BAREKENG: Journal of Mathematics and Its Applications
September 2024 Volume 18 Issue 3 Page 1729–1738
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss3pp1729-1738

DERIVATION ON SEVERAL RINGS

Abdiel Bellamy Thomas[1*], Nikken Prima Puspita[2], Fitriani[3]

[1,2]Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro
Jl. Prof. Jacub Rais, Semarang, 50275, Indonesia

[3]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung
Jl. Prof. Dr. Ir. Soemantri Brojonegoro No. 1, Bandar Lampung, 35141, Indonesia

*Corresponding author's e-mail: * **abdiel16@students.undip.ac.id***

Article History:

Received: 17th February 2024
Revised: 5th May 2024
Accepted: 7th June 2024
Published: 1st September 2024

Keywords:

Ring;
Derivation;
Ideal;
Cartesian product;
Derivation on ring

ABSTRACT

Research on ring derivation is one of the studies that is quite popular among algebra lovers. The definition of the derivation on the ring is motivated by the derivation in calculus which has Leibniz's rule. The purpose of this paper is to show some of the derivation properties on several rings, namely divisor rings, cartesian product rings, and factor rings. Let R be a commutative ring with multiplicative identity and A the set of multiplicative closed that has non-zero divisor. In this paper, we have shown some results of derivation on ring theory. If d is a ring derivation of R and R_A is a divisor ring of R, we can construct $\delta_A: R_A \to R_A$, where $\delta_A\left(\frac{r}{a}\right) = \frac{d(r)\cdot a - r\cdot d(a)}{a^2}$ for all $\frac{r}{a} \in R_A$, then the map δ_A is a derivation on R_A. The concept of embedding one ring into another ring can be used so that the ring of constant of d, namely R^d, is a subring of the divisor ring R_A. Related to the ideal on ring theory, if I is an ideal of R, then $\bar{d}: R/I \to R/I$ where $\bar{d}(a + I) \longmapsto d(a) + I$ is also a derivation on the ring R/I. The last result in this paper comes from the ring of cartesian product, take R_i be a ring with derivation $d_i: R_i \to R_i$ for $i \in \mathbb{N}$. The cartesian product ring $\prod_{i=1}^{n} R_i$ have a derivation ring defined by $\prod_{i=1}^{n} d_i : \prod_{i=1}^{n} R_i \to \prod_{i=1}^{n} R_i$, $\prod_{i=1}^{n} d_i\left((r_1, r_2, \dots, r_n)\right) = \left(d_1(r_1), d_2(r_2), \dots, d_n(r_n)\right)$ for any $(r_1, r_2, \dots, r_n) \in \prod_{i=1}^{n} R_i$.

How to cite this article:

A. B. Thomas, N. P. Puspita and Fitriani., "DERIVATION ON SEVERAL RINGS," *BAREKENG: J. Math. & App.*, vol. 18, iss. 3, pp. 1729-1738, September, 2024.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

Ring theory is a structure in abstract algebra whose existence is very important in mathematics. Almost every branch of mathematics and non-mathematics uses it indirectly, one of which is a derivative of Calculus. In mathematics, especially in calculus, the concept of derivation is well known. The derivative was first thought of by a British mathematician and physicist, whose name is Sir Isaac Newton (1642-1727), and a German mathematician, whose name is Gottfried Wilhelm Leibniz (1646-1716).

The concept of the derivative in calculus uses polynomial rings $\mathbb{Z}[x]$ as a domain and codomain of the derivative function defined for each $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ with $a_i \in \mathbb{Z}$ and $i = 0,1,2,\dots,n$ where $n \in \mathbb{N} \cup \{0\}$ holds $\frac{d}{dx}\big(f(x)\big) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in \mathbb{Z}[x]$. There is the Leibniz's rule that for every $f(x), g(x) \in \mathbb{Z}[x]$ holds $\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big) \in \mathbb{Z}[x]$ and $\frac{d}{dx}\big(f(x) \cdot g(x)\big) = \frac{d}{dx}\big(f(x)\big) \cdot g(x) + f(x) \cdot \frac{d}{dx}\big(g(x)\big) \in \mathbb{Z}[x]$, as described in [1]. This shows that the existence of derivatives is inseparable from the ring theory. This is because the concept of derivation was first introduced on a ring, namely a polynomial ring.

Based on [2], a ring R is a system consisting of a non-empty set R and two binary operations in R, called addition $(+)$ and multiplication $(\cdot)$ such that $(R, +)$ is an abelian group, $(R,\cdot)$ is a semigroup, and the distributive law applies between multiplication and addition operations. There are many types of rings that we already know, for example, divisor rings, cartesian product rings, factor rings, idempotent rings, and many more. It can be seen that the derivative in calculus is a mapping from one ring to another ring and there is Leibniz's rule. This is the initial motivation for defining the ring derivation.

According to [3], [4], and [5], a map $d: R \to R$ is a derivation on the ring R if for all $a, b \in R$ a map d satisfies $d(a + b) = d(a) + d(b)$ and $d(a \cdot b) = d(a) \cdot b + a \cdot d(b)$. One example of the derivation is the derivation on the polynomial ring $\mathbb{Z}[x]$. Let a polynomial ring $\mathbb{Z}[x]$ and the mapping $d: \mathbb{Z}[x] \to \mathbb{Z}[x]$ with the definition $d\big(p(x)\big) = d(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$ for all $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$ is a derivation on the polynomial ring $\mathbb{Z}[x]$ as explained in [6]. Another easy-to-understand example of derivation is the derivation which uses lie product. Let R be a noncommutative ring. For a fixed element $a \in R$ is defined $d: R \to R$ by $d(r) = [a, r] = ar - ra$ for all $r \in R$. If we check, it can be seen that the mapping d is a derivation because for all $r_1, r_2 \in R$ we obtain $d(r_1 + r_2) = [a, r_1 + r_2] = a(r_1 + r_2) - (r_1 + r_2)a = ar_1 + ar_2 - r_1a - r_2a = ar_1 - r_1a + ar_2 - r_2a = d(r_1) + d(r_2)$ and $d(r_1r_2) = [a, r_1r_2] = ar_1r_2 - r_1r_2a = ar_1r_2 - r_1ar_2 + r_1ar_2 - r_1r_2a = [a, r_1]r_2 + r_1[a.r_2] = d(r_1)r_2 + r_1d(r_2)$. That means the mapping d is a derivation called the inner derivation of R which is associated with a, see [5]. Several types of special rings can be found, namely divisor rings, cartesian product rings, and factor rings. For each of these rings, a general derivation can be defined that will always apply. In this paper, we will discuss each derivation on the divisor ring, the cartesian product ring, and the factor ring. We will also discuss the concept of inserting one ring into another ring, where for example a ring R with derivation d has a ring of constant of d, namely R^d, then R^d is a subring of the divisor ring R_A where A is a multiplicatively closed set of R.

2. RESEARCH METHODS

This study focuses on the derivation properties of the divisor ring, the cartesian product ring, and the factor ring. At the beginning of the research, we conducted a literature study regarding rings, a derivation on ring, an ideal-d, and several theorems as contained in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], and [12]. After that, we construct a conjecture regarding the properties of the derivation in the predetermined rings. In the last step, we will prove some of the properties that we have built. In building a derivation property of some of these rings, it is necessary to first introduce the process of constructing divisor rings, the theorem of necessary and sufficient conditions of a subring, a ring of constant, and an ideal-d.

There is a type of ring, namely the divisor ring obtained from a ring with a multiplicatively closed set which is explained in the following proposition.

Proposition 1. [7] *Let R be a commutative ring with unity. If A is a multiplicatively closed set of R without zero element and zero divisor, then there is a commutative ring R_A with unity which contains R as a subring of R_A. Furthermore, for all element in A is a unit in R_A.*

A subset S of a ring R with the same operations as a ring R that also satisfies every axiom of the ring is called a subring, see [7]. To make it easier to prove a subring, there is a theorem of necessary and sufficient conditions for a subring as follows.

Theorem 1. [8], [12] *Let S be a non-empty set in a ring $(R,+,\cdot)$. The set S is a subring of R if and only if for all $s_1, s_2 \in S$ satisfies $s_1 - s_2 \in S$ and $s_1 \cdot s_2 \in S$.*

In the introduction, we have known the definition of derivation. Suppose a ring R has the derivation d. For any $c \in R$, if $d(c) = 0_R$, then c is called a constant element of d. The set containing the constant elements of d will further form a subring of R or commonly referred to as the ring of constant of d as explained in the following theorem.

Theorem 2. [9], [10] *Let R be a ring and $d: R \to R$ a derivation. We have the set of constants of d, denote by $R^d = \{r \in R | d(r) = 0_R\}$. The set R^d is a subring of R, we call it the ring of constant of d and $1_R \in R^d$.*

In the ring, we know the ideal concept which is a special subring of a ring. For example, ring R with the derivation d and ideal I. If I is an ideal in R and d is an inner derivation of R, then the derivation d satisfies $d(I) \subseteq I$. However, if d is not an inner derivation, then the derivation d does not necessarily satisfy to $d(I) \subseteq I$. Therefore, this problem raises the idea of defining an ideal I that satisfy $d(I) \subseteq I$ as follows.

Definition 1. [5], [11] Let a ring R with unity, ideal I in R, and a derivation d of R. The ideal I is called d-ideal if it satisfies $d(I) \subseteq I$.

After understanding some of the definitions and theorems needed in the discussion, the last step that will be taken is to make a conjecture and prove the conjecture to be a theorem or proposition.

3. RESULTS AND DISCUSSION

The derivation discussed in this paper is the derivation on rings, especially the derivation of the divisor ring, cartesian product ring, factor ring, and idempotent ring.

3.1 Derivation on The Divisor Ring

The quotient has been identified. Suppose a ring R with a multiplicatively closed set A has a derivation of d. From **Proposition 1**, we find that there is a divisor ring $R_A = \left\{\frac{r}{a} \middle| r \in R, a \in A\right\}$. In this ring, a mapping $\delta_A: R_A \to R_A$ can be defined, furthermore, this mapping is a derivation of the divisor ring.

Theorem 3. *Let R be a commutative ring with unity and A is a multiplicatively closed of R without zero element and zero divisors. If a map $d: R \to R$ is a derivation on ring R, then a map $\delta_A: R_A \to R_A$ with the definition $\delta_A\left(\frac{r}{a}\right) = \frac{d(r)\cdot a - r\cdot d(a)}{a^2}$ for all $\frac{r}{a} \in R_A$ is a derivation of the divisor ring R_A.*

Proof. From **Proposition 1**, we can define $R_A = \left\{\frac{r}{a} \middle| r \in R, a \in A\right\}$ as a commutative divisor ring with unity. We need to prove that δ_A is a derivation on the ring R_A. First, we will show that δ_A is well-defined and closed. For any $r_1, r_2, a_1, a_2 \in R$ with $r_1 = r_2, a_1 = a_2$, we have $r_1 \cdot a_2 = r_2 \cdot a_1 \Leftrightarrow \frac{r_1}{a_1} = \frac{r_2}{a_2}$ where $\frac{r_1}{a_1}, \frac{r_2}{a_2} \in R_A$. We can get

$$\begin{aligned}
\delta_A\left(\frac{r_1}{a_1}\right) &= \frac{d(r_1)\cdot a_1 - r_1\cdot d(a_1)}{a_1^2} && \text{[Definition of } \delta_A\text{]} \\
&= \frac{d(r_2)\cdot a_2 - r_2\cdot d(a_2)}{a_2^2} && \text{[Because } d \text{ is a mapping and } (R,+,\cdot) \text{ is a ring]} \\
&= \delta_A\left(\frac{r_2}{a_2}\right) && \text{[Definition of } \delta_A\text{]}
\end{aligned}$$

And for any $\frac{r}{a} \in R_A$, we have $\delta_A\left(\frac{r}{a}\right) = \frac{d(r)\cdot a - r\cdot d(a)}{a^2} \in R_A$ because $r \in R$, $a \in A \subset R$, a mapping d is a derivation of R, $(R,+,\cdot)$ is a ring, and $(A,\cdot)$ is multiplicatively closed. This means that δ_A is a mapping because it is well defined and closed.

Next, for any $\frac{r_1}{a_1}, \frac{r_2}{a_2} \in R_A$, we have

1. $$\begin{aligned}
\delta_A\left(\frac{r_1}{a_1} + \frac{r_2}{a_2}\right) &= \delta_A\left(\frac{r_1\cdot a_2 + r_2\cdot a_1}{a_1\cdot a_2}\right) \\
&= \frac{d(r_1\cdot a_2 + r_2\cdot a_1)\cdot a_1\cdot a_2 - (r_1\cdot a_2 + r_2\cdot a_1)\cdot d(a_1\cdot a_2)}{(a_1\cdot a_2)^2} \\
&= \frac{\big(d(r_1\cdot a_2) + d(r_2\cdot a_1)\big)\cdot a_1\cdot a_2 - r_1\cdot a_2\cdot d(a_1\cdot a_2) - r_2\cdot a_1\cdot d(a_1\cdot a_2)}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1\cdot a_2)\cdot a_1\cdot a_2 + d(r_2\cdot a_1)\cdot a_1\cdot a_2 - r_1\cdot a_2\cdot\big(d(a_1)\cdot a_2 + a_1\cdot d(a_2)\big) - r_2\cdot a_1\cdot\big(d(a_1)\cdot a_2 + a_1\cdot d(a_2)\big)}{a_1^2\cdot a_2^2} \\
&= \frac{\big(d(r_1)\cdot a_2 + r_1\cdot d(a_2)\big)\cdot a_1\cdot a_2 + \big(d(r_2)\cdot a_1 + r_2\cdot d(a_1)\big)\cdot a_1\cdot a_2 - r_1\cdot a_2\cdot\big(d(a_1)\cdot a_2 + a_1\cdot d(a_2)\big) - r_2\cdot a_1\cdot\big(d(a_1)\cdot a_2 + a_1\cdot d(a_2)\big)}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot a_1\cdot a_2^2 + r_1\cdot d(a_2)\cdot a_1\cdot a_2 + d(r_2)\cdot a_2\cdot a_1^2 + r_2\cdot d(a_1)\cdot a_1\cdot a_2 - r_1\cdot d(a_1)\cdot a_2^2 - r_1\cdot d(a_2)\cdot a_1\cdot a_2 - r_2\cdot d(a_1)\cdot a_1\cdot a_2 - r_2\cdot d(a_2)\cdot a_1^2}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot a_1\cdot a_2^2 - r_1\cdot d(a_1)\cdot a_2^2 + d(r_2)\cdot a_2\cdot a_1^2 - r_2\cdot d(a_2)\cdot a_1^2}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot a_1 - r_1\cdot d(a_1)}{a_1^2}\cdot\frac{a_2}{a_2}\cdot\frac{a_2}{a_2} + \frac{d(r_2)\cdot a_2 - r_2\cdot d(a_2)}{a_2^2}\cdot\frac{a_1}{a_1}\cdot\frac{a_1}{a_1} \\
&= \frac{d(r_1)\cdot a_1 - r_1\cdot d(a_1)}{a_1^2} + \frac{d(r_2)\cdot a_2 - r_2\cdot d(a_2)}{a_2^2} \\
&= \delta_A\left(\frac{r_1}{a_1}\right) + \delta_A\left(\frac{r_2}{a_2}\right)
\end{aligned}$$

2. $$\begin{aligned}
\delta_A\left(\frac{r_1}{a_1}\cdot\frac{r_2}{a_2}\right) &= \delta_A\left(\frac{r_1\cdot r_2}{a_1\cdot a_2}\right) \\
&= \frac{d(r_1\cdot r_2)\cdot a_1\cdot a_2 - d(a_1\cdot a_2)\cdot r_1\cdot r_2}{(a_1\cdot a_2)^2} \\
&= \frac{\big(d(r_1)\cdot r_2 + r_1\cdot d(r_2)\big)\cdot a_1\cdot a_2 - \big(d(a_1)\cdot a_2 + a_1\cdot d(a_2)\big)\cdot r_1\cdot r_2}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot r_2\cdot a_1\cdot a_2 + r_1\cdot d(r_2)\cdot a_1\cdot a_2 - d(a_1)\cdot a_2\cdot r_1\cdot r_2 - a_1\cdot d(a_2)\cdot r_1\cdot r_2}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot a_1\cdot r_2\cdot a_2 - r_1\cdot d(a_1)\cdot r_2\cdot a_2}{a_1^2\cdot a_2^2} + \frac{r_1\cdot a_1\cdot d(r_2)\cdot a_2 - r_1\cdot a_1\cdot r_2\cdot d(a_2)}{a_1^2\cdot a_2^2} \\
&= \frac{d(r_1)\cdot a_1 - r_1\cdot d(a_1)}{a_1^2}\cdot\frac{r_2}{a_2}\cdot\frac{a_2}{a_2} + \frac{r_1}{a_1}\cdot\frac{a_1}{a_1}\cdot\frac{d(r_2)\cdot a_2 - r_2\cdot d(a_2)}{a_2^2} \\
&= \frac{d(r_1)\cdot a_1 - r_1\cdot d(a_1)}{a_1^2}\cdot\frac{r_2}{a_2} + \frac{r_1}{a_1}\cdot\frac{d(r_2)\cdot a_2 - r_2\cdot d(a_2)}{a_2^2} \\
&= \delta_A\left(\frac{r_1}{a_1}\right)\cdot\frac{r_2}{a_2} + \frac{r_1}{a_1}\cdot\delta_A\left(\frac{r_2}{a_2}\right)
\end{aligned}$$

It is proved that the mapping $\delta_A: R_A \to R_A$ is a derivation of the divisor ring R_A. ■

From **Theorem 3**, we can define the derivation of any divisor ring. Here we give the following example of **Theorem 3**.

Example 1. Let $R = \mathbb{Z}[x]$ be a polynomial ring and $A = \mathbb{Z}[x]\backslash\{0\}$ be the set of multiplicatively closed. We can define a derivation on a polynomial ring, i.e., $d: \mathbb{Z}[x] \to \mathbb{Z}[x]$, for all $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$, $d\big(p(x)\big) = a_1 + 2\cdot a_2x + \cdots + n\cdot a_nx^{n-1}$. From **Proposition 1**, we can construct a commutative divisor ring $R_A = \left\{\frac{p(x)}{q(x)}\middle| p(x) \in R, q(x) \in A\right\}$. Then, from **Theorem 3**, we can define a derivation $\delta_A: R_A \to R_A$ for all $\frac{p(x)}{q(x)} \in R_A$ define $\delta_A\left(\frac{p(x)}{q(x)}\right) = \frac{d\big(p(x)\big)\cdot q(x) - p(x)\cdot d\big(q(x)\big)}{q(x)^2}$.

In the following theorem, we will explain the relationship between the ring of constant of d and the ring of constant of δ_A where the ring of constant of d is a subring of the ring of constant of δ_A. Further explanation is shown in the following theorem.

Theorem 4. *Let R be a commutative ring with unity, a derivation $d: R \to R$, and R_A be a divisor ring with derivation $\delta_A: R_A \to R_A$ with the definition $\delta_A\left(\frac{r}{a}\right) = \frac{d(r)\cdot a - r\cdot d(a)}{a^2}$ for all $\frac{r}{a} \in R_A$. If $R^d = \{r \in R | d(r) = 0_R\}$ is a ring of constant of d and $R_A^{\delta_A} = \left\{\frac{r}{a} \in R_A \middle| \delta_A\left(\frac{r}{a}\right) = 0_{R_A}\right\}$ is a ring of constant of δ_A, then ring R^d is a subring of ring $R_A^{\delta_A}$.*

Proof. We need to prove that the ring R^d is a subring of ring $R_A^{\delta_A}$. We can form a mapping $\alpha: R \to R_A$ with the definition $\alpha(r) = \frac{r}{1_R}$ for all $r \in R$. For all $r_1, r_2 \in R$, we obtain

1. $\alpha(r_1 + r_2) = \frac{r_1+r_2}{1_R} = \frac{r_1}{1_R} + \frac{r_2}{1_R} = \alpha(r_1) + \alpha(r_2)$
2. $\alpha(r_1 \cdot r_2) = \frac{r_1 \cdot r_2}{1_R} = \frac{r_1 \cdot r_2}{1_R \cdot 1_R} = \frac{r_1}{1_R} \cdot \frac{r_2}{1_R} = \alpha(r_1) \cdot \alpha(r_2)$

It can be concluded that α is a ring homomorphism. Now, we will show that α is a ring monomorphism. For any $r_1, r_2 \in R$ where $\alpha(r_1) = \alpha(r_2)$. From this, we can get

$$\alpha(r_1) = \alpha(r_2) \Longrightarrow \frac{r_1}{1_R} = \frac{r_2}{1_R} \Longrightarrow r_1 \cdot 1_R = r_2 \cdot 1_R \Longrightarrow r_1 = r_2$$

Now, we know that $r_1 = r_2$. Thus, we can say that α is a ring monomorphism and it is proved that ring R can be embedded into the R_A. Because of any ring monomorphism $\alpha: R \to R_A$, element $r \in R$ can be considered as element $\frac{r}{1_R} \in R_A$. We know that $R^d \subseteq R$ and $R_A^{\delta_A} \subseteq R_A$. So, it can be obtained that for all $c_1, c_2 \in R^d$

1. $c_1 - c_2 = \frac{c_1}{1_R} - \frac{c_2}{1_R} = \frac{c_1 - c_2}{1_R} \in R_A$

   Because R^d is the ring of constant of d, that means ring R^d is a subring of the ring R. It can be obtained from **Theorem 1** that $c_1 - c_2 \in R^d$ or $d(c_1 - c_2) = 0_R$. So, we can get

   $$\delta_A\left(\frac{c_1 - c_2}{1_R}\right) = \frac{d(c_1 - c_2) \cdot 1_R - (c_1 - c_2) \cdot d(1_R)}{1_R^2} = \frac{0_R \cdot 1_R - (c_1 - c_2) \cdot 0_R}{1_R} = \frac{0_R}{1_R} = 0_{R_A}$$

   It can be concluded that $\frac{c_1 - c_2}{1_R} \in R_A^{\delta_A}$
2. $c_1 \cdot c_2 = \frac{c_1}{1_R} \cdot \frac{c_2}{1_R} = \frac{c_1 \cdot c_2}{1_R} \in R_A$

   Because R^d is the ring of constant of d, that means ring R^d is a subring of the ring R. From **Theorem 1** we have $c_1 \cdot c_2 \in R^d$ or $d(c_1 \cdot c_2) = 0_R$. We can get that

   $$\delta_A\left(\frac{c_1 \cdot c_2}{1_R}\right) = \frac{d(c_1 \cdot c_2) \cdot 1_R - c_1 \cdot c_2 \cdot d(1_R)}{1_R^2} = \frac{0_R \cdot 1_R - c_1 \cdot c_2 \cdot 0_R}{1_R} = \frac{0_R}{1_R} = 0_{R_A}$$

   It can be concluded that $\frac{c_1 \cdot c_2}{1_R} \in R_A^{\delta_A}$

Because for all $c_1, c_2 \in R^d$ we get $c_1 - c_2, c_1 \cdot c_2 \in R^d$ and element $c_1 - c_2, c_1 \cdot c_2 \in R^d$ can be considered as element $\frac{c_1 - c_2}{1_R}, \frac{c_1 \cdot c_2}{1_R} \in R_A^{\delta_A}$, then ring R^d can be embedded into the $R_A^{\delta_A}$ or we can say that the ring R^d is a subring of ring $R_A^{\delta_A}$. ■

An example is given to clarify **Theorem 4**.

Example 2. Let a polynomial ring $R = \mathbb{Z}[x]$ and the set of multiplicatively closed $A = \mathbb{Z}[x] \backslash \{0\}$. We can define a derivation on a polynomial ring, i.e., $d: \mathbb{Z}[x] \to \mathbb{Z}[x]$ for all $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ define $d(p(x)) = a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^{n-1}$. Assume that the ring of constant of d is $\mathbb{Z}[x]^d = \{p(x) \in \mathbb{Z}[x] | d(p(x)) = 0\}$. Note that ring $\mathbb{Z}$ is subring of $\mathbb{Z}[x]$ and for all $p(x) = a_0 \in \mathbb{Z}$ where $p(x) = a_0 + 0x + \cdots + 0x^n$ we get $d(p(x)) = 0$ or $\mathbb{Z} \subseteq \mathbb{Z}[x]^d$. From the definition d, we have for any $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]^d$ where $d(p(x)) = 0 \Leftrightarrow a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^{n-1} = 0$. In $\mathbb{Z}[x]$, any non-empty set $B = \{x^n | n \in \mathbb{Z}^+\} = \{1, x, x^2, \ldots, x^n\} \subseteq \mathbb{Z}[x]$ such that for all $p(x) \in \mathbb{Z}[x]$ there is $a_i \in \mathbb{Z}$ and $x^i \in B$ such that $p(x) = \sum_{i=0}^{n} a_i x^i$ with $i \in \mathbb{N} \cup \{0\}$. Besides that, if for any $a_i \in \mathbb{Z}, x^i \in B$ for $i \in \mathbb{Z}^+$ holds $\sum_{i=0}^{n} a_i x^i = 0$, then the only value of a_i that satisfies is 0 or we can write $a_0 = a_1 = \cdots = a_n = 0$. That means B is a base of $\mathbb{Z}[x]$ and $\mathbb{Z}$-module $\mathbb{Z}[x]$ is a free module. Because $a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^n = 0$ and $\mathbb{Z}[x]$ is a free module, then $a_1 = 2 \cdot a_2 = \cdots = n \cdot a_n = 0$. We know that $\mathbb{Z}$ is a domain integral, so we have $a_1 = a_2 = \cdots = a_n = 0$. Substitute $a_1 = a_2 = \cdots = a_n = 0$ to $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ to get $p(x) = a_0$. Because $a_0 \in \mathbb{Z}$, then $\mathbb{Z}[x]^d \subseteq \mathbb{Z}$. In conclusion, the ring of constant of d is $\mathbb{Z}$. From **Proposition 1**, we can construct a commutative divisor ring $R_A = \left\{\frac{p(x)}{q(x)} \middle| p(x) \in \mathbb{Z}[x], q(x) \in A\right\}$. So, from **Theorem 3**, we can define a derivation $\delta: R_A \to R_A$ for all $\frac{p(x)}{q(x)} \in R_A$ as $\delta\left(\frac{p(x)}{q(x)}\right) = \frac{d(p(x)) \cdot q(x) - p(x) \cdot d(q(x))}{q(x)^2}$. Assume that the ring of constant of δ is $R_A^{\delta_A} = \left\{\frac{p(x)}{q(x)} \middle| \delta\left(\frac{p(x)}{q(x)}\right) = 0_{R_A}, p(x) \in \mathbb{Z}[x], q(x) \in \mathbb{Z}[x] \backslash \{0\}\right\}$. Based on **Theorem 4**, it can be obtained that the ring of constant of d, i.e., $\mathbb{Z}[x]^d$, is a subring of the ring of constant of δ, i.e., $R_A^{\delta_A}$.

3.2 Derivation on The Cartesian Product Ring

Furthermore, there is a ring that is obtained from the cartesian product of several other rings, which is called the cartesian product ring. In the following theorem, we will show a derivation on the cartesian product ring.

Theorem 5. *Let R_i be a commutative ring with unity for $i = 1,2,\dots,n$ and $(\prod_{i=1}^{n} R_i, +,\cdot)$ be a cartesian product ring where $i \in \mathbb{N}$. If a map $d_i: R_i \to R_i$ is a derivation on the ring R_i for $i = 1,2,\dots,n$, then a map $\prod_{i=1}^{n} d_i : \prod_{i=1}^{n} R_i \to \prod_{i=1}^{n} R_i$ with the definition $\prod_{i=1}^{n} d_i \left((r_1, r_2, \dots, r_n)\right) = \left(d_1(r_1), d_2(r_2), \dots, d_n(r_n)\right)$ for all $(r_1, r_2, \dots, r_n) \in \prod_{i=1}^{n} R_i$ is a derivation of the Cartesian product ring $\prod_{i=1}^{n} R_i$ where $i \in \mathbb{N}$.*

Proof. We have a ring R_i, cartesian product ring $\prod_{i=1}^{n} R_i$, and derivation on the ring R_i, *i.e.*, $d_i: R_i \to R_i$, for $i = 1,2,\dots,n$. It will be shown that $\prod_{i=1}^{n} d_i$ is a derivation of the cartesian product ring $\prod_{i=1}^{n} R_i$ where $i \in \mathbb{N}$. First, we will show that $\prod_{i=1}^{n} d_i$ is well-defined and closed. For all $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \prod_{i=1}^{n} R_i$ where $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$, that means $a_i = b_i$ for $i = 1,2,\dots,n$. We can get

$$\begin{aligned}\prod_{i=1}^{n} d_i (a_1, a_2, \dots, a_n) &= \left(d_1(a_1), d_2(a_2), \dots, d_n(a_n)\right) && \text{[Definition of } \textstyle\prod_{i=1}^{n} d_i\text{]}\\ &= \left(d_1(b_1), d_2(b_2), \dots, d_n(b_n)\right) && \text{[Because } d \text{ is a mapping]}\\ &= \prod_{i=1}^{n} d_i (b_1, b_2, \dots, b_n) && \text{[Definition of } \textstyle\prod_{i=1}^{n} d_i\text{]}\end{aligned}$$

And for all $(a_1, a_2, \dots, a_n) \in \prod_{i=1}^{n} R_i$, we have $\prod_{i=1}^{n} d_i (a_1, a_2, \dots, a_n) = \left(d_1(a_1), d_2(a_2), \dots, d_n(a_n)\right) \in \prod_{i=1}^{n} R_i$ because d_i is a derivation of R_i for $i = 1,2,\dots,n$. This means that $\prod_{i=1}^{n} d_i$ for $i = 1,2,\dots,n$ is a mapping because it is well-defined and closed.

Next, for all $x = (a_1, a_2, \dots, a_n), y = (b_1, b_2, \dots, b_n) \in \prod_{i=1}^{n} R_i$, we obtain

1. $$\begin{aligned}\prod_{i=1}^{n} d_i (x + y) &= \prod_{i=1}^{n} d_i (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n)\\ &= \prod_{i=1}^{n} d_i \left((a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)\right)\\ &= \left(d_1(a_1 + b_1), d_2(a_2 + b_2), \dots, d_n(a_n + b_n)\right)\\ &= \left(d_1(a_1) + d_1(b_1), d_2(a_2) + d_2(b_2), \dots, d_n(a_n) + d_n(b_n)\right)\\ &= \left(d_1(a_1), d_2(a_2), \dots, d_n(a_n)\right) + \left(d_1(b_1), d_2(b_2), \dots, d_n(b_n)\right)\\ &= \prod_{i=1}^{n} d_i \left((a_1, a_2, \dots, a_n)\right) + \prod_{i=1}^{n} d_i \left((b_1, b_2, \dots, b_n)\right)\\ &= \prod_{i=1}^{n} d_i (x) + \prod_{i=1}^{n} d_i (y)\end{aligned}$$
2. $$\begin{aligned}\prod_{i=1}^{n} d_i (x \cdot y) &= \prod_{i=1}^{n} d_i \left((a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n)\right)\\ &= \prod_{i=1}^{n} d_i \left((a_1 \cdot b_1, a_2 \cdot b_2, \dots, a_n \cdot b_n)\right)\\ &= \left(d_1(a_1 \cdot b_1), d_2(a_2 \cdot b_2), \dots, d_n(a_n \cdot b_n)\right)\\ &= \left(d_1(a_1) \cdot b_1 + a_1 \cdot d(b_1), d_2(a_2) \cdot b_2 + a_2 \cdot d(b_2), \dots, d_n(a_n) \cdot b_n + a_n \cdot d(b_n)\right)\\ &= \left(d_1(a_1) \cdot b_1, d_2(a_2) \cdot b_2, \dots, d_n(a_n) \cdot b_n\right) + \left(a_1 \cdot d_1(b_1), a_2 \cdot d(b_2), \dots, a_n \cdot d(b_n)\right)\\ &= \left(d_1(a_1), d_2(a_2), \dots, d_n(a_n)\right) \cdot (b_1, b_2, \dots, b_n) + (a_1, a_2, \dots, a_n) \cdot \left(d_1(b_1), d_2(b_2), \dots, d_n(b_n)\right)\\ &= \prod_{i=1}^{n} d_i \left((a_1, a_2, \dots, a_n)\right) \cdot (b_1, b_2, \dots, b_n) + (a_1, a_2, \dots, a_n) \cdot \prod_{i=1}^{n} d_i \left((b_1, b_2, \dots, b_n)\right)\\ &= \prod_{i=1}^{n} d_i (x) \cdot y + x \cdot \prod_{i=1}^{n} d_i (y)\end{aligned}$$

From the explanation above, it can be concluded that the mapping $\prod_{i=1}^{n} d_i : \prod_{i=1}^{n} R_i \to \prod_{i=1}^{n} R_i$ is a derivation on the cartesian product ring $\prod_{i=1}^{n} R_i$. ■

The following is an example of the application of **Theorem 5** to a cartesian product ring of two polynomial rings.

Example 3. Let a polynomial ring $(\mathbb{R}[x], +,\cdot)$ and the derivation $d_1: \mathbb{R}[x] \to \mathbb{R}[x]$ with the definition $d_1(p(x)) = a_1 + 2 \cdot a_2 x + \dots + n \cdot a_n x^{n-1}$ for all $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \in \mathbb{R}[x]$ define Then given a polynomial ring $(\mathbb{Z}[x], +,\cdot)$ and the derivation $d_2: \mathbb{Z}[x] \to \mathbb{Z}[x]$ with the definition $d_2(q(x)) = b_1 + 2 \cdot b_2 x + \dots + n \cdot b_n x^{n-1}$for all $q(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n \in \mathbb{Z}[x]$. It is possible to construct a cartesian product ring $\mathbb{R}[x] \times \mathbb{Z}[x] = \{(p(x), q(x)) | p(x) \in \mathbb{R}[x], q(x) \in \mathbb{Z}[x]\}$. From **Theorem 5**, we can form a mapping $d_1 \times d_2: \mathbb{R}[x] \times \mathbb{Z}[x] \to \mathbb{R}[x] \times \mathbb{Z}[x]$ for all $(f(x), g(x)) \in \mathbb{R}[x] \times \mathbb{Z}[x]$ define $d_1 \times d_2 \left((f(x), g(x))\right) = \left(d_1(f(x)), d_2(g(x))\right)$ which is the derivation on the cartesian product ring $\mathbb{R}[x] \times \mathbb{Z}[x]$.

3.3 Derivation on The Factor Ring

In this subsection, we will discuss the derivation on factor rings. The factor ring used here is a factor ring constructed from a ring and d-ideal. The properties of the derivation on the factor ring are described in the following theorem.

Theorem 6. *Let a ring R with unity, a map $d: R \to R$ is a derivation on ring R, and set $I \subseteq R$ is a d-ideal of R. A map $\bar{d}: R/I \to R/I$ with the definition $\bar{d}(a+I) = d(a) + I$ for all $a + I \in R/I$ is a derivation on the factor ring R/I.*

Proof. We will show that $\bar{d}$ is a derivation on the factor ring $(R/I, \bar{+}, \bar{\cdot})$. Because we have a ring R and I is a d-ideal in R, so we can construct a factor ring $R/I = \{\bar{r} = r + I | r \in R\}$. First, we will show that $\bar{d}$ is well-defined and closed. For any $\bar{a} = a + I, \bar{b} = b + I \in R/I$ where $\bar{a} = \bar{b} \Leftrightarrow a - b \in I$. That means $d(a-b) \in I$. So, we can get

$$\begin{aligned} \bar{d}(\bar{a}) &= \bar{d}(a+I) && [\text{Because } \bar{a} = a + I] \\ &= d(a) + I && [\text{Definition of } \bar{d}] \\ &= d(b) + I && [\text{Because } d(a) - d(b) = d(a-b) \in I] \\ &= \bar{d}(b+I) && [\text{Definition of } \bar{d}] \\ &= \bar{d}(\bar{b}) && [\text{Because } \bar{b} = b + I] \end{aligned}$$

And for any $\bar{a} = a + I \in R/I$, we have $\bar{d}(\bar{a}) = d(a) + I \in R/I$ because $a \in R$ and d is a derivation on R. This means that $\bar{d}$ is a mapping because it is well defined and closed.

Next, from the definition of $\bar{d}$ and for all $\bar{r_1}, \bar{r_2} \in R/I$ where $\bar{r_1} = r_1 + I$ and $\bar{r_2} = r_2 + I$, we obtain

1. $$\begin{aligned} \bar{d}(\bar{r_1} \bar{+} \bar{r_2}) &= \bar{d}(r_1 + I \bar{+} r_2 + I) \\ &= \bar{d}((r_1 + r_2) + I) \\ &= d(r_1 + r_2) + I \\ &= d(r_1) + d(r_2) + I \\ &= d(r_1) + I \bar{+} d(r_2) + I \\ &= \bar{d}(r_1 + I) \bar{+} \bar{d}(r_2 + I) \\ &= \bar{d}(\bar{r_1}) \bar{+} \bar{d}(\bar{r_2}) \end{aligned}$$
2. $$\begin{aligned} \bar{d}(\bar{r_1} \bar{\cdot} \bar{r_2}) &= \bar{d}((r_1 + I) \bar{\cdot} (r_2 + I)) \\ &= \bar{d}((r_1 \cdot r_2) + I) \\ &= d(r_1 \cdot r_2) + I \\ &= d(r_1) \cdot r_2 + r_1 \cdot d(r_2) + I \\ &= d(r_1) \cdot r_2 + I \bar{+} r_1 \cdot d(r_2) + I \\ &= (d(r_1) + I) \bar{\cdot} (r_2 + I) \bar{+} (r_1 + I) \bar{\cdot} (d(r_2) + I) \\ &= \bar{d}(r_1 + I) \bar{\cdot} (r_2 + I) \bar{+} (r_1 + I) \bar{\cdot} \bar{d}(r_2 + I) \\ &= \bar{d}(\bar{r_1}) \bar{\cdot} \bar{r_2} \bar{+} \bar{r_1} \bar{\cdot} \bar{d}(\bar{r_2}) \end{aligned}$$

From the description, it is clear that $\bar{d}$ is the derivation on the factor ring R/I. ■

The following is an example of applying the **Theorem 6**.

Example 4. Given a ring matrix $M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z} \right\}$ with addition and multiplication operations on matrix with the derivation $d: M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$ where for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$, $d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$. An ideal of $M_2(\mathbb{Z})$, namely $I = \left\{ \begin{bmatrix} e & f \\ g & h \end{bmatrix} \middle| e, f, g, h \in 2\mathbb{Z} \right\}$. A factor ring $M_2(\mathbb{Z})/I$ can be formed with the definition of addition operation " $\bar{+}$ " and multiplication operation " $\bar{\cdot}$ " where for each $A + I, B + I \in M_2(\mathbb{Z})$, define $(A + I) \bar{+} (B + I) = (A + B) + I$ and $(A + I) \bar{\cdot} (B + I) = (A \cdot B) + I$. We can define a function $\bar{d}: M_2(\mathbb{Z}) \to M_2(\mathbb{Z})$ where $\bar{d}(A + I) = d(A) + I$ for all $A + I \in M_2(\mathbb{Z})/I$. From **Theorem 6**, a mapping $\bar{d}$ is a derivation on the factor ring $M_2(\mathbb{Z})/I$ because for all $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I \in M_2(\mathbb{Z})/I$ we obtain

1. $$\begin{aligned} \bar{d}\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I \bar{+} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right) &= \bar{d}\left(\begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} + I\right) \\ &= d\left(\begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}\right) + I \end{aligned}$$

$$= \begin{bmatrix} 0 & -a_2 - b_2 \\ -a_3 - b_3 & 0 \end{bmatrix} + I$$
$$= \begin{bmatrix} 0 & -a_2 \\ -a_3 & 0 \end{bmatrix} + I \bar{+} \begin{bmatrix} 0 & -b_2 \\ -b_3 & 0 \end{bmatrix} + I$$
$$= d\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) + I \bar{+} d\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right) + I$$
$$= \bar{d}\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I\right) \bar{+} \bar{d}\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right)$$

2. $$\bar{d}\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I \bar{\cdot} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right) = \bar{d}\left(\begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix} + I\right)$$
$$= d\left(\begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix}\right) + I$$
$$= \begin{bmatrix} 0 & -a_1b_2 - a_2b_4 \\ -a_3b_1 - a_4b_3 & 0 \end{bmatrix} + I$$
$$= \left(\begin{bmatrix} 0 & -a_2 \\ -a_3 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} 0 & -b_2 \\ -b_3 & 0 \end{bmatrix}\right) + I$$
$$= \begin{bmatrix} 0 & -a_2 \\ -a_3 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I \bar{+} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} 0 & -b_2 \\ -b_3 & 0 \end{bmatrix} + I$$
$$= \left(\begin{bmatrix} 0 & -a_2 \\ -a_3 & 0 \end{bmatrix} + I \bar{\cdot} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right) \bar{+} \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I \bar{\cdot} \begin{bmatrix} 0 & -b_2 \\ -b_3 & 0 \end{bmatrix} + I\right)$$
$$= \left(\bar{d}\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I\right) \bar{\cdot} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right) \bar{+} \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + I \bar{\cdot} \bar{d}\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + I\right)\right)$$

Theorem 7. *Let a ring R with unity, a derivation on ring R with definition $d: R \to R, (\forall r \in R) r \mapsto d(r)$, and the ideal $I \subseteq R$. In the factor ring R/I, we can construct a derivation $\bar{d}: R/I \to R/I$ with the definition $\bar{d}(a + I) = d(a) + I$ for all $a + I \in R/I$. If $R^d = \{r \in R | d(r) = 0_R\}$ is a ring of constant of d and $I \subseteq R^d$, then factor ring $R^d/I = \{a + I \in R/I | a \in R^d\}$ is a subring of ring of constant of $\bar{d}$.*

Proof. We can show that if ring $R^d = \{r \in R | d(r) = 0_R\}$ is a ring of constant of the derivation d and $I \subseteq R^d$, then factor ring $R^d/I = \{a + I \in R/I | a \in R^d\}$ is a subring of ring of constant of the derivation $\bar{d}$. Assume that the set $(R/I)^{\bar{d}} = \{r + I \in R/I | \bar{d}(r + I) = 0_R + I\}$ is a ring of constant of the derivation $\bar{d}$. We have the fact that the set I is ideal of R. Because of $I \subseteq R^d$ and R^d is a ring, then I is an ideal of R^d too. So, we can construct the set $R^d/I = \{a + I \in R/I | a \in R^d\}$. First, for all $a + I \in R^d/I \subseteq R/I$ we know $a + I \in R/I$. Because we have $\bar{d}$ as a derivation of factor ring R/I, we get $\bar{d}(a + I) = d(a) + I \in R/I$. Because element $a \in R^d$ and ring R^d is a ring of constant of the derivation d, so $\bar{d}(a + I) = d(a) + I = 0_R + I$. It's mean that $a + I \in (R/I)^{\bar{d}}$ or $R^d/I \subseteq (R/I)^{\bar{d}}$. And then, for all $a + I, b + I \in R^d/I \subseteq R/I$ we get $a + I \bar{+} (-b) + I = (a - b) + I \in R^d/I$ and $(a + I) \bar{\cdot} (b + I) = a \cdot b + I \in R^d/I$ because $a, b \in R^d$ and R^d is a ring of constant of the derivation d. Based on **Theorem 1**, it can be concluded that R^d/I is a subring of the ring of constant of the derivation $\bar{d}$, i.e. $(R/I)^{\bar{d}}$. ■

4. CONCLUSIONS

A commutative ring R with unit elements along with a set A is provided with a derivation $d: R \to R$ will have a derivation on the divisor ring defined by $\delta_A: R_A \to R_A$, *for all* $\frac{r}{a} \in R_A, \delta\left(\frac{r}{a}\right) = \frac{d(r) \cdot a - r \cdot d(a)}{a^2}$ with $R_A = \left\{\frac{r}{a} \middle| r \in R, a \in A\right\}$. The derivation of the divisor ring also has several properties such as if $R^d = \{r \in R | d(r) = 0_R\}$ is a subring of R or as a ring constant of d and $R_A^{\delta_A} = \left\{\frac{r}{a} \in R_A \middle| \delta_A\left(\frac{r}{a}\right) = 0_{R_A}\right\}$ is a subring of R_A or as a ring of constant of δ_A, then ring R^d is a subring of ring $R_A^{\delta_A}$. It is also possible to construct a ring derivation on the cartesian product ring by definition $\prod_{i=1}^n d_i : \prod_{i=1}^n R_i \to \prod_{i=1}^n R_i$ for all $(r_1, r_2, \dots, r_n) \in \prod_{i=1}^n R_i$, $\prod_{i=1}^n d_i\left((r_1, r_2, \dots, r_n)\right) = \left(d_1(r_1), d_2(r_2), \dots, d_n(r_n)\right)$ and derivation on factor ring by definition $\bar{d}: R/I \to R/I$ for all $a + I \in R/I, \bar{d}(a + I) = d(a) + I$.

REFERENCES

[1] V. E. Tarasov, "Leibniz Rule and Fractional Derivatives of Power Functions," *J. Comput. Nonlinear Dyn.*, vol. 11, no. 3, pp. 1–4, 2016, doi: 10.1115/1.4031364.

[2] E. A. Behrens and C. Reis, *Ring theory*. Academic Press, 1972.

[3] M. Ashraf, S. Ali, and C. Haetinger, "On Derivations in Rings and Their Applications," *The Aligarh Bull. of Maths.*, vol. 25, no. 2, 2006.

[4] M. Bronstein, "Symbolic integration I: Transcendental functions," *Comput. Math. with Appl.*, vol. 33, no. 7, 1997, doi: 10.1016/s0898-1221(97)84595-6.

[5] C. R. Hajarnavis, "An Introduction to Noncommutative Noetherian Rings," *Bull. London Math. Soc.*, vol. 23, no. 1, 1991, doi: 10.1112/blms/23.1.91.

[6] I. Ernanto, "Sifat-sifat Ring Faktor yang Dilengkapi Derivasi," *J. Fundam. Math. Appl.*, vol. 1, no. 1, 2018, doi: 10.14710/jfma.v1i1.3.

[7] S. Wahyuni, I. E. Wijayanti, D. A. Yuwaningsih, and A. D. Hartanto, *Teori ring dan modul*. UGM PRESS, 2021.

[8] I. K. Wolf and L. J. Goldstein, "Abstract Algebra: A First Course.," *Am. Math. Mon.*, vol. 82, no. 8, 1975, doi: 10.2307/2319828.

[9] A. Nowicki and J. M. Strelcyn, "Genesrators of rings of constants for some diagonal derivations in polynomial rings," *J. Pure Appl. Algebr.*, vol. 101, no. 2, 1995, doi: 10.1016/0022-4049(94)00011-7.

[10] P. Jedrzejewicz, "Rings of constants of polynomial derivations and p-bases," in *Analytic and Algebraic Geometry*, 2013. doi: 10.18778/7969-017-6.06.

[11] M. R. Helmi, H. Marubayashi, and A. Ueda, "Differential polynomial rings which are generalized Asano prime rings," *Indian J. Pure Appl. Math.*, vol. 44, no. 5, 2013, doi: 10.1007/s13226-013-0035-6.

[12] Rasiman, M. R. Rubowo, and A. S. Pramasdyahsari, "Teori Ring," *Maret*, 2018.