,1\}$  with  $w \neq u$  and  $w \neq v$  such that  $uvw \in U(\mathbb{Z}_7)$ . Based on the adjacency condition, for example, take  $u = 2, v = 3$ , and  $w = 4$  to obtain  $2 \cdot 3 \cdot 4 = 3 \in U(\mathbb{Z}_7)$ . Thus, vertex 2 and vertex 3 are adjacent. Based on the adjacency condition of the graph  $TU2(\mathbb{Z}_7)$ , the edge set  $E(TU2(\mathbb{Z}_7)) = \{(\bar{2}, \bar{3}), (\bar{2}, \bar{4}), (\bar{2}, \bar{5}), (\bar{2}, \bar{6}), (\bar{3}, \bar{4}), (\bar{3}, \bar{5}), (\bar{3}, \bar{6}), (\bar{4}, \bar{5}), (\bar{4}, \bar{6}), (\bar{5}, \bar{6})\}$ . Therefore, we get the graph of  $UG2(\mathbb{Z}_7)$  and  $TU2(\mathbb{Z}_7)$  as shown in **Figure 3**.



**Figure 3.** (a) The Graph of  $UG2(\mathbb{Z}_7)$ , (b) The Graph of  $TU2(\mathbb{Z}_7)$

**Figure 3** in (a) is a disconnected graph of the unit graph of type II of ring  $\mathbb{Z}_7$  which has 6 vertices and 2 edges, while (b) is a connected graph of the triple unit graph of type II of ring  $\mathbb{Z}_7$  which has 5 vertices and 10 edges.

### 3. RESULTS AND DISCUSSION

In this article, we will develop an algorithm to construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  using Python programming. The result of the algorithm is as follows.

#### 3.1 Algorithm for Constructing of $TU2(\mathbb{Z}_n)$

First, the algorithm to construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  is parameterized by the unit elements. The vertex set of the triple unit graph of type II is a no-zero and non-unit element in  $\mathbb{Z}_n$ . The triple unit graph of type II of ring  $\mathbb{Z}_n$  is a simple graph denoted by  $TU2(\mathbb{Z}_n)$  with vertex set  $V(TU2(\mathbb{Z}_n))$ . In  $TU2(\mathbb{Z}_n)$ , two distinct vertices,  $u$  and  $v$ , are adjacent if and only if there exists an element  $w \in \mathbb{Z}_n - \{0, 1\}$  with  $w \neq u$  and  $w \neq v$ , such that  $uvw \in U(\mathbb{Z}_n)$ , where  $n$  is a non-negative integer and  $U(\mathbb{Z}_n)$  is the set of all unit elements in  $\mathbb{Z}_n$ . The recursive algorithm to construct  $TU2(\mathbb{Z}_n)$  is called TripleUnitGraphofTypeII.

Then, import the library to draw a graph in Python programming.

```

TripleUnitGraphofTypeII
Import library networkx and matplotlib.pyplot
Step 1:
for  $a, b \in \mathbb{Z}_n$  do
    if  $ab = 1$  then
    add  $a$  to  $U(\mathbb{Z}_n)$ ;
    end if;
end for;
Step 2:
for  $u, v, w \in \mathbb{Z}_n - \{0,1\}$  do
    add  $u$  to  $V(TU2(\mathbb{Z}_n))$ ;
    if  $u \neq v$  and  $v \neq w$  and  $u \neq w$  then
    if  $uvw \in U(\mathbb{Z}_n)$  then
        add  $u \sim v$  to  $E(TU2(\mathbb{Z}_n))$ ;
    end if;
    end if;
end for;
Step 3:
Draw  $TU2(\mathbb{Z}_n)$ ;
end;

```

The following are the steps to constructing a graph using the algorithm:

1. Step 1, check the algorithm by looping through each element of the ring  $\mathbb{Z}_n$ . For each element of the ring  $a, b \in \mathbb{Z}_n$ , check if  $ab = 1$ . If so, then  $a$  is a unit element of the ring  $\mathbb{Z}_n$ , add  $a$  to  $U(\mathbb{Z}_n)$ .
2. Step 2, the algorithm checks by looping through each element  $u, v, w \in \mathbb{Z}_n$ , excluding 0 and 1. Then, add  $u$  to the vertex set  $V(TU2(\mathbb{Z}_n))$ . Check for adjacency conditions, if  $u \neq v, v \neq w, u \neq w$ , and if  $uvw \in U(\mathbb{Z}_n)$ , add an edge between  $u$  and  $v$  to  $E(TU2(\mathbb{Z}_n))$ . Otherwise, return to the looping process in Step 2. Now, we have the vertex set and the edge set.
3. Step 3, draw the result of  $TU2(\mathbb{Z}_n)$  from the vertex and edge set elements obtained in Step 2.

### 3.2 Constructing of $TU2(\mathbb{Z}_n)$

Some examples will be given to see how this algorithm works in the ring  $\mathbb{Z}_n$ . This algorithm will be parameterized by an order  $n$  of the ring  $\mathbb{Z}_n$ .

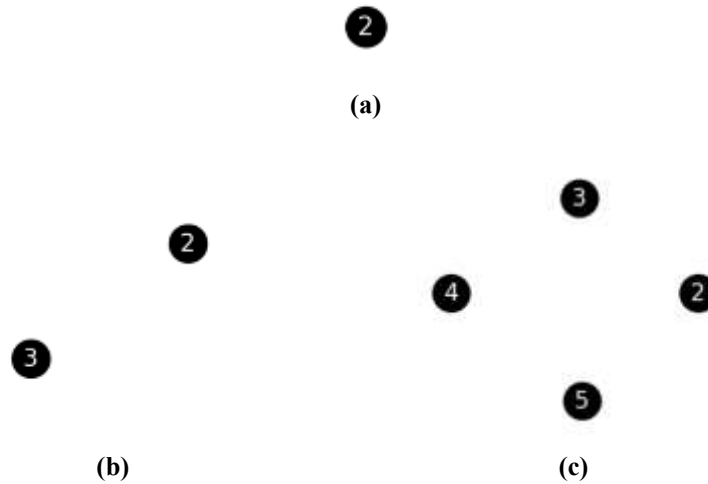
**Example 2.** Construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 3, 4, 6$ .

For  $n = 3$ , then  $\mathbb{Z}_3 = \{0,1,2\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_3$ . If  $ab = 1$ , add  $a$  to  $U(\mathbb{Z}_3)$ . Thus, we obtain  $U(\mathbb{Z}_3) = \{1,2\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_3 - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_3))$ . Hence, we get  $V(TU2(\mathbb{Z}_3)) = \{2\}$ . Since there is only one vertex, it does not satisfy the conditions for  $u, v, w \in \mathbb{Z}_3 - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_3)$ . Therefore, the looping process stops at this step and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_3)$  in **Figure 4**.

For  $n = 4$ , then  $\mathbb{Z}_4 = \{0,1,2,3\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_4$ . If  $ab = 1$ , then add  $a$  to  $U(\mathbb{Z}_4)$ . Thus, we obtain  $U(\mathbb{Z}_4) = \{1,3\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_4 - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_4))$ . Hence, we get  $V(TU2(\mathbb{Z}_4)) = \{2,3\}$ . Since there are only two vertices, none of them satisfy the conditions for  $u, v, w \in \mathbb{Z}_4 - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_4)$ . Therefore, no adjacent vertices exist between vertices in  $V(TU2(\mathbb{Z}_4))$ . Hence, the looping process stops at this step and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_4)$  in **Figure 4**.

For  $n = 6$ , then  $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_6$ ,

if  $ab = 1$  then add  $a$  to  $U(\mathbb{Z}_6)$ . Thus, we obtain  $U(\mathbb{Z}_6) = \{1,5\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_6 - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_6))$ . Hence, we get  $V(TU2(\mathbb{Z}_6)) = \{2,3,4,5\}$ . Next, it checks for each vertex in  $V(TU2(\mathbb{Z}_6))$  whether it satisfies the adjacency conditions for  $u, v, w \in \mathbb{Z}_6 - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_6)$ . Suppose when  $u = 2, v = 3$ , and  $w = 4$ , then  $uvw = 0 \notin U(\mathbb{Z}_6)$ , indicating that  $u$  and  $v$  are not adjacent. Consequently,  $u$  and  $w$  are also not adjacent, and similarly for  $v$  and  $w$ . Hence, no edges are added between 2, 3, and 4. The looping process continues until every vertex in  $V(TU2(\mathbb{Z}_6))$  exhibits its adjacency or not. Subsequently, the looping process stops and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_6)$  in **Figure 4**.

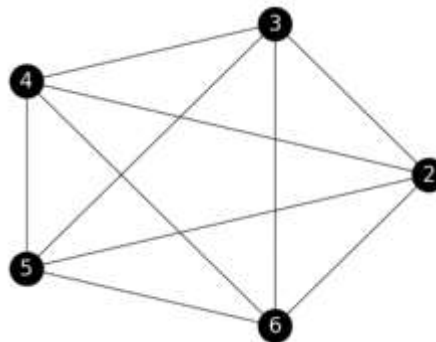


**Figure 4.** (a)  $TU2(\mathbb{Z}_3)$ , (b)  $TU2(\mathbb{Z}_4)$ , (c)  $TU2(\mathbb{Z}_6)$

**Figure 4** is the result of the construction of the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 3,4,6$ .  $TU2(\mathbb{Z}_3), TU2(\mathbb{Z}_4)$ , and  $TU2(\mathbb{Z}_6)$  are empty graphs with no edge set but have a vertex set.

**Example 3.** Construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 7$ .

For  $n = 7$ , then  $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_7$ . If  $ab = 1$  then add  $a$  to  $U(\mathbb{Z}_7)$ . Thus, we obtain  $U(\mathbb{Z}_7) = \{1,2,3,4,5,6\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_7 - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_7))$ . Hence, we get  $V(TU2(\mathbb{Z}_7)) = \{2,3,4,5,6\}$ . Next, it checks for each vertex in  $V(TU2(\mathbb{Z}_7))$  whether it satisfies the adjacency conditions for  $u, v, w \in \mathbb{Z}_7 - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_7)$ . Suppose when  $u = 4, v = 5$ , and  $w = 6$ , then  $uvw = 6 \in U(\mathbb{Z}_7)$ , so  $u$  and  $v$  will be adjacent. Since  $u$  and  $v$  are adjacent,  $u$  and  $w$  will be adjacent, and  $v$  and  $w$  will also be adjacent, so add edges between 4, 5, and 6. Then, repeat the looping process. Suppose when  $u = 5, v = 2$ , and  $w = 3$ , then  $uvw = 2 \in U(\mathbb{Z}_7)$ , so  $u$  and  $v$  will be adjacent. Since  $u$  and  $v$  are adjacent,  $u$  and  $w$  will be adjacent, and  $v$  and  $w$  will also be adjacent, so add edges between 5, 2, and 3. The looping process continues until every vertex in  $V(TU2(\mathbb{Z}_7))$  exhibits its adjacency or not. Subsequently, the looping process stops and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_7)$  in **Figure 5**.



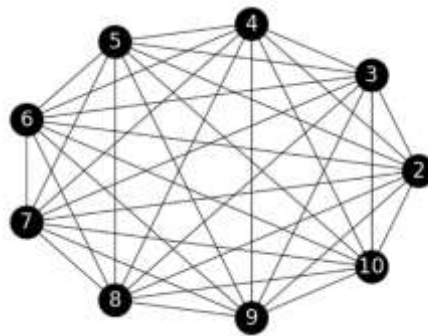
**Figure 5.**  $TU2(\mathbb{Z}_7)$

**Figure 5** is the result of the construction of the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 7$ .  $TU2(\mathbb{Z}_7)$

is a connected graph and a complete graph  $K_5$  which has 5 vertices and 10 edges.

**Example 4.** Construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 11$ .

For  $n = 11$ , then  $\mathbb{Z}_{11} = \{0,1,2,4,5,6,7,8,9,10\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_{11}$ . If  $ab = 1$  then add  $a$  to  $U(\mathbb{Z}_{11})$ . Thus, we obtain  $U(\mathbb{Z}_{11}) = \{1,2,3,4,5,6,7,8,9,10\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_{11} - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_{11}))$ . Hence, we get  $V(TU2(\mathbb{Z}_{11})) = \{2,3,4,5,6,7,8,9,10\}$ . Next, it checks for each vertex in  $V(TU2(\mathbb{Z}_{11}))$  whether it satisfies the adjacency conditions for  $u, v, w \in \mathbb{Z}_{11} - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_{11})$ . Suppose when  $u = 2, v = 3$ , and  $w = 4$ , then  $uvw = 2 \in U(\mathbb{Z}_{11})$ , so  $u$  and  $v$  will be adjacent. Since  $u$  and  $v$  are adjacent,  $u$  and  $w$  will be adjacent, and  $v$  and  $w$  will also be adjacent, so add edges between 2, 3, and 4. Then, repeat the looping process. Suppose when  $u = 4, v = 3$ , and  $w = 8$ , then  $uvw = 6 \in U(\mathbb{Z}_{11})$ , so  $u$  and  $v$  will be adjacent. Since  $u$  and  $v$  are adjacent,  $u$  and  $w$  will be adjacent, and  $v$  and  $w$  will also be adjacent, so add edges between 4, 3, and 8. The looping process continues until every vertex in  $V(TU2(\mathbb{Z}_{11}))$  exhibits its adjacency or not. Subsequently, the looping process stops and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_{11})$  in **Figure 6**.



**Figure 6.**  $TU2(\mathbb{Z}_{11})$

**Figure 6** is the result of the construction of the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 11$ .  $TU2(\mathbb{Z}_{11})$  is a connected graph and a complete graph  $K_9$  which has 9 vertices and 36 edges.

**Example 5.** Construct the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 16$ .

For  $n = 16$ , then  $\mathbb{Z}_{16} = \{0,1,2,4,5,6,7,8,9,10,11,12,13,14,15\}$ . In Step 1, by checking the algorithm for each element  $a, b \in \mathbb{Z}_{16}$ . If  $ab = 1$  then add  $a$  to  $U(\mathbb{Z}_{16})$ . Thus, we obtain  $U(\mathbb{Z}_{16}) = \{1,3,5,7,9,11,13,15\}$ . Then in Step 2, the algorithm loops through each element  $u, v, w \in \mathbb{Z}_{16} - \{0,1\}$ . Add  $u$  to be the vertex set  $V(TU2(\mathbb{Z}_{16}))$ . Hence, we get  $V(TU2(\mathbb{Z}_{16})) = \{2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ . Next, it checks for each vertex in  $V(TU2(\mathbb{Z}_{16}))$  whether it satisfies the adjacency conditions for  $u, v, w \in \mathbb{Z}_{16} - \{0,1\}, u \neq v, v \neq w, u \neq w$ , and  $uvw \in U(\mathbb{Z}_{16})$ . Suppose when  $u = 5, v = 9$ , and  $w = 7$ , then  $uvw = 11 \in U(\mathbb{Z}_{16})$ , so  $u$  and  $v$  will be adjacent. Since  $u$  and  $v$  are adjacent,  $u$  and  $w$  will be adjacent, and  $v$  and  $w$  will also be adjacent, so add edges between 5, 9, and 7. Then, repeat the looping process. Suppose when  $u = 4, v = 6$ , and  $w = 8$ , then  $uvw = 0 \notin U(\mathbb{Z}_{16})$ , indicating that  $u$  and  $v$  are not adjacent. Consequently,  $u$  and  $w$  are also not adjacent, and similarly for  $v$  and  $w$ . Hence, no edges are added between 4, 6, and 8. The looping process continues until every vertex in  $V(TU2(\mathbb{Z}_{16}))$  exhibits its adjacency or not. Subsequently, the looping process stops and we proceed to Step 3 to draw the result of  $TU2(\mathbb{Z}_{16})$  in **Figure 7**.



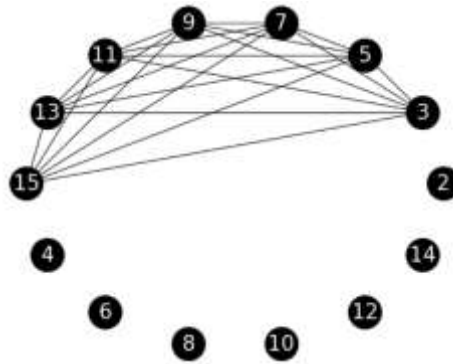


Figure 7.  $TU2(\mathbb{Z}_{16})$

Figure 7 is the result of the construction of the triple unit graph of type II of ring  $\mathbb{Z}_n$  for  $n = 16$ .  $TU2(\mathbb{Z}_{16})$  is a disconnected graph which has 14 vertices and 21 edges.

### 3.3 Observation of The Characteristics of $TU2(\mathbb{Z}_n)$

There are several observations for the resulting properties of the triple unit graph of type II of ring  $\mathbb{Z}_n$ .

1. If  $3 \leq n \leq 6$  and  $n \neq 5$ ,  $TU2(\mathbb{Z}_n)$  it is an empty graph.
2. If  $n$  is prime and  $n \geq 5$ , then  $TU2(\mathbb{Z}_n)$  it is a complete graph, a connected graph, and a Hamiltonian graph.
3. If  $n$  is prime and  $n \geq 5$ , then  $TU2(\mathbb{Z}_n)$  it is a  $(n - 3)$ -regular graph.
4. If  $n$  is not prime and  $n \geq 4$ , then  $TU2(\mathbb{Z}_n)$  it is a disconnected graph.
5. If  $TU2(\mathbb{Z}_n)$  is a connected graph, then  $diam(TU2(\mathbb{Z}_n)) = 1$ .
6. If  $TU2(\mathbb{Z}_n)$  is a connected graph, then  $g(TU2(\mathbb{Z}_n)) = 3$ .

The results of these observations are useful as assumptions for properties that apply to the triple unit graph of type II of ring  $\mathbb{Z}_n$  that still need to be proven algebraically.

## 4. CONCLUSIONS

In this paper, we have defined triple unit graphs of type II of ring  $\mathbb{Z}_n$  and described an algorithm to construct the triple unit graphs of type II using Python programming language. The examples described show how the algorithm works on the given ring  $\mathbb{Z}_n$ . With the help of the Python program to construct the triple unit graphs of type II, it is very easy for researchers to assume the properties of the triple unit graphs of type II of the ring  $\mathbb{Z}_n$  because it only takes a short time to construct graphs of various ring variations.

## ACKNOWLEDGMENT

We gratefully acknowledge the support from the Institute for Research and Community Service (LPPM) Universitas Sebelas Maret through PNPB 2024 with contract number 194.2/UN27.22/PT.01.03/2024.

## REFERENCES

- [1] I. Beck, "Coloring of Commutative Rings," *J Algebra*, vol. 116, no. 1, pp. 208–226, Jul. 1988, doi: 10.1016/0021-8693(88)90202-5.
- [2] D. D. Anderson and M. Naseer, "Beck's Coloring of a Commutative Ring," *J Algebra*, vol. 159, no. 2, pp. 500–514, Aug.

- 1993, doi: 10.1006/jabr.1993.1171.
- [3] D. F. Anderson and P. S. Livingston, "The Zero-Divisor Graph of a Commutative Ring," *J Algebra*, vol. 217, no. 2, pp. 434–447, Jul. 1999, doi: 10.1006/jabr.1998.7840.
- [4] S. Akbari, D. Kiani, F. Mohammadi, and S. Moradi, "The Total Graph And Regular Graph of a Commutative Ring," *J Pure Appl Algebra*, vol. 213, no. 12, pp. 2224–2228, Dec. 2009, doi: 10.1016/j.jpaa.2009.03.013.
- [5] H. Mariiaa, V. Y. Kurniawan, and Sutrima, "Zero Annihilator Graph of Semiring of Matrices Over Boolean Semiring," in *AIP Conference Proceedings*, Oct. 2021, p. 020012. doi: 10.1063/5.0039557.
- [6] A. Rahmawati, V. Y. Kurniawan, and S. Wibowo, "The Diameter of Annihilator Ideal Graph of  $\mathbb{Z}_n$ ," in *AIP Conference Proceedings*, Oct. 2021, p. 020020. doi: 10.1063/5.0039847.
- [7] E. A. Osba, S. Al-Addasi, and B. Al-Khamaiseh, "Some Properties of the Zero-Divisor Graph for the Ring of Gaussian Integers Modulo  $n$ ," *Glasgow Mathematical Journal*, vol. 53, no. 2, pp. 391–399, May 2011, doi: 10.1017/S0017089511000024.
- [8] R. P. Grimaldi, "Graphs from Rings," in *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989)*, 1990, pp. 95–103.
- [9] N. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, "Unit Graphs Associated With Rings," *Commun Algebra*, vol. 38, no. 8, pp. 2851–2871, Aug. 2010, doi: 10.1080/00927870903095574.
- [10] B. Satyanarayana, D. Srinivasulu, and M. Bhavanari, "A Note On Unit Graph of a Ring," *IJRAR- International Journal of Research and Analytical Reviews*, vol. 6, no. 1, pp. 84y–87y, Mar. 2019.
- [11] B. Satyanarayana, D. Srivasulu, and M. Bhavanari, "Unit Graph of Type-2," *TWMS J. APP. AND ENG. MATH.*, vol. 12, no. 2, pp. 497–504, 2022.
- [12] E. Y. ÇELİKEL, "The Triple Zero Graph of a Commutative Ring," *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, vol. 70, no. 2, pp. 653–663, Dec. 2021, doi: 10.31801/cfsuasmas.786804.
- [13] A. Suharto, *Fundamental Bahasa Pemrograman Python*, 1st ed. Purbalingga: CV. Eureka Media Aksara, 2023.
- [14] N. Kansal, B. Kaur, P. Garg, and D. Sinha, "On Another Class of Strongly Perfect Graphs," *Mathematics*, vol. 10, no. 12, p. 2014, Jun. 2022, doi: 10.3390/math10122014.
- [15] Y. Tian and L. Li, "Comments on the Clique Number of Zero-Divisor Graphs of  $\mathbb{Z}_n$ ," *Journal of Mathematics*, vol. 2022, pp. 1–11, Mar. 2022, doi: 10.1155/2022/6591317.
- [16] T. Hartati and V. Y. Kurniawan, "Construct The Triple Nilpotent Graph of Ring Using Python," in *AIP Conference Proceedings*, Oct. 2023, p. 020004. doi: 10.1063/5.0154928.
- [17] V. Y. Kurniawan, B. Purboutomo, S. Sutrima, and N. A. Kurdhi, "Connectivity of The Triple Idempotent Graph of Ring  $\mathbb{Z}_n$ ," in *(IJCSAM) International Journal of Computing Science and Applied Mathematics*, Apr. 2024, pp. 34–37.
- [18] R. Meinawati, V. Y. Kurniawan, and N. A. Kurdhi, "Algorithm for Constructing Total Graph of Commutative Ring," in *JTAM (Jurnal Teori dan Aplikasi Matematika)*, Apr. 2024, pp. 351–363. doi: 10.31764/jtam.v8i2.19850.
- [19] P. Wulandari, V. Y. Kurniawan, and N. A. Kurdhi, "Construct The Triple Zero Graph of Ring  $\mathbb{Z}_n$  Using Python," in *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, Universitas Pattimura, Mar. 2024, pp. 0507–0516. doi: 10.30598/barekengvol18iss1pp0507-0516.
- [20] Marsudi, *Teori Graf*, 1st ed. Malang: UB Press, 2016.
- [21] G. Chartrand and P. Zhang, *A First Course in Graph Theory*. Mineola: Dover Publications, 2012.