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DIFFERENCE EQUATION FOR AUSTRALIAN SHEEP BLOWFLIES GROWTH

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ABSTRACT

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The population of Australian sheep blowflies, Lucilia cuprina, in Australia is of concern to many researchers because it causes several problems. These problems occur in the sheep industry where there is a term "flystrike" in the industry. Flystrike is a fly attack on sheep that causes myiasis on the sheep's skin, affecting the quality and quantity of wool. In the worst cases, the sheep may die if not treated. This issue has attracted researcher to conduct a population control study of fly growth to suppress flystrike in the Australian sheep industry. In this paper, fly growth will be approached using a difference equation to better represent the industry's situation. This equation will be analyzed using its approximate solution that is obtained through linearization of perturbation method, Cardano's formula, and Galois solution's method. By studying fly growth, Australian sheep farmers may find it easier to handle and prevent fly infestations using the solution.

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1. INTRODUCTION

The progress of human civilization is influenced by the progress of the application of mathematics, which uses mathematical concepts to solve problems in everyday real life. Mathematical modeling is one of the implementations of applied mathematics that deals with real-life problems expressed in mathematical symbols through systematic steps. One of the problems that can be modeled using mathematics is the prediction of population growth. In creating a population model, several parameters are required that can be predicted according to population growth. In this paper, the population model is considered in discrete time intervals, so that each age group describes the same time interval. When modeling the population growth of living things, it will be closer to the actual situation by assuming that individuals can reproduce if they have reached adulthood. Every individual needs a process to grow and develop from birth until reaching the fertile period in order to reproduce. In the mathematical model, this is called a delay time. In the real world, population growth does not always increase infinitely, nor does it always decrease. A delay time can cause the population to decrease at one point in time, but then increase again, resulting in the form of oscillations in population growth. Another approach is to assume limited resources and a biological situation that leads to a logistic model (see [1]). Thus, in the logistic model, the population is limited by the population density, resulting in an infinite number of populations (see [2]). The results given by the Logistic Model are considered to be the closest to the actual results.

About sixty six years ago, A. J. Nicholson was the first to published a study concerning how populations may accommodate themselves to changed conditions in laboratory populations of Australian sheep blowflies, *Lucilia cuprina* (see [3]). May (see [4]) and Barnes-Fulford (see [5]) re-described research by its Nicholson's research more easily understood. In Australia, this species caused particular problems in the sheep industry, with farms consisting of thousands of acres accompanied by huge with very large fly populations. Of the several groups of flies that exist, there are groups of flies that not only cause disease but can harm humans. The activity of "blowflies" is an extrapolation of normal fly activity that extends activity from dead animals to live animals. One of the cases that harm humans is the attack of blowflies on livestock, namely sheep in Australia. This attack is detrimental to sheep farmers in Australia because it affects the health of their sheep and causes death. These fly attacks on sheep are called "flystrike" and result in animal mortality and large treatment and prevention costs, estimated at \$100-150 million per year in Australia. The most influential factor in these attacks is the warm and humid environment, especially in spring, and late summer or autumn, in Australia (see [6]). Sheep that have recovered from the attack will also experience a decline in condition and the quality of their wool will not be as good as before, resulting in a decrease in the quality and quantity of wool. Initially, this fly causes skin myiasis in living sheep. Myiasis is the invasion of maggots or flies into body tissue, causing damage to healthy tissue. This fly breeds mainly on carcasses. Sheep skin myiasis in Australia is higher due to the level of susceptibility. If infested sheep are not euthanized and the development of maggot continues, sheep mortality will occur. An estimate made in 1938 suggested that the annual cost of sheep skin myiasis in Australia in terms of crop losses and preventative measures amounted to £4,000,000 (see [7]). The problem of fly control in Australia is approached in two main ways: protection of sheep against infestation and investigation of ways to reduce fly breeding. Recently, Colvin et al (see [8]) revealed reports incidence and control practices on blowfly strike in sheep between 2003 and 2019. There are several studies on the analysis of population dynamics growth based on various models such as Rashkovsky and Margaliot (see [9]) which gave a fuzzy modeling approach to transform the verbal description of several enthologists into a well-defined mathematical model, Hutchinson (see [10]) that first introduced differential logistic delay equation for general population, and then Nicholson (see [11]) which modified the equation to be applied on the growth of *Lucilia cuprina*. Some research about Nicholson's blowflies can be seen in [3], [9] , [11]–[17].

Population dynamics in an ecosystem are not always close to or away from the carrying capacity of the environment. From [5], there is a model that uses that term i.e. *discrete logistic equation*,

$$X_{n+1} - X_n = rX_n - \frac{r}{K}X_n^2$$

(see [5]), where X_{n+1} and X_n represent the number of population at discrete time $n+1$ and n, respectively. The notation r and K represent the difference between rate of birth and death and the carrying capacity, respectively. However, there are population dynamics where the individuals cannot reproduce (give birth/lay eggs) continuously throughout their lives. Therefore, this phenomenon makes population dynamics in growing will be delayed. Delay time can cause a decreasing population, but then an increasing occurs, resulting oscillations in graphic of population growth. It means that the rate of population growth not only depends on the population size at the current time t, but also depends on the population size at the previous time $t - t_d$ and the number of populations at initial conditions are the same i.e.

$X_0 = X_1 = \cdots = X_{t_d}$. The delay time version of the previous equation is

$$X_{n+1} - X_n = rX_n - \frac{r}{K} X_n X_{n-t_d}$$

where $t_d > 0$ (see [5]), which is called *difference equation*.

In this paper, we employ the useful method that is a linear perturbation method to get the solution of the difference equation. As on perturbation theory, the solution that will be obtained is an approximation, see for example [18], [19]. We will focus only on the solution for a particular delay time i.e. $0 \leq t_d \leq 3$ and then analyzing the growth characteristics of *Lucilia cuprina*, especially for $t_d = 0{,}1$. The time for the fly grows from a laid egg to be an adult fly is t_d days depending on the surrounding environmental conditions. We assume that every unit of time throughout this discussion accounts t_d and another assumption is the current density position is affected by the presence of eggs in the past.

2. RESEARCH METHODS

Consider the discrete logistic equation,

$$X_{n+1} - X_n = rX_n - \frac{r}{K} X_n^2 \tag{1}$$

The delay time version of the previous equation is

$$X_{n+1} - X_n = rX_n - \frac{r}{K} X_n X_{n-t_d} \tag{2}$$

where $t_d > 0$ (see [5]), which is called *difference equation with delay time*. We determine the approximate solution for difference equation with $0 \leq t_d \leq 3$, which is obtained using the linearization of perturbation method, particularly by analyzing the displacement around the equilibrium solution, i.e., carrying capacity K. The solutions for $t_d = 0$ and $t_d = 1$ will be explicitly analyzed to describe the growth of Australian sheep blowflies, *Lucilia cuprina*. The solutions for $t_d = 2$ and $t_d = 3$ will be obtained using the same method but with extra tools, namely Cardano's formula and Galois solution's method. The key point to find the solution in this case ($t_d = 2$ and $t_d = 3$) is solving cubic and quartic polynomials. Cardano's formula stated that the zeros of cubic polynomial (has degree 3),

$s(x) = ax^3 + bx^2 + cx + d$, are

$$x_1 = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$x_2 = -\frac{b}{3a} + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

and

$$x_3 = -\frac{b}{3a} + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

where $p = -\frac{b^2}{3a^2} + \frac{c}{a}$ and $q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$. The method of Galois solution (see [20]) stated that the zeros of quartic polynomial (has degree 4) with rational coefficient i.e. $s(x) = x^4 + ax^3 + bx^2 + cx + d$, can be obtained after applying the following steps:

1. Substitute $x = y - \frac{a}{4}$ into the polynomial to get $Y = y^4 + \alpha y^2 + \beta y + \gamma$.
2. Let y_1, y_2, y_3, and y_4 are the zeros of Y and let $h = (z - w)(z - w')(z - w'')$, where
$$w = (y_1 + y_2)(y_3 + y_4),$$
$$w' = (y_1 + y_3)(y_2 + y_4),$$
$$w'' = (y_1 + y_4)(y_2 + y_3)$$
Solve the polynomial $h = z^3 - 2\alpha z^2 + (\alpha^2 - 4\gamma)z + \beta^2$.
3. Use $y_1 + y_2 + y_3 + y_4 = 0$ and the zeros of h to get the values of y_1, y_2, y_3, and y_4.
4. The zeros of $x^4 + ax^3 + bx^2 + cx + d$ are $x_1 = y_1 - \frac{a}{4}$, $x_2 = y_2 - \frac{a}{4}$, $x_3 = y_3 - \frac{a}{4}$, and $x_4 = y_4 - \frac{a}{4}$.

3. RESULTS AND DISCUSSION

Given a positive integer α and a real number β, define a polynomial $P_{\alpha,\beta}(x) = x^{\alpha+1} - x^{\alpha} + \beta$ of degree $\alpha + 1$. Now, we use the linearization of perturbation method particularly analyzing displacement around the equilibrium solution, i.e., carrying capacity K.

Let $X_n = K + \epsilon h_n$ for $0 < \epsilon \ll 1$ and $|\epsilon h_n| \ll K$. Substitute this X_n into Equation (2) such that
$$h_{n+1} - h_n = -rh_{n-1} - \frac{r\epsilon}{K} h_n h_{n-t_d}$$
We can neglect the nonlinear term $-\frac{r\epsilon}{K} h_n h_{n-t_d}$ because of the assumption of ϵh_n. Therefore, we have
$$h_{n+1} - h_n + rh_{n-t_d} = 0$$
It seems like a recurrence relation of degree $t_d + 1$ where the coefficient of h_{n-i} is zero, for each $1 \le i \le t_d + 1$. By substituting $h_n = x^n$, we have $P_{t_d,r}(x)$ as a characteristic polynomial of the recurrence relation. If $z_1, z_2, \dots, z_{t_d+1}$ are zeros of the polynomial, then the general solution of the recurrence relation is
$$h_n = \alpha_1 z_1^n + \alpha_2 z_2^n + \dots + \alpha_{t_d+1} z_{t_d+1}^n$$
for some constant $\alpha_1, \alpha_2, \dots, \alpha_{t_d+1}$. However, the special one can be obtained using the initial values that should be given. In other words, the solution of X_n is gained after we find all zeros of $P_{t_d,r}(x)$. According to Abel-Ruffini theorem (see [21]), there is no solution in radicals to general $P_{t_d,r}(x)$ for $t_d \ge 4$, but $P_{t_d,r}(x)$ has a beautiful form because $x = c$ is always the root of $P_{t_d,c^{t_d}(1-c)}(c) = 0$. It means that for $t_d = 4$, even if it is a quintic polynomial (has degree 5), we still can obtain all radical solutions of the polynomial after we solve the quartic form depressed from the quintic.

3.1 Case $t_d = 0$

It is easy to calculate that the root of $P_{0,r}(x)$ is only $x = 1 - r$. Therefore, we obtain $h_n = C(1 - r)^n$, for some constant C, such that $X_n = K + \epsilon C(1 - r)^n$ where $0 < \epsilon \ll 1$. Furthermore, we use an initial condition X_0 that is the amount of first population of flies such that we obtain the approximate solution of Equation (2) for $t_d = 0$ is
$$X_n = X_0(1 - r)^n + K(1 - (1 - r)^n) \tag{3}$$
There are certain questions about the solution such as what are characteristics cling to the curve of Equation (3) and how it compared to the curve of Equation (2) for $t_d = 0$. We determine them by fixing X_0 and K, and varying r. Let $\phi = 1 - r$, then:

- For $r < 0$, we have $\phi > 0$. In this case, the rate birth is strictly less than the rate of death, so that the amount of population should be decreasing in increasing unit of time n. It is obvious that $\phi^n(\phi-1) > 0$ and since $X_0 < K$ then for all units time $n \in \mathbb{N}$,

$$X_0\phi^n(\phi-1) + K < K\phi^n(\phi-1) + K \;\Rightarrow\; X_{n+1} < X_n$$

- For $r = 0$, we have $\phi = 1$. The rate birth is equivalent to the rate of death, so that there is no change for the amount of population of all units of time n i.e. $X_n = X_0$.
- For $0 < r < 1$, we have $0 < \phi < 1$. We consider that for all units of time $n \in \mathbb{N}$,

$$\phi^n(X_0 - K) + K < \phi^{n+1}(X_0 - K) + K < K \;\Rightarrow\; X_n < X_{n+1} < K$$

  It means that the growth is exponentially increasing and asymptotically close to K. This growth is often called as *logistic growth* and the equilibrium K is stable.
- For $r = 1$, we have $\phi = 0$. The rate of birth and the rate of death differ by 1 and it is obvious that $X_n = K$ for $n \in \mathbb{N}$. In other words, the amount of population for all units of time $n \in \mathbb{N}$ is always in stable condition K.
- For $r > 1$, we have $\phi < 0$. The term ϕ^n is depend on the parity of n, so that it can be divided by two cases :
  - If n is even integer, then $\phi^n > 0$. Therefore,

$$X_n = X_0\phi^n + K(1-\phi^n) = (X_0 - K)\phi^n + K < K.$$

    It means that the amount of population is always under K while unit of time n is even integer.
  - If n is odd integer, then $\phi^n < 0$. Therefore,

$$X_n = X_0\phi^n + K(1-\phi^n) = (X_0 - K)\phi^n + K > K.$$

    It means that the amount of population is always beyond K while unit of time n is odd integer.

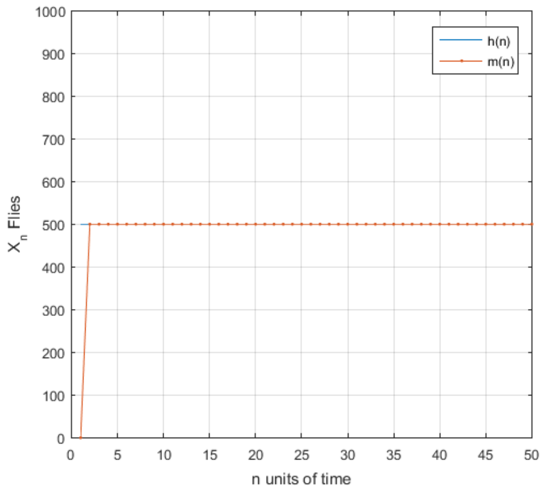
  Those cases show that it is an oscillation growth around equilibrium K for all units of time n.
- For $r = 2$, we have $\phi = -1$ and then $X_n = X_0(-1)^n + K(1-(-1)^n)$. It is obvious that

  $X_n = X_0 < K$, when n is even integer and $X_n = -X_0 + 2K > K$, when n is odd integer. Those values are always constant such that the oscillation forms a 2-cycle (when the same amount of population always repeated after two units of time).

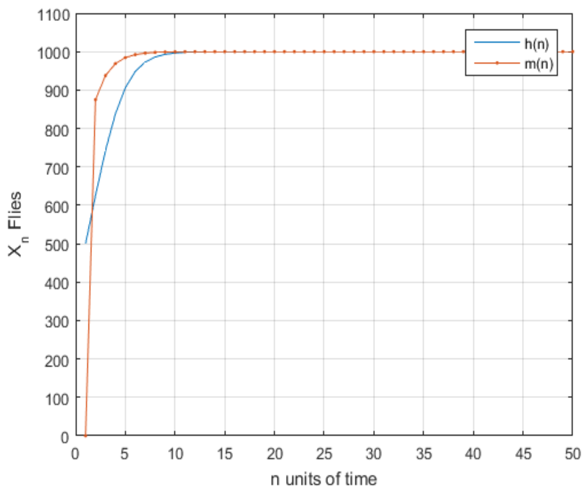
Equation (3) also shows that the amplitude of oscillation in the curve will increase for large r due to

$$\lim_{r\to\infty} X_n = \begin{cases} K + s, & s \to +\infty, n \text{ is odd} \\ K + s, & s \to -\infty, n \text{ is even} \end{cases}$$

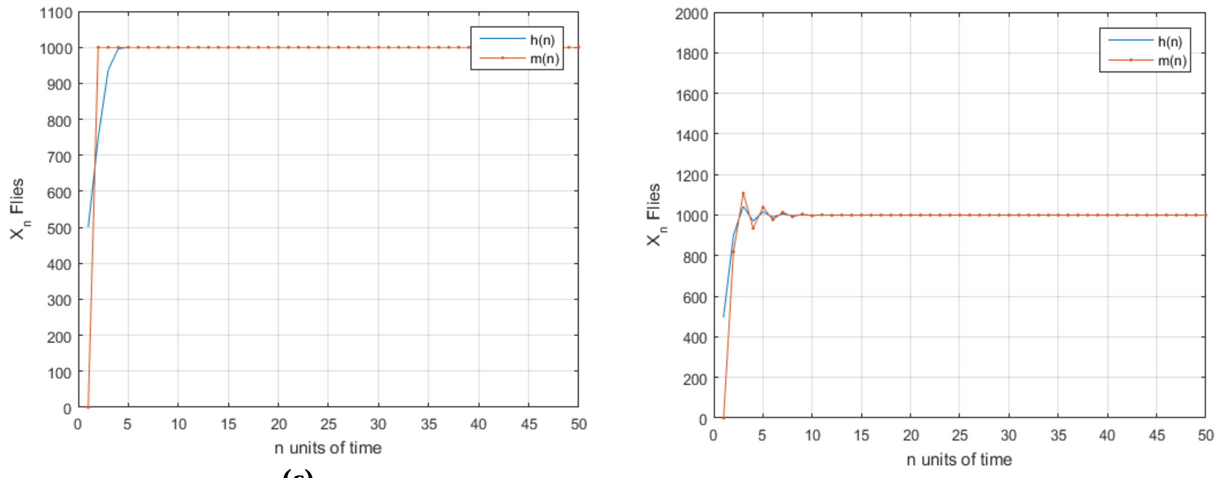
In **Figure 1** are presented the population of flies based on the **Equation (2)** for $t_d = 0$ and **Equation (3)** for n unit time.



(a)



(b)



(c) (d)

Figure 1. In this simulation, it is used Matlab software and the input parameters are $K = 1000$ and $X_0 = 500$ for each figure (a) $r = 0$, (b) $r = 0.5$, (c) $r = 1$, and (d) $r = 1.6$. The function $h(n)$ and $m(n)$ represent the graphic of Equation (2) for $t_d = 0$ and Equation (3), respectively.

3.2 Case $t_d = 1$

In this case, since $P_{1,r}(x)$ is a quadratic polynomial, then it is also easy to calculate that the zeros of $P_{1,r}(x)$ are $x_1 = \frac{1+\sqrt{1-4r}}{2}$ and $x_2 = \frac{1-\sqrt{1-4r}}{2}$. Therefore,

$$h_n = C_1\left(\frac{1+\sqrt{1-4r}}{2}\right)^n + C_2\left(\frac{1-\sqrt{1-4r}}{2}\right)^n$$

for some constants C_1 and C_2, such that

$$X_n = K + \epsilon\left[C_1\left(\frac{1+\sqrt{1-4r}}{2}\right)^n + C_2\left(\frac{1-\sqrt{1-4r}}{2}\right)^n\right]$$

where $0 < \epsilon \ll 1$. Recall that $X_0 = X_1 = \mu$ as initial conditions based on the delayed time. So, using these initial conditions, we obtain $C_1 = \frac{\mu-K}{2\epsilon}\left(1 + \frac{1}{\sqrt{1-4r}}\right)$ and $C_2 = \frac{\mu-K}{2\epsilon}\left(1 - \frac{1}{\sqrt{1-4r}}\right)$, where $r \neq \frac{1}{4}$. Therefore, the approximate solution of Equation (2) for $t_d = 1$ is

$$X_n = K + \frac{\mu - K}{2^{n+1}\Delta^{\frac{1}{2}}}\left[\left(1+\Delta^{\frac{1}{2}}\right)^{n+1} - \left(1-\Delta^{\frac{1}{2}}\right)^{n+1}\right] \tag{4}$$

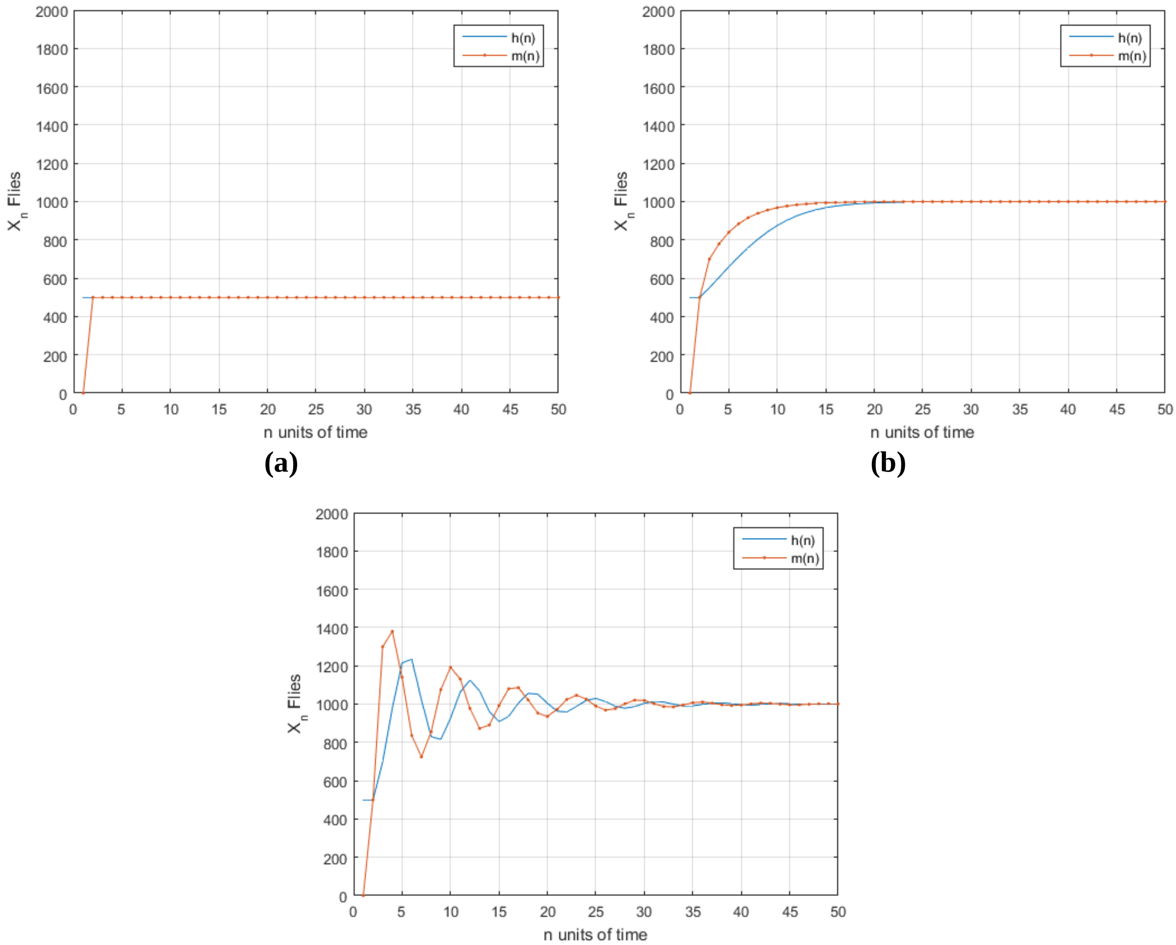
where $\Delta = 1 - 4r$. It is obvious that X_n is unbounded whenever r goes to $\frac{1}{4}$. For other values of r, there are some characteristics of the curve of Equation (4) by fixing μ and K as follows:

- For $r = 0$, it is obvious that $X_n = \mu$ for any positive integer $n \geq 2$. It means that there is no change for the amount of *Lucilia cuprina*'s population of all units of time $n \geq 2$.
- For $0 < r < \frac{1}{4}$, then $0 < \Delta < 1$. In holds true that $\left(1+\sqrt{\Delta}\right)^n > \left(1-\sqrt{\Delta}\right)^n$ and $\mu - K < 0$, then $X_n < K$ for any positive integer $n \geq 2$. On the other hand, $\lim_{n\to\infty} X_n = K$ so that the amount of *Lucilia cuprina*'s population grows exponentially close to the equilibrium K for large number unit of time n. In other words, this is called as logistic growth and the equilibrium K is stable.
- For $\frac{1}{4} < r < 1$ such that $0 < -\Delta < 3$, then

$$X_n = K + \frac{\mu - K}{2^{n+1}(-\Delta)^{\frac{1}{2}}}\left[\left((-\Delta)^{\frac{1}{2}} - i\right)\left(1 + i(-\Delta)^{\frac{1}{2}}\right)^n + \left((-\Delta)^{\frac{1}{2}} + i\right)\left(1 - i(-\Delta)^{\frac{1}{2}}\right)^n\right] \tag{5}$$

By changing **Equation (5)** into a polar coordinate, we have $X_n = K + \frac{2(\mu-K)r^{\frac{n+1}{2}}\cos\alpha}{(-\Delta)^{1/2}}$, where $\alpha = n\tan^{-1}\left((-\Delta)^{1/2}\right) - \tan^{-1}\left(\frac{1}{(-\Delta)^{1/2}}\right)$. It is not hard to know that there is $\alpha = 0$ such that $X_n < K$. Suppose that $\cos\alpha \geq 0$ for any $0 < -\Delta < 3$. Then $\alpha \geq 2n\pi$. In fact, $0 < n\tan^{-1}\left((-\Delta)^{1/2}\right) < \frac{n\pi}{3}$ and $-\frac{\pi}{6} < -\cot^{-1}\left((-\Delta)^{1/2}\right) < -\frac{\pi}{2}$ such that $-\frac{\pi}{6} < \alpha < \frac{(2n-3)\pi}{6}$. However, the inequality $2n\pi \leq \alpha < \frac{(2n-3)\pi}{6}$ is impossible. Therefore, there is at least one $-\Delta \in (0,3)$ such that $\cos\alpha < 0$ and it implies that $X_n > K$. Hence, the *Lucilia cuprina*'s population growth contains an oscillation around equilibrium K and this equilibrium is stable due to $X_n \to K$ for large number unit of time n.

In **Figure 2**, we present the population of flies based on **Equation (2)** for $t_d = 1$ and **Equation (4)** for n unit time.



(a) (b)

(c)

Figure 2. In this simulation, it is used Matlab software and the input parameters are $K = 1000$ and $X_0 = X_1 = 500$ for each figure (a) $r = 0$, (b) $r = 0.2$, and (c) $r = 0.8$. The function $h(n)$ and $m(n)$ represent the graphic of Equation (2) for $t_d = 1$ and Equation (4), respectively.

3.3 Case $t_d = 2$

By Cardano's formula, the zeros of $P_{2,r}(x)$ are

$$x_1 = \frac{1}{3} + \sqrt[3]{-\frac{r}{2} + \frac{1}{27} + \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}} + \sqrt[3]{-\frac{r}{2} + \frac{1}{27} - \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}}$$

$$x_2 = \frac{1}{3} + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{r}{2} + \frac{1}{27} + \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}} + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{r}{2} + \frac{1}{27} - \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}}$$

and

$$x_3 = \frac{1}{3} + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{r}{2} + \frac{1}{27} + \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}} + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\sqrt[3]{-\frac{r}{2} + \frac{1}{27} - \sqrt{\frac{\left(r - \frac{2}{27}\right)^2}{4} - \frac{1}{729}}}$$

so the general solution is

$$X_n = K + \epsilon(\ell_1 x_1^n + \ell_2 x_2^n + \ell_3 x_3^n)$$

Recall that $X_0 = X_1 = X_2 = Z$ as initial conditions based on the delay of time. Those initial conditions give us the following linear system in three variables,

$$\ell_1 + \ell_2 + \ell_3 = \frac{Z - K}{\epsilon}$$

$$\ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 = \frac{Z - K}{\epsilon}$$

and

$$\ell_1 x_1^2 + \ell_2 x_2^2 + \ell_3 x_3^2 = \frac{Z - K}{\epsilon}$$

The system can be considered as the following

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = \begin{bmatrix} \frac{Z - K}{\epsilon} \\ \frac{Z - K}{\epsilon} \\ \frac{Z - K}{\epsilon} \end{bmatrix}$$

We can obtain the solutions of the linear system by doing reduced row echelon of the following augmented 3×3 transposed Vandermonde matrix of the linear system,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{Z - K}{\epsilon} \\ x_1 & x_2 & x_3 & \frac{Z - K}{\epsilon} \\ x_1^2 & x_2^2 & x_3^2 & \frac{Z - K}{\epsilon} \end{array}\right]$$

Therefore, we obtain

$$\ell_1 = \big(1 + (1 - x_1)(x_2 - x_1)\big)\frac{(x_2 - 1)}{(x_2 - x_1)}\frac{Z - K}{\epsilon}$$

$$\ell_2 = \left(1 - \frac{(1-x_2)}{(x_3-x_2)}\right)\frac{(1-x_1)}{(x_2-x_1)}\frac{Z-K}{\epsilon}$$

and

$$\ell_3 = \frac{(1-x_1)(1-x_2)}{(x_3-x_1)(x_3-x_2)}\frac{Z-K}{\epsilon}$$

We finally obtain the approximate solution of **Equation (2)** for $t_d = 2$ is

$$X_n = K + (Z-K)\left\{\begin{aligned}&\left(1+(1-x_1)(x_2-x_1)\right)\frac{(x_2-1)x_1^n}{(x_2-x_1)} + \left(1 - \frac{(1-x_2)}{(x_3-x_2)}\right)\frac{(1-x_1)x_2^n}{(x_2-x_1)}\\ &+ \frac{(1-x_1)(1-x_2)x_3^n}{(x_3-x_1)(x_3-x_2)}\end{aligned}\right\}$$

3.4 Case $t_d = 3$

In this case, we cannot give the general solutions for **Equation (2)** because our methods, which will be used to solve $P_{3,r}(x)$, only hold for certain values of r. First, let $r = a^3(1-a)$ for any $a \in \mathbb{R}$. It is clear that the zeros of $P_{3,a^3(1-a)}(x)$ are

$$x_1 = a$$

$$x_2 = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} - \frac{(a-1)}{3}$$

$$x_3 = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} + \left(-\frac{3}{2} - \frac{3i\sqrt{3}}{2}\right)\sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} - \frac{(a-1)}{3}$$

and

$$x_4 = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} + \left(-\frac{3}{2} + \frac{3i\sqrt{3}}{2}\right)\sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{(a^2-2a-2)^3}{5832}}} - \frac{(a-1)}{3}$$

where $b = \frac{20a^3-31a^2-5a-2}{27}$. Here we have the general solution

$$X_n = K + \epsilon(s_1a^n + s_2x_2^n + s_3x_3^n + s_4x_4^n)$$

Recall that $X_0 = X_1 = X_2 = X_3 = Z$ as initial conditions based on the delayed time. Those conditions give us the following linear system in four variables,

$$s_1 + s_2 + s_3 + s_4 = \frac{Z-K}{\epsilon}$$

$$s_1a + s_2x_2 + s_3x_3 + s_4x_4 = \frac{Z-K}{\epsilon}$$

$$s_1a^2 + s_2x_2^2 + s_3x_3^2 + s_4x_4^2 = \frac{Z-K}{\epsilon}$$

and

$$s_1a^3 + s_2x_2^3 + s_3x_3^3 + s_4x_4^3 = \frac{Z-K}{\epsilon}$$

By doing the reduced row echelon, we obtain the solutions of the linear system as follows,

$$s_1 = \frac{Z-K}{\epsilon} - s_2 - s_3 - s_4,$$

$$s_2 = \left(1 - \frac{(1+x_3+x_4-x_2)}{(x_3-x_2)(x_4-x_2)}\right)\frac{(1-a)}{(x_2-a)}\frac{Z-K}{\epsilon}$$

$$s_3 = \frac{(1-a)(1-x_2)(x_4-1)}{(x_3-x_2)(x_3-a)(x_4-x_3)}\frac{Z-K}{\epsilon}$$

and

$$s_4 = \frac{(1-a)(1-x_2)(1-x_3)}{(x_4-x_2)(x_4-a)(x_4-x_3)}\frac{Z-K}{\epsilon}$$

Therefore, the approximate solution of **Equation (2)** for $t_d = 3$ is the following,

$$X_n = K + (Z-K)\left\{\begin{array}{l} s_1 a^n + \left(1 - \frac{(1+x_3+x_4-x_2)}{(x_3-x_2)(x_4-x_2)}\right)\frac{(1-a)}{(x_2-a)}x_2^n + \frac{(1-a)(1-x_2)(x_4-1)}{(x_3-x_2)(x_3-a)(x_4-x_3)}x_3^n \\ + \frac{(1-a)(1-x_2)(1-x_3)}{(x_4-x_2)(x_4-a)(x_4-x_3)}x_4^n \end{array}\right\} \quad (6)$$

Second, let r is any rational number. We will use the Galois solution's method for $P_{3,r}(x)$. Now, substitute $x = y + \frac{1}{4}$, yielding $P'_{3,r}(y) = y^4 - \frac{6}{16}y^2 - \frac{3}{32}y + \left(r - \frac{3}{256}\right)$. Let y_1, y_2, y_3, and y_4 are the zeros of $P'_{3,r}(y)$. Let $L \in \mathbb{Q}[y]$ be the polynomial

$$L = \big(y - (y_1+y_2)(y_3+y_4)\big)\big(y - (y_1+y_3)(y_2+y_4)\big)\big(y - (y_1+y_4)(y_2+y_3)\big)$$

where

$$L = y^3 + \frac{3}{4}y^2 + \left(\frac{48}{256} - 4r\right)y + \frac{9}{1024}$$

The last L can be solved by Cardano's formula. Therefore,

$$(y_1+y_2)(y_3+y_4) = \triangle_+ + \triangle_- - \frac{1}{4}$$

$$(y_1+y_3)(y_2+y_4) = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}$$

and

$$(y_1+y_4)(y_2+y_3) = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}$$

where $\triangle_+ := \sqrt[3]{\frac{1}{2}\left(\frac{7}{1024} - r\right) + \sqrt{\frac{\left(\frac{7}{1024}-r\right)^2}{4} - \frac{64r^3}{27}}}$ and $\triangle_- := \sqrt[3]{\frac{1}{2}\left(\frac{7}{1024} - r\right) - \sqrt{\frac{\left(\frac{7}{1024}-r\right)^2}{4} - \frac{64r^3}{27}}}$. By using the fact that the sum of all zeros of $P'_{3,r}(y)$ is zero, we obtain

$$y_1 = -\frac{i}{2}\left(\begin{array}{l} \sqrt{\triangle_+ + \triangle_- - \frac{1}{4}} + \sqrt{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}} \\ + \sqrt{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}} \end{array}\right)$$

$$y_2 = \frac{i}{2}\left(\sqrt{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}} - \sqrt{\triangle_+ + \triangle_- - \frac{1}{4}} \right.$$
$$\left. - \sqrt{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}}\right)$$

$$y_3 = \frac{i}{2}\left(\sqrt{\triangle_+ + \triangle_- - \frac{1}{4}} - \sqrt{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}} \right.$$
$$\left. + \sqrt{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}}\right)$$

and

$$y_4 = \frac{i}{2}\left(\sqrt{\triangle_+ + \triangle_- - \frac{1}{4}} + \sqrt{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}} \right.$$
$$\left. - \sqrt{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\triangle_+ + \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\triangle_- - \frac{1}{4}}\right)$$

So, the zeros of $P_{3,r}(x)$ are

$$x_1 = y_1 + \frac{1}{4}, \quad x_2 = y_2 + \frac{1}{4}, \quad x_3 = y_3 + \frac{1}{4}, \quad x_4 = y_4 + \frac{1}{4}$$

We obtain the general solution

$$X_n = K + \epsilon(v_1 x_1^n + v_2 x_2^n + v_3 x_3^n + v_4 x_4^n)$$

Recall that $X_0 = X_1 = X_2 = X_3 = Z$ as initial conditions based on the delayed time. Those conditions give us the following linear system in four variables,

$$v_1 + v_2 + v_3 + v_4 = \frac{Z-K}{\epsilon}$$
$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 = \frac{Z-K}{\epsilon}$$
$$v_1x_1^2 + v_2x_2^2 + v_3x_3^2 + v_4x_4^2 = \frac{Z-K}{\epsilon}$$

and

$$v_1x_1^3 + v_2x_2^3 + v_3x_3^3 + v_4x_4^3 = \frac{Z-K}{\epsilon}$$

By doing the reduced row echelon, we obtain the solutions of the linear system as follows,

$$v_1 = \frac{Z-K}{\epsilon} - v_2 - v_3 - v_4$$
$$v_2 = \left(1 - \frac{(1 + x_3 + x_4 - x_2)}{(x_3 - x_2)(x_4 - x_2)}\right)\frac{(1-x_1)}{(x_2-x_1)}\frac{Z-K}{\epsilon}$$
$$v_3 = \frac{(1-x_1)(1-x_2)(x_4-1)}{(x_3-x_2)(x_3-x_1)(x_4-x_3)}\frac{Z-K}{\epsilon}$$
$$v_4 = \frac{(1-x_1)(1-x_2)(1-x_3)}{(x_4-x_2)(x_4-x_1)(x_4-x_3)}\frac{Z-K}{\epsilon}$$

Therefore, the approximate solution of **Equation (2)** for $t_d = 3$, where $r \in \mathbb{Q}$, is the following,

$$X_n = K + (Z - K)\left\{\begin{matrix} v_1 x_1^n + \left(1 - \dfrac{(1 + x_3 + x_4 - x_2)}{(x_3 - x_2)(x_4 - x_2)}\right)\dfrac{(1 - x_1)}{(x_2 - x_1)}x_2^n + \dfrac{(1 - x_1)(1 - x_2)(x_4 - 1)}{(x_3 - x_2)(x_3 - x_1)(x_4 - x_3)}x_3^n \\ + \dfrac{(1 - x_1)(1 - x_2)(1 - x_3)}{(x_4 - x_2)(x_4 - x_1)(x_4 - x_3)}x_4^n \end{matrix}\right\} \quad (7)$$

The solution of the model for $t_d = 0,1,2$ has covered all values of parameter r whereas for $t_d = 3$, the solutions only cover two possibilities of r, which are $r = a^3(1 - a)$ for any $a \in \mathbb{R}$ and $r \in \mathbb{Q}$. According to Abel-Ruffini's Theorem, we cannot write the solution of the model for $t_d > 3$, so the numerical method may be used to solve it.

4. CONCLUSIONS

The results obtained in the discussion are the approximate solutions of the difference equation at certain values of delay time, i.e., $t_d = 0,1,2,3$, and the analysis of **Equation (3)** and **Equation (4)** which show that the growth characteristics of the solutions closely approximate those of the related equations. This research has not provided a handling solution for the problem of Australian Sheep Blowflies, *Lucilia cuprina*, but the approximate solutions that have been obtained and some analysis can be a reference to other researchers to study the growth characteristics of the flies in certain period of time. Moreover, the readers that are interested in this topic can directly apply our results to solve the *Lucillia cuprina*'s problem deeply or determine the rest case that is not covered in this paper (i.e. the solution of the model when $t_d = 3$ and $r \neq a^3(1 - a)$ for $a \in \mathbb{R}$ and $r \notin \mathbb{Q}$).

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