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THE NUMERICAL APPROXIMATION OF STATIONARY WAVE SOLUTIONS FOR TWO-COMPONENT SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS BY USING GENERALIZATION PETVIASHVILI METHOD

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ABSTRACT

The Petviashvili method is a numerical method for obtaining fundamental solitary wave solutions of stationary scalar nonlinear wave equations with power-law nonlinearity: $Mu + u^p = 0$, where M is a positive definite and self-adjoint operator and p is constant. Due to the case being a system of solitary nonlinear wave equations, we generalize the Petviashvili method. We apply this generalized method for a two-component system of Nonlinear Schrödinger Equations (NLSE) for 2-D.

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1. INTRODUCTION

Nonlinear waves and vortices are often described by partial differential equations, whose solutions cannot be found analytically even in a space of one dimension. As it is clear to all, solitons play a crucial role in diverse areas of scientific disciplines such as plasma physics, quantum electronics, and nonlinear optics, as they are valuable tools for understanding the dynamics of nonlinear phenomena modeled [1]. The Korteweg-de Vries (KdV) equation is one of the nonlinear partial differential equations that has a key role in wave physics and many other disciplines [2]. For most nonlinear wave equations arising in physical applications, their solitary wave solutions can be obtained only numerically. Numerical computations are used to approximate various solutions, including stationary solutions [3]. Some numerical methods can be applied to find the solutions of nonlinear wave equations, but most of the recent studies focus on the so-called imaginary-time evolution method (ITEM), also referred to as the normalized gradient flow method (see in [4], [5], [6], [7], [8]). However, ITEM cannot specify the propagation constant μ of solitary wave and is instead computed using the available approximation to the stationary solution at each iteration. One numerical method that can be used to seek a solitary wave with a specified propagation constant rather than with a specified power is called the Newton method [8]. The numerical method used to determine the stationary waves of the NLS equation is the Petviashvili iteration method. One example of it is the NLS equation generated by the theory of Bose-Einstein condensation. However, the equation of this theory is the NLS equation with potential function. The existence of a potential function causes modification in the Petviashvili iteration method [9]. The modified method is known to be very fast and also to be able to converge to both fundamental and excited-state solitary waves [10]. On the other hand, there is an effective numerical for computing solitary wave solutions quickly in a space of two dimensions proposed by V. I. Petviashvili in the context of the Kadomtsev-Petviashvili equation with positive dispersion in 1976 [11]. Petviasvili's numerical method was applied to numerous nonlinear problems in modern mathematical physics to find the solitary waves [12].

Petviashvili's method is one of the most popular schemes to obtain solitary waves as the solutions of nonlinear stationary wave equations [11]. This method is well suited to handle a large class of equations and can easily be adapted to further constraints and components. It has some advantages such as the elegant way to find the stationary solutions for nonlinear solitary waves. The algorithm is not complicated and can be implemented easily. Furthermore, this method has quick convergence and good accuracy, which is the smallest error limit 10^{-10} [13]. Several recent interest research communities study the application of Bose-Einstein condensation by using the Petviashvili method. However, they applied the Petviashvili method to find a stationary wave solution of nonlinear solitary waves for single equation. Then, the purpose of this research is to obtain the stationary wave solution for system equations. We walk on the specific equation, namely, a two-component system of Nonlinear Schrödinger Equation (NLSE).

This research is organized as follows. First, we recast the original Petviashvili method into an equivalent form. All subsequent analyses will be carried out for that equivalent formulation of the Petviashvili method. We give the generalization for single and system wave equations in the second part and also show the convergence of this scheme. Third, presents the numerical computation results for 2-D and its discussion. There is a two-component system of nonlinear Schrödinger equations and its system contains a small positive parameter value (ε). Finally, we have the concluding remarks.

2. RESEARCH METHODS

In this part, we explain about NLSE that given by

$$\begin{cases} iU_t + \nabla^2 U + \mu_1 |U|^{p-1}U + \beta |V|^2 U = 0 \\ iV_t + \nabla^2 V + \mu_2 |V|^{q-1}V + \beta |U|^2 V = 0 \\ x \in \mathbb{R}^n, t > 0 \\ U = U(x,t), V = V(x,t) \in \mathbb{C} \\ U(x,t), V(x,t) \to 0, as\ |x| \to \infty, t > 0 \end{cases} \quad (1)$$

where $\mu_j > 0$ are positive constants, $n \leq 3$, and $\beta \in \mathbb{R}$ is a coupling constant, $p, q > 2$. The system above has applications in many physical problems, especially in nonlinear optics. Physically, the solution (U, V)

denotes the two-component beam in Kerr-like photorefractive media [1]. The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β is the interaction between two components of the beam. As $\beta > 0$, the interaction is attractive, but the interaction is repulsive if $\beta < 0$. To obtain solitary wave solutions of the system above, we may set the system of solitary wave solutions [14] for system equations as follows:

$$\begin{cases} U(x,t) = e^{i\lambda_1 t}u(x) \\ V(x,t) = e^{i\lambda_2 t}v(x) \end{cases} \tag{2}$$

Then, we may transform the system to steady-state two coupled nonlinear Schrodinger equations that are given by

$$\begin{cases} \nabla^2 u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \in \mathbb{R}^n \\ \nabla^2 v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \in \mathbb{R}^n \\ u, v > 0 \in \mathbb{R}^n, u, v(x) \to 0, as\ |x| \to +\infty. \end{cases} \tag{3}$$

The existence of ground state (i.e. least energy) solutions of the system may depend on the coupling constant β. When β is positive but sufficiently small, the system has a ground-state solution (u, v). On the other hand, as β becomes negative, there is no ground-state solution to it. In this research, we show the existence of solitary wave solutions if β is negative and $|\beta|$ is positive and small enough.

In this part, we first remodel the Petviashvili method into a different form, but the form is still equivalent to the original method, yielding the Petviashvili Iteration Method [11]. It is said to be equivalent because to reformulate its scheme, it is needed to apply the Taylor expansion to take the linear part and needs some algebraic manipulations just to make a different form. The reformulation of the original Petviashvili method will be discussed in detail for our generalization of the Pethviashvili method in this section.

2.1 Generalized Petviashvili Method for Single Equation

Before we show the details of a generalized Peviashvili method we would like to introduce why the single scheme of this method needs to be reconstructed to be equivalent form over the original scheme. We have two underlying reasons of **Equation (13)** below is preferred over the original Petviashvili method for the subsequent generalization of the method. First, the generalization Petviashvili method can select the value of the new parameter $\Delta\tau$ to control its convergence. Second, it is free from difficulties that there exist in the original Petviashvili method (i.e, if $\frac{\langle u_n, u_n^p \rangle}{\langle u_n, Mu_n \rangle} < 0$) in the original scheme, then it cannot be raised to non-integer power unless u_{n+1} is complex-valued. Due to the form in the generalized Petviashvili method is linear, it does not have the parentheses part power γ that is mentioned in the original Petviashvili scheme.

The first point of a relation that will be crucial thing for this case is the eigenvalue problem. It has eigenvalue-eigenfunction pair. The form of the following is obtained by linearizing the fundamental solitary wave solutions of stationary scalar nonlinear wave equations with power-law nonlinearity $Mu + u^p = 0$, where M is a positive definite and self-adjoint operator and p is a constant.

$$Lu = (p-1)Mu \tag{4}$$

or, similarly by

$$M^{-1}Lu = (p-1)u \tag{5}$$

Hence, we have u is an eigenfunction of the operator $M^{-1}L$ and $p-1$ is eigenfunction of $M^{-1}L$. Since our process always needs the eigenvalue-eigenfunction pair, then **Equation (5)** is the key relation of this problem. Based on **Equation (5)**, we describe a reason why we need a generalization for this method. We will walk on the system equations and we want to obtain the stationary wave solution for nonlinear solitary waves of two-component system with self-focusing cubic of the nonlinear Schrödinger equation by using the Petviashvili method. However, there are some problems. First, the eigenvalue cannot be found directly. Second, the case is system equations. The last problem is we consider several parameters such as coupling constant parameter β which is located in the interaction of two-component u and v. The coupling constant β is the interaction between two beam components and it influences the behavior of the wave solutions. The

important thing is to find the convergence solutions for this problem by having a suitable definite positive and self-adjoint operator. Thus, for the two-component equation for single equation (first equation of System **Equation (3)**) we have a coupling constant and two-component u and v in one term, then we have a linearized operator as follows

$$L = -M + 3\mu u^2 + \beta v^2 \tag{6}$$

Then,

$$Lu = -Mu + 3\mu u^3 + \beta u v^2 = 2(M - \beta v^2)u \neq cont \cdot Mu$$

The equation above is the main problem of this case. As a solution, we should find a suitable positive definite and self-adjoint operator to process the algorithm. Suppose that M in **Equation (5)** is given by a positive definite and self-adjoint operator. Since we do not know whether $(M - \beta v^2)$ is invertible or not, we should approximate it by replacing it with the simplest ansatz. Thus **Equation (4)** can be approached by using

$$Lu \approx \alpha N u \tag{7}$$

where N is

$$N = c - \nabla^2 \tag{8}$$

Here c is to be determined from the condition that the "vector" Nu_n be "aligned along" "vector" Lu_n as closely as possible and it is obtained numerically. Therefore, we require that

$$\frac{\langle Nu_n, Lu_n\rangle^2}{\langle Nu_n, Nu_n\rangle\langle Lu_n, Lu_n\rangle} = maximum \tag{9}$$

Differentiating the left-hand solutions (l.h.s) of the above condition with respect to c and setting the result to zero, one obtains

$$\frac{\langle N_c u_n, Lu_n\rangle}{\langle N_c u_n, Nu_n\rangle} = \frac{\langle Nu_n, Lu_n\rangle}{\langle Nu_n, Nu_n\rangle} \tag{10}$$

where $N_c \equiv \frac{\partial N}{\partial x} = 1$. Then, it yields the value for c at the n-th iteration:

$$c_n = \frac{\langle u_n, Lu_n\rangle\langle \nabla^2 u_n, \nabla^2 u_n\rangle - \langle \nabla^2 u_n, Lu_n\rangle\langle u_n, \nabla^2 u_n\rangle}{\langle u_n, Lu_n\rangle\langle u_n, \nabla^2 u_n\rangle - \langle \nabla^2 u_n, Lu_n\rangle\langle u_n, u_n\rangle} \tag{11}$$

It is straightforward to verify that for equations with power-law nonlinearity with M, **Equation (10)** yields $c = \mu$ and hence $N = M$.

Now that N has been determined from **Equation (7)** and **Equation (10)**, the approximate eigenvalue α in **Equation (7)** can be found from

$$\alpha_n = \frac{\langle u_n, Lu_n\rangle}{\langle u_n, Nu_n\rangle} \tag{12}$$

Refer to the explanation above, we then construct the following counterpart of the algorithm that is described in detail below by **Equation (13)**

$$u_{n+1} - u_n = \left(N^{-1}(L_1 u)_n - \gamma \frac{\langle u_n, (Lu)_n\rangle}{\langle u_n, Mu_n\rangle} u_n\right)\Delta\tau \tag{13}$$

We have the optimum γ as the following

$$\gamma = 1 + \frac{1}{\alpha\Delta\tau} \tag{14}$$

In **Equation (13)**, $L_1 u = -Nu_n + u_n^p$. Thus, **Equation (8)**, **Equation (11)**, **Equation (12),** and **Equation (13)** supply all necessary information for the application of the generalized Petviashvili method.

Furthermore, we want to describe the details of how to change the original algorithm Petviashvili Iteration Method **[12]** into a different form, said **Equation (13)**. We begin by denoting the stationary equation whose solitary wave is what we want to find. The scheme for single equation is a famous method to solve the nonlinear solitary wave, but rarely for the system equation. As the challenge, in this article, we would like to construct the single scheme to be a system scheme and apply it to the two-component system of NLSE. For the basic, as background, we show the details process to recast the scheme for single equation. In the first part, we introduce the notation below

$$L_0 u = -M + u^{p-1} = 0 \tag{15}$$

where M is a definite positive and self-adjoint operator and u is the exact solitary wave. By applying some algebraic manipulations, then we obtain

$$\begin{aligned} u_{n+1} - u_n &= M^{-1}u_n^p \left(\frac{\langle u_n, u_n^p\rangle}{\langle u_n, Mu_n\rangle}\right)^{-\gamma} - u_n \\ &= \left(u_n - M^{-1}Mu_n + M^{-1}u_n^p\right)\left(\frac{\langle u_n, Mu_n\rangle + \langle u_n, -Mu_n + u_n^p\rangle}{\langle u_n, Mu_n\rangle}\right)^{-\gamma} - u_n \\ &= (u_n - M^{-1}(L_0u)_n)\left(1 + \frac{\langle u_n, (L_0u)_n\rangle}{\langle u_n, Mu_n\rangle}\right)^{-\gamma} - u_n \end{aligned} \tag{16}$$

where

$$(L_0u)_n \equiv -Mu_n + u_n^p$$

A good result of numerical approximation is obtained if we can find the numerical solution near the exact solution. We hope the above scheme is close to the exact solution u, then, we may linearize by substituting **Equation (7)** into **Equation (16)** and the result is in the following

$$u_n = u + \tilde{u}_n, \|\tilde{u}_n\| \ll |u| \tag{17}$$

We apply the Taylor expansion and we just consider the term of order $O(\tilde{u}_n)$. In this step, we obtain the left-hand side is

$$\begin{aligned} u_{n+1} - u_n &= u + \tilde{u}_{n+1} - u - \tilde{u}_n \\ &= \tilde{u}_{n+1} - \tilde{u}_n \end{aligned}$$

Here we should substitute **Equation (17)** into $(L_0u)_n = -Mu_n + u_n^p$ to obtain the linearized term for r.h.s. Here we choose $p = 3$. Hence, we obtain

$$\begin{aligned} (L_0u)_n &= -M(u + \tilde{u}_n) + (u + \tilde{u}_n)^3 = -Mu + u^3 + \tilde{u}_n(-M + 3\tilde{u}^2) \\ &= L_0u + L\tilde{u} \end{aligned}$$

we need the information above to linearize the r.h.s. of **Equation (6)**. Thus, we obtain the right-hand side of the following

$$\left(M^{-1}L\tilde{u}_n - \gamma\frac{\langle u_n, L\tilde{u}_n\rangle}{\langle u_n, Mu\rangle}u\right)\Delta\tau$$

or for both sides, we can rewrite it to be

$$u_{n+1} + \tilde{u}_n = \left(M^{-1}L\tilde{u}_n - \gamma\frac{\langle u_n, L\tilde{u}_n\rangle}{\langle u_n, Mu\rangle}u\right)\Delta\tau \tag{18}$$

with the optimum value of $\Delta\tau$ on **Equation (20)**

$$\Delta\tau = 1 \tag{19}$$

where L is the linearized operator of **Equation (15)**

$$L\tilde{u}_n \equiv (-M + pu^{p-1})\tilde{u}_n \tag{20}$$

Refer to **[10]**, **Equation (18)** is interpreted as the explicit Euler discretization of the following continuous linear flow

$$\tilde{u}_\tau = M^{-1}L\tilde{u} - \gamma\frac{\langle u, L\tilde{u}\rangle}{\langle u_n, Mu\rangle}u \tag{21}$$

where τ is the auxiliary (nonphysical) ”time” variable. According to **[10]** we may ”delinearize” the above continuous flow to be

$$\tilde{u}_\tau = M^{-1}L_0\tilde{u} - \gamma\frac{\langle \bar{u}, L\bar{u}\rangle}{\langle \bar{u}, M\bar{u}\rangle}\bar{u} \tag{22}$$

Since u depends on τ, then u is namely the ”current” approximation to the exact solitary wave u. That is, if one linearizes **Equation (22)** via a continuous analog of **Equation (17)**, one will obtain **Equation (21)** **[10]**. Therefore, we need to discretize **Equation (22)** in time using the explicit Euler method and it is a generalized of the original Petviashvili method. Its scheme is shown in the following

$$u_{n+1} + u_n = \left(M^{-1}(L_0u)_n - \gamma\frac{\langle u_n, (L_0u)_n\rangle}{\langle u_n, Mu\rangle}u\right)\Delta\tau \tag{23}$$

Equation (23) with $\Delta\tau = 1$ is equivalent to the original Petviashvili algorithm for single equation. Indeed, **Equation (8)** still can converge to the solitary wave u without $\Delta\tau$ parameter. However, we put this parameter into the scheme because it can make the solutions converge quickly. As a reminder of this research, **Equation (23)** is also equivalent to the Petviashvili method when given $\Delta\tau \approx 1$. Moreover, we should replace the definite positive and self-adjoint operator M by the simplest ansatz N to obtain the approximate eigenvalue-eigenfunction pair first, see **Equation (13)**. **Equation (23)** represents the generalized Petviashvili method for single and acts as clues to develop the generalized scheme for system equations proposed in this chapter, which is one of two main results of this research and will be based on this reformulated version of the original algorithm.

2.2 Generalized Petviashvili Method for System Equations

This section contains the first main result of this study. Namely, we will show how the Petviashvili method can be generalized for a system equation of the form

$$L_0\begin{pmatrix}u\\v\end{pmatrix} \equiv -M\begin{pmatrix}u\\v\end{pmatrix} + \mu_j\begin{pmatrix}u^3\\v^3\end{pmatrix} + \beta\begin{pmatrix}uv^2\\vu^2\end{pmatrix} = 0, \qquad j = 1,2 \tag{24}$$

where $\mu_j > 0$’s are positive constants, $n \leq 3$, and β is a coupling constant. Next, we will simulate it in Chapter 3.

Recall that one of the key results of this section is **Equation (4)**. We will seek to obtain a counterpart of **Equation (4)** for **Equation (24)**. In the aforementioned physical applications, M is given. The linearized operator L in this case is

$$L = -M\begin{pmatrix}u\\v\end{pmatrix} + 3\mu_j\begin{pmatrix}u^2\\v^2\end{pmatrix} + \beta\begin{pmatrix}v^2\\u^2\end{pmatrix}, \qquad j = 1,2 \tag{25}$$

and hence

$$L\begin{pmatrix}u\\v\end{pmatrix} = -M\begin{pmatrix}u\\v\end{pmatrix} + 3\mu_j\begin{pmatrix}u^3\\v^3\end{pmatrix} + \beta\begin{pmatrix}uv^2\\vu^2\end{pmatrix} \tag{26}$$

$$= 2\left(M - \beta\begin{pmatrix} v^2 & 0 \\ 0 & u^2 \end{pmatrix}\right)\begin{pmatrix} u \\ v \end{pmatrix} \neq const.\, M\begin{pmatrix} u \\ v \end{pmatrix} \tag{27}$$

Thus, an exact counterpart for a general stationary wave in **Equation (25)** cannot be obtained. We don't know

$$\left(M - \beta\begin{pmatrix} v^2 & 0 \\ 0 & u^2 \end{pmatrix}\right)$$

is invertible or not. As a solution to the above problem, we propose to seek such a positive definite and self-adjoint operator N that the counterpart of **Equation (24)** would approximately hold:

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} N\begin{pmatrix} u \\ v \end{pmatrix} \tag{28}$$

Here both N and the constants α_1, α_2 remain to be determined. Given such, α_1, and α_2, then we construct the counterpart of the algorithm for the system equation. For the easy way, we follow **Equation (18)** and **Equation (19)** to construct the generalization for the system scheme.

For the system, we show how the operator N and constants α_1, α_2 in **Equation (28)** can be determined efficiently. It should be noted that we cannot give the most general recipe in this regard, simply because there are infinitely many possibilities here, as it will become clear as we proceed. Instead, we will consider in detail only one typical case that arises in many applications and will show how N can be found for it. At the end of this subsection, we will also briefly comment on another example of finding N.

This research aims to obtain the solitary wave solution of the system of nonlinear solitary wave equations for a two-component system of NLSE. We should construct the generalization of the Petviasvili method first to solve the case of two-component system of nonlinear Schrödinger equations. Let us construct the algorithm for system equations and find out the optimal value γ. One substitutes the following decomposition of $\tilde{u}_n$ and $\tilde{v}_n$ into **Equation (18)**:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}_n (\mathbf{x}) = \begin{pmatrix} \alpha_n & 0 \\ 0 & b_n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(\mathbf{x}) + \begin{pmatrix} z \\ s \end{pmatrix}_n (\mathbf{x}) \tag{29}$$

where a_n and b_n are scalars (i.e., not a function of **x**), z_n and s_n are chosen to be orthogonal to N_u and N_v at every iteration:

$$\left\langle \begin{pmatrix} z \\ s \end{pmatrix}_n, M\begin{pmatrix} u \\ v \end{pmatrix}_n \right\rangle = 0 \; or \; \left\langle M\begin{pmatrix} z \\ s \end{pmatrix}_n, \begin{pmatrix} u \\ v \end{pmatrix}_n \right\rangle = 0, \forall n \tag{30}$$

We know that is the key relation, which is to establish its counterpart for a more general equation of **Equation (24).** It will correspondingly be one of the key steps in the generalization of the Petviashvili method. Since we have used the fact that the linearized operator L is self-adjoint operator, then note that from **Equation (18)** there follow the orthogonality relations

$$\left\langle \begin{pmatrix} z \\ s \end{pmatrix}_n, L\begin{pmatrix} u \\ v \end{pmatrix}_n \right\rangle = 0 \; or \; \left\langle L\begin{pmatrix} z \\ s \end{pmatrix}_n, \begin{pmatrix} u \\ v \end{pmatrix}_n \right\rangle = 0, \forall n \tag{31}$$

We continue with the analysis of the evolution of the error $\tilde{u}_n$ with n. Substituting decomposition **Equation (37)** into **Equation (18)**, using relation **Equation (4)** and the second of the orthogonality conditions **Equation (31)**. However, we know that we cannot find the eigenvalues directly in this case, so we should replace $p - 1$ and $q - 1$ by $\alpha_1$and α_2 respectively, where α_1 and α_2 are the approximation of eigenvalues. Then, one obtains:

$$\begin{pmatrix} a_{n+1} - a_n & 0 \\ 0 & b_{n+1} - b_n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} z_{n+1} - z_n & 0 \\ 0 & s_{n+1} - s_n \end{pmatrix} = M^{-1}L\begin{pmatrix} z_{n+1} - z_n & 0 \\ 0 & s_{n+1} - s_n \end{pmatrix}\Delta\tau$$
$$+\begin{pmatrix} a_{n+1} - a_n & 0 \\ 0 & b_{n+1} - b_n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\left(1 - \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\right)\Delta\tau \tag{32}$$

Then, taking the inner product of this equation with Mu and Mv and using the orthogonality conditions **Equation (30)** and **Equation (31)**, one gets

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \left(1 + \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \left(1 - \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right) \right) \Delta\tau \tag{33}$$

Then when

$$\gamma_1 = 1 + \frac{1}{\alpha_1 \Delta\tau} \tag{34}$$

$$\gamma_2 = 1 + \frac{1}{\alpha_2 \Delta\tau} \tag{35}$$

$a_{n+1} = b_{n+1} = 0$, i.e. the component of the error $\tilde{u}_{n+1}, \tilde{v}_{n+1}$ "along" the eigenfunction u, v respectively are zero (in the order $O(\tilde{u}_n)$ and $O(\tilde{v}_n)$, no matter what this component was at the n-th iteration. Note that for $\Delta\tau = 1$, formula **Equation (34)** and **Equation (35)** yield the optimal value of generalization Petviashvili method. Here, $\alpha_1$and α_2 are eigenvalues.

To find the generalization Petviashvili method for system equation we may have an extension of **Equation (21)** to the vector case:

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}_\tau = N^{-1} L \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \sum_{k=1}^{2} \gamma_k \frac{\langle \vec{e}_k, L \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rangle}{\langle \vec{e}_k, N\vec{e}_k \rangle} \vec{e}_k \tag{36}$$

$$\gamma_k = 1 + \frac{1}{\alpha_k \Delta_\tau}, \alpha_k = \frac{\langle \vec{e}_k, L\vec{e}_k \rangle}{\langle \vec{e}_k, N\vec{e}_k \rangle}, k = 1,2 \tag{37}$$

where N is a self-adjoint operator, a positive definite matrix operator, whose form will be discussed shortly. For the notation on the following, we follow the paper of **[10]** where $\vec{e}_k$ and α_k are the approximate eigenvectors and eigenvalues of $N^{-1}L$:

$$L\vec{e}_k \approx \alpha_k N\vec{e}_k \tag{38}$$

Now, we discuss the computationally efficient choice of operator N and vector $\vec{e}_k$. This choice makes the generalization of the method to the case of system equations nontrivial. Hence, it constitutes an important technical result of this section. The form of N that we advocate

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, N_k = c_k - d_k \nabla^2, k = 1,2 \tag{39}$$

Let us now show how $c_{1,2}$ and $d_{1,2}$ can be computed while assuming a general form of the eigenvector $\vec{e}_k$, and then will argue that one can and should take $\vec{e}_k = (u, v)^T$. Let

$$L \equiv \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \vec{e}_1 \equiv \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} \tag{40}$$

where each of L_{ij} is a self-adjoint operator. In a similar way in **[13]** we require that

$$\frac{\langle N\vec{e}_1, L\vec{e}_1 \rangle^2}{\langle N\vec{e}_1, N\vec{e}_1 \rangle \langle L\vec{e}_1, L\vec{e}_1 \rangle} = maximum \tag{41}$$

and then we find the equations for $c_{1,2}$ and $d_{1,2}$ by setting the derivatives of the l.h.s with respect to these parameters to zero. Thus, similarly to **Equation (25)**, one obtains

$$\frac{\langle N_r\vec{e}_1, L\vec{e}_1 \rangle}{\langle N_r\vec{e}_1, N\vec{e}_1 \rangle} = \frac{\langle N\vec{e}_1, L\vec{e}_1 \rangle}{\langle N\vec{e}_1, N\vec{e}_1 \rangle}, N_r \equiv \frac{\partial N}{\partial r}, r = c_1, c_2, d_1, d_2 \tag{42}$$

The solution of **Equation (42)** can most easily be found by using a similarly way for one equation in Section 2.1. Equating the l.h.s with $r = c_k$ to those with the corresponding $r = d_k$ yields:

$$\eta_k \equiv \frac{c_k}{d_k} = \frac{\langle \left(\langle \nabla^2 e_{k1}, \sum_{j=1}^{2} L_{kj} e_{j1} \rangle e_{k1} - \langle e_{k1}, \sum_{j=1}^{2} L_{kj} e_{j1} \rangle \nabla^2 e_{k1} \right), \nabla^2 e_{k1} \rangle}{\langle \left(\langle \nabla^2 e_{k1}, \sum_{j=1}^{2} L_{kj} e_{j1} \rangle e_{k1} - \langle e_{k1}, \sum_{j=1}^{2} L_{kj} e_{j1} \rangle \nabla^2 e_{k1} \right), e_{k1} \rangle} \tag{43}$$

where $k = 1,2$.

Then, equating the l.h.s's of **Equation (41)** with $r = c_1$ and $r = c_2$ yields

$$\frac{d_2}{d_1} = \frac{\langle e_{11}, (\eta_1 - \nabla^2)e_{11}\rangle\langle e_{21}, \sum_{j=1}^{2} L_{2j}e_{j1}\rangle}{\langle e_{21}, (\eta_2 - \nabla^2)e_{21}\rangle\langle e_{11}, \sum_{j=1}^{2} L_{1j}e_{j1}\rangle} \tag{44}$$

We now discuss the choice of the eigenvectors $\vec{e}_1$ and $\vec{e}_2$. First, we note that since these eigenvectors enter **Equation (36)** on equal footing, it might seem that it would be "more correct" to replace the l.h.s of **Equation (41)** by

$$\sum_{k=1}^{2} \frac{\langle N\vec{e}_k, L\vec{e}_k\rangle^2}{\langle N\vec{e}_k, N\vec{e}_k\rangle\langle L\vec{e}_k, L\vec{e}_k\rangle} \tag{45}$$

However, in particular, the corresponding counterpart of **Equation (42)** becomes a truly nonlinear system for c_1, c_2, d_1, d_2 and hence cannot be easily solved. Therefore, we continue to use the results obtained from (41). Next, a reasonable, although not the most general, choice for $\vec{e}_1$ is

$$\vec{e}_1 = \begin{pmatrix} u \\ \rho_{21} v \end{pmatrix} \tag{46}$$

Then, $\vec{e}_2$ can be written by

$$\vec{e}_2 = \begin{pmatrix} \rho_{21} u \\ v \end{pmatrix} \tag{47}$$

where ρ_{21} is determined from the orthogonality condition $\vec{e}_1$ toward $N\vec{e}_1$, then we can compute

$$\rho_{21} = -\rho_{21} \frac{\langle v, N_2 v\rangle}{\langle u, N_1 u\rangle} \tag{48}$$

where N_1 and N_2 are from **Equation (38)**, **Equation (43)**, and **Equation (44)** for each given value ρ_{21} by simply taking

$$\rho_{21} = 1 \tag{49}$$

We now state the algorithm of the generalized Petviashvili method for coupled nonlinear wave equations, which is obtained by "delinearizing" **Equation (44)**:

$$\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} u \\ v \end{pmatrix}_n + \left[N^{-1}\left(L_0 \begin{pmatrix} u \\ v \end{pmatrix} \right)_n - \sum_{k=1}^{2} \gamma_k \frac{\langle \vec{e}_{k,n}, \left(L_0 \begin{pmatrix} u \\ v \end{pmatrix} \right)_n \rangle}{\langle \vec{e}_k, N\vec{e}_{k,n}\rangle} \right] \Delta\tau, \tag{50}$$

where $\vec{e}_{k,n}$ are computed using the components u_n, v_n at each iteration, and N and γ_k are computed iteratively until the solution reaches a prescribed accuracy 10^{-3}. Iteration scheme above along with the details of the calculation of N and $\vec{e}_k$ (**Equation (39)**, **Equation (43)**, **Equation (44)**, and **Equation (46)**- **Equation (49)**) is the main result of this section.

3. RESULTS AND DISCUSSION

3.1 Two-Component System of Nonlinear Schrödinger Equations

To obtain the stationary solution, we should set the initial iterative data close to the solution. We fix this initial iterative data by trying the suitable form in the numerical simulation. We let some parameter values and components inside it. Then, we fix

$$\begin{cases} u_0 = 2e^{-(x^2+y^2)}(\epsilon + x + y) \\ u_0 = 1.5e^{-(x^2+y^2)}(\epsilon + x + y) \end{cases}. \tag{51}$$

Previously, we do not have a unique value for ϵ. There are two different values, namely, ϵ_1 and ϵ_2 for x and y respectively. However, we get the divergence solution directly. Hence, we try to modify by swapping

and changing some components inside, then we obtain the initial iterative data as above. To find an efficient number of iterations, we may choose the good parameter values and make some trials and errors first until we obtain the best parameter values. Then, the numerical computation results are shown on the following table

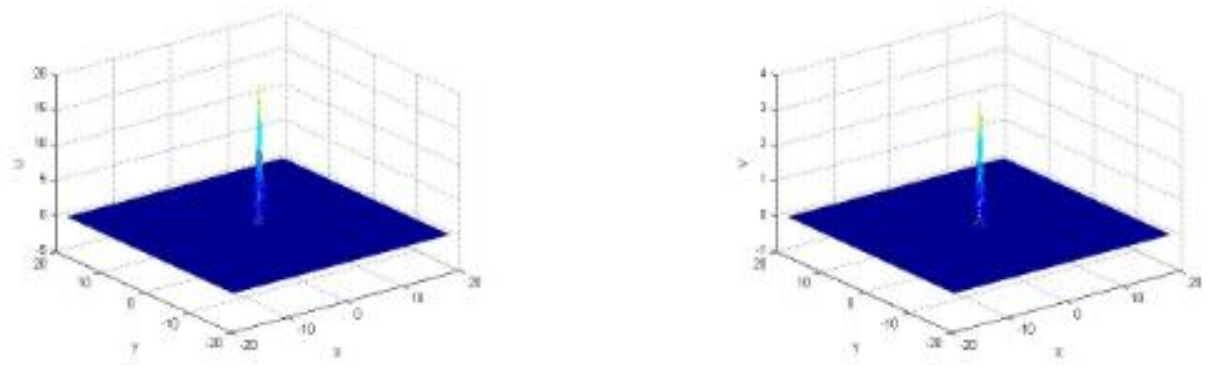
Table 1. Table of Parameter Value when $\beta > 0$

$\lambda_{1,2}$	$\mu_{1,2}$	ϵ	$\Delta\tau$	β
1.89 ; 8.5	0.1 ; 1.9	0.001	0.9	0.05

λ_1 and λ_2 are fixed from [15]. It is the stiffest condition, which mentioned three conditions, there are mildly stiff, stiffer, and stiffest. The stiffest condition can produce the convergence of solitary wave solutions faster than others. Meanwhile, numerically, the value of μ_1, μ_1 are chosen by trial and error many times to obtain the best value of $\mu_{1,2}$.

As the computation results, we obtain a pair convergence solution for a two-component system equation for u and v. We obtain one bump solution for both. The supports are getting narrow and the high peak of the u-solution is higher than the v-solution. For instance, we found that $\Delta\tau = 0.9$, the convergence rate of our method converges in 80 iterations. It is the fastest convergence with the optimal value of $\Delta\tau = 0.9$. Otherwise, the solutions are convergence slowly or divergence. The numerical solutions are shown in **Figure. 1**.

The solution exists since the interaction between two-component is attractive. By applying the theorem 4.1 in [16], which is the extension, we show the uniqueness of solitary wave solution numerically when fixed β to be positive small enough and $\lambda_{1,2}$ has a similar value. To satisfy the condition of its theorem, we set $\lambda_1 = \lambda_2 = 8.5$.



(a) **(b)**

Figure 1. The Stationary Wave Solution by Table 1 (a) The Stationary Wave Solution of u and (b) The Stationary Wave Solution of v

Choose β is positive and small enough. We obtain the convergence results even the β value tends to zero. However, the solution is blow-up when β is really small and close to zero. We show our results in **Table 2**.

Table 2. Numerical Results for The Uniqueness of Solution with

β	Number of Iterations	High Peak of u-solution	High Peak of v-solution
0.06	62	10^{10}	10
0.05	76	10^{10}	10
0.04	108	10^{10}	10^2
0.03	97	10^{10}	10^2
0.01	292	10^{10}	10^2
0.002	180	10^{10}	10^2
0.001	214	10^{10}	10^3
$\mathbf{10^{-8}}$	159	10^{10}	10^3
$\mathbf{10^{-10}}$	282	Blow-up Solution	Blow-up Solution

Moreover, we have the extension condition based on the refer theorem [16]. We will show our numerical results with $\lambda_1 \neq \lambda_2$, but β values are still positive enough and decrease. One obtains in Table 3,

In this case, we still get the convergence solution with one bump solution for u and v even though $\lambda_{1,2}$ has a different value. Instead, we have a suitable range larger than the previous results of β to obtain the solitary wave solution. On the other hand, when the value of β tends to zero, the solution is not blow-up but it exists and tends to $\delta(x)$. The solution tends to $\delta(x)$ means that everywhere the value is zero, except at zero point, the value is maximal. The looks of this result are the same as in Figure. 1, just had a change of peaks with different values.

The opposite results shown by Table 3 produce the divergence solution for both u and v fastly, if we simulate $\beta < 0$ directly.

Table 3. Numerical Results for The Uniqueness of Solution ($\lambda_1 = 7.89$; $\lambda_2 = 8.5$)

β	Number of Iterations	High Peak of u-solution	High Peak of v-solution
0.05	74	10^{10}	10^{1}
0.04	108	10^{10}	10^{2}
0.03	97	10^{10}	10^{2}
0.01	292	10^{11}	10^{2}
0.002	180	10^{12}	10^{2}
0.001	214	10^{13}	10^{2}
$\mathbf{10^{-8}}$	235	10^{14}	10^{3}
$\mathbf{10^{-10}}$	232	10^{15}	10^{3}
$\mathbf{10^{-12}}$	108	10^{15}	10^{3}
$\mathbf{10^{-16}}$	119	10^{15}	10^{3}
$\mathbf{10^{-22}}$	106	10^{15}	10^{3}
$\mathbf{10^{-32}}$	106		

It is caused by the interaction between u and v is repulsive, then we cannot find the solution easily. However, we may obtain the existence of the u and v solution even when β is negative, namely, given a special condition. We have a certain condition to get it by using theorem 1 in [17]. Given a condition

$$\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} < \sin\frac{\pi}{k}$$

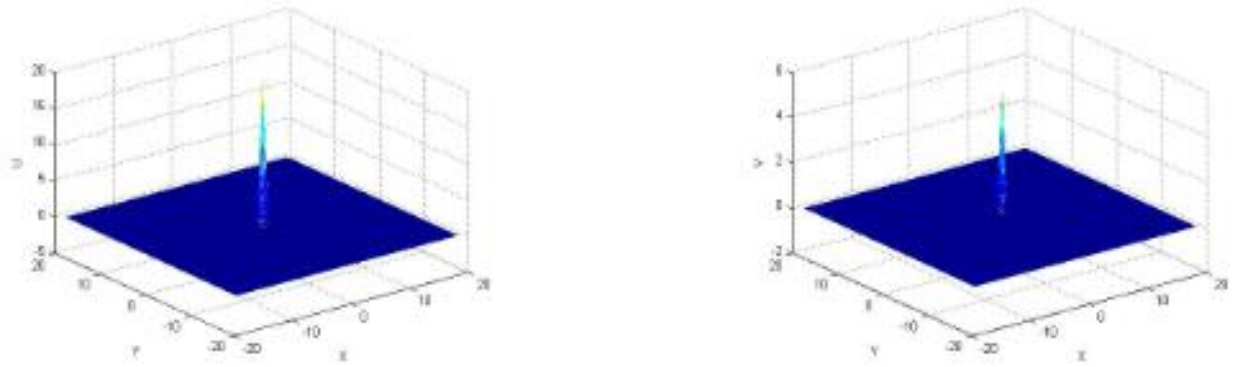
on $n = 2$, assume k is a positive integer $k > 2$, let $\beta < 0$, and assume that $\lambda_1 < \lambda_2$. Then, we obtain the solitary wave solution by fixing parameter values in Table 3. By Table 3 we get the graphs for u-solution and v-solution in Figure 2.

Table 4. Table of parameter value when $\beta < 0$

$\lambda_{1,2}$	$\mu_{1,2}$	ϵ	$\Delta\tau$	β
7.89 ; 11	0.1 ; 1.9	0.001	0.9	-0.05

Based on Table 4 with $\beta < 0$, we have the optimum parameters. The solution is convergence with 568 iterations. We have λ_1 is 7.89 and 11 for λ_2 or we can say λ_1 is less than λ_2. This is also the key to the convergence of the solution. The solution convergence only in this condition. However, they have a suitable value of λ_2, it is $11 \leq \lambda_2 \leq 11.85$. Otherwise, the solutions are divergent for all conditions.

One bump of u-solution and v-solution has a similar form to the case $\beta > 0$. Both of them are getting narrower for the support. The high peak of solutions is between 10^{10} and 10^{15} for u and for v is started from 10^{2} until 10^{4}. They have different high peak solutions, but they have the same form.



(a) **(b)**

Figure 2. The Stationary Wave Solution by Table 3 (a) The Stationary Wave Solution of u and (b) The Stationary Wave Solution of v

4. CONCLUSIONS

In this research, we obtained the following two main results. The results are as follows below.

1. The main result here was finding a way in which all the required parameters of the iteration scheme can be computed by explicit expressions. The algorithm is given by **Equation (50)**, **Equation (39)**, **Equation (43)**, **Equation (44)**, and **Equation (46)-(49)**.
2. The numerical simulation with the generalized Petviashvili method cannot be applied to Two-Component System of NLSE with a small positive parameter ε^2. It can be observed when the coupling constants β that shows the existential of the solitary wave solutions for nonlinear stationary wave equations. If β is positive and small enough, then the stationary wave exists and is unique. If β is negative, then the stationary wave exists by providing a certain condition.

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