BAREKENG: Journal of Mathematics and Its Applications
June 2024 Volume 18 Issue 2 Page 1307–1316
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss2pp1307-1316

THE CLEANNESS OF THE SUBRINGS OF $M_2(\mathbb{Z}_P)$

Shinta Nur Alfiana [1*], Nikken Prima Puspita [2], Widowati [3]

[1,2,3] *Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro*
Jln. Prof Jacub Rais, Tembalang, Semarang, Central Java 50725, Indonesia

*Corresponding author's e-mail: * **shintanura@students.undip.ac.id***

ABSTRACT

Let $(R, +, \cdot)$ be a ring. Ring R is said to be a clean ring if every element of R can be expressed as the sum of a unit and an idempotent element. Furthermore, there are r-clean rings. An r-clean ring is a generalization of a clean ring. In an r-clean ring, all of its elements can be represented as the sum of a regular element and an idempotent element. Moreover, strongly r-clean rings were introduced. A strongly r-clean ring is a ring where every element of the ring can be expressed as the sum of a regular and an idempotent element, and the multiplication of that regular and idempotent is commutative. On the other hand, there is a ring of the set of 2×2 matrices over ring R denotes by $M_2(R)$. In this paper, we will discuss the cleanness properties, especially strongly r-clean of the subring of $M_2(\mathbb{Z}_P)$. The aim of this paper is to find the characteristics of strongly r-clean of the subring of $M_2(\mathbb{Z}_P)$. Here, we assumed that $M_2(\mathbb{Z}_P)$ is a ring of matrix over $\mathbb{Z}_P$.

Article History:

Received: 1st February 2024
Revised: 29th March 2024
Accepted: 4th May 2024
Published: 1st June 2024

Keywords:

Regular Element;
r-Clean;
Strongly r-Clean;
The subrings of $M_2(\mathbb{Z}_P)$.

How to cite this article:

S. N. Alfiana, N. P. Puspita and Widowati, "THE CLEANNESS OF THE SUBRINGS $M_2(\mathbb{Z}_P)$", *BAREKENG: J. Math. & App.*, vol. 18, iss. 2, pp. 1307-1316, June, 2024

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*
Research Article • Open Access

1. INTRODUCTION

Let $(R, +, \cdot)$ be a ring. An element a in R is clean if a can be represented as the sum of a unit and an idempotent element or we can write as $a = u + e$ where u is a unit element in R and e is an idempotent element in R. A ring R is called a clean ring if every element of R is clean. Clean rings were introduced by W. K. Nicholson in [1]. Zhang and Camilo have studied clean endomorphism rings and unit regular rings [2]. Kar and Das introduced clean semiring in their study [3]. Furthermore, [4] introduced strongly clean rings. An element a in ring R is said to be strongly clean if $a = u + e$ and $ue = eu$. Many authors have studied strongly clean rings, for example is [5].

In 2013, Ashrafi and Nasibi introduced an r-clean ring and generalized a clean element to an r-clean element [6], [7]. An element a in R is r-clean if $a = r + e$ where r is a regular element in R and e is an idempotent element in R. A ring R is said to be r-clean ring if every element of R is r-clean. In [8] have studied about r-clean and group rings and [9] have studied about r-clean ideals. Moreover, Sharma and Singh introduced strongly r-clean rings [10]. An element a in R is called strongly r-clean if $a = r + e$ and $re = er$, and ring R is said to be a strongly r-clean ring if every element of R is a strongly r-clean. According to [10], clearly, every strongly r-clean is an r-clean, but the converse is not always true. In [10], we also know that a strongly r-clean is an r-clean ring, and the converse holds if the ring is abelian.

In addition to r-clean rings, there are many extensions of clean rings. In 2010, [11] extended clean rings and introduced the concept of f-clean rings. Many studies have discussed the f-clean ring, for example, [12], [13], and [14]. Moreover, there are studies about m-clean rings, for example, [15] and [16].

Let $M_2(R)$ be a ring of matrix 2×2 over R. In fact, we know that $M_2(R)$ is not Abelian. Even though there is a ring of matrix that is a strongly r-clean ring, for example $M_2(\mathbb{Z}_P)$ where $\mathbb{Z}_P$ is the set of modulo integer P (prime number). Motivated by the conditions of $M_2(\mathbb{Z}_P)$, we will find the characteristics of the cleanness subrings of $M_2(\mathbb{Z}_P)$. First, we establish the subrings of $M_2(\mathbb{Z}_P)$. Furthermore, we present the regular elements of the ring of matrix. In the end, we find the subrings of $M_2(\mathbb{Z}_P)$ which can be considered as a strongly r-clean ring and the relation with the cleanness concept on the other conditions. Throughout this article, the set of all regular elements of a ring R is denoted by $Reg(R)$, the set of all idempotent elements of a ring R is denoted by $Id(R)$, and the ring of matrix 2×2 over R denoted by $M_2(R)$. In the general condition of the cleanness properties in ring theory, we did not have the relation between clean ring, r-clean ring, strongly clean, and strongly r-clean of the subring of a ring. In this paper, we want to explain the characteristics of the cleanness of the subring of matrix rings over $\mathbb{Z}_P$.

2. RESEARCH METHODS

This study used a literature review as the research method. First, we need to study ring theory, regular elements, idempotent elements, clean rings theory, r-clean rings theory, and strongly r-clean rings theory. The authors studied ring theory through books, i.e., [17], [18]. Additionally, the authors also learn about clean rings, r-clean rings, and strongly r-clean rings by using related research articles. The next step is to identify the regular elements in the ring of the matrix $\mathbf{2 \times 2}$. We establish the subrings of $\boldsymbol{M_2}(\mathbb{Z}_{\boldsymbol{P}})$ and identify the cleanness of the subrings of $\boldsymbol{M_2}(\mathbb{Z}_{\boldsymbol{P}})$. The state of the art of this research is as follows.

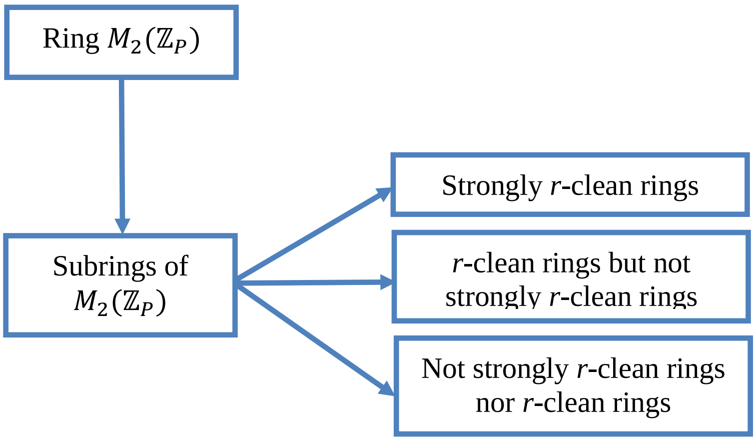

Figure 1. State of the Art

3. RESULTS AND DISCUSSION

Let $(R,+,\cdot)$ be a ring. The set of 2×2 matrices over ring R, denotes $M_2(R)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}: a,b,c,d\in R\right\}$ is a ring under the addition and multiplication matrix operation, and we can write it as ring $(M_2(R),+,\cdot)$. We know that $(\mathbb{Z}_P,+_P,\cdot_P)$ is a field that means $\mathbb{Z}_P$ is a commutative ring and non-zero elements of $\mathbb{Z}_P$ have an inverse of the multiplication operation. Therefore, we can express the set of matrices over $\mathbb{Z}_P$ as a ring with unity $(M_2(\mathbb{Z}_P),+,\cdot)$. Moreover, we will discuss the cleanness of the subring of $M_2(\mathbb{Z}_P)$. We start by the regular elements of the ring of matrix 2×2 to help in understanding the concept of strongly r-clean rings on the subring of $M_2(\mathbb{Z}_P)$.

3.1 The Regular Elements in the Ring of Matrix 2 × 2

In 1936, Von Neumann described the regular elements [19]. An element $r\in R$ is said to be a regular element in ring R if there exists an element $y\in R$ such that $ryr=r$. Additionally, we will explain the properties of regular elements in $M_2(R)$.

Proposition 1. *Let R be a unity and commutative ring, and $M_2(R)$ is a ring. For element $P=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in M_2(R)$, we have:*

i. *If $ad-bc\neq 0$ and there exist $\begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1}\\ -c(ad-bc)^{-1} & a(ad-bc)^{-1}\end{bmatrix}\in M_2(R)$, then P is a regular element.*
ii. *If $ad-bc=0$ where $a\neq 0$ and there exist $\begin{bmatrix} a^{-1} & 0\\ 0 & 0\end{bmatrix}\in M_2(R)$, then P is a regular element.*
iii. *If $ad-bc=0$ where $b\neq 0$ and there exist $\begin{bmatrix} 0 & 0\\ b^{-1} & 0\end{bmatrix}\in M_2(R)$, then P is a regular element.*
iv. *If $ad-bc=0$ where $c\neq 0$ and there exist $\begin{bmatrix} 0 & c^{-1}\\ 0 & 0\end{bmatrix}\in M_2(R)$, then P is a regular element.*
v. *If $ad-bc=0$ where $d\neq 0$ and there exist $\begin{bmatrix} 0 & 0\\ 0 & d^{-1}\end{bmatrix}\in M_2(R)$, then P is a regular element.*
vi. *If $ad-bc=0$ where $a=b=c=d=0$, then P is a regular element.*

Proof. We know R is a unity and commutative ring and $M_2(R)$ is a ring of the matrix over R. We have some facts as follows.

i. Suppose $ad-bc\neq 0$ and there exist $\begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1}\\ -c(ad-bc)^{-1} & a(ad-bc)^{-1}\end{bmatrix}\in M_2(R)$. Let $Y=\begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1}\\ -c(ad-bc)^{-1} & a(ad-bc)^{-1}\end{bmatrix}$. Then,

$$\begin{aligned} PYP &= \begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1}\\ -c(ad-bc)^{-1} & a(ad-bc)^{-1}\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\ &= \begin{bmatrix} (ad-bc)(ad-bc)^{-1} & (-ab+ab)(ad-bc)^{-1}\\ (cd-cd)(ad-bc)^{-1} & (ad-bc)(ad-bc)^{-1}\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\ &= \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\\ &= \begin{bmatrix} a & b\\ c & d\end{bmatrix}\\ &= P\end{aligned}$$

Thus, P is a regular element.

ii. The second case, we have $ad-bc=0$ where $a\neq 0$ and there exist $\begin{bmatrix} a^{-1} & 0\\ 0 & 0\end{bmatrix}$. Since $a,b,c,d\in R$ and there exist $\begin{bmatrix} a^{-1} & 0\\ 0 & 0\end{bmatrix}$ so $a^{-1}\in R$, then $d=bc(a^{-1})$.

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} a^{-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix} = \begin{bmatrix} aa^{-1} & 0\\ ca^{-1} & 0\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ ca^{-1} & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ ca^{-1}a & ca^{-1}b \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Hence, $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a regular element.

For the case where $c = 0, P$ is also a regular ring. The following explanation proves this statement. We have a condition

$$ad - bc = 0$$

where $a \neq 0$ and $c = 0$,

$$ad - bc = 0$$
$$ad - b(0) = 0$$
$$ad - 0 = 0$$
$$ad = 0$$

Since $a \neq 0$ hence $d = 0$.

Now we want to show $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is a regular.

The reason that we have the condition that exists $\begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix} \in M_2(R)$, then

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Clearly P is a regular.

The proof of point (iii), (iv), and (v) are analog with (ii).

vi. If $a = b = c = d = 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For any $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M_2(R)$ we have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a regular element.■

From the **Proposition 1**, we can determine the conditions of elements in a ring of the matrix 2×2 over a commutative ring which giving the relation of unit and regular elements. By applying this proposition, we can easily identify an element on the ring $M_2(\mathbb{Z}_P)$ that is a regular or not by using the properties of the field $\mathbb{Z}_P$. Here we give the following example of the **Proposition 1**.

Example 1 Given a ring $(\mathbb{Z}_3, +_3, \cdot_3)$ and $(M_2(\mathbb{Z}_3), +, \cdot)$. An element $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \in M_2(\mathbb{Z}_3)$ is a regular element. Here for $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ we have $ad - bc = 0$, $a \neq 0$ and there exist $\begin{bmatrix} 2^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Z}_3)$ such that $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$. Clearly, $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \in M_2(\mathbb{Z}_3)$ is a regular element.

Now, we give an example that there is subring $M_2(\mathbb{Z}_P)$ is not a regular ring.

Example 2 Given a ring $(\mathbb{Z}_2, +_2, \cdot_2)$ and ring $(T_2(\mathbb{Z}_2), +, \cdot)$ where $(T_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{Z}_2 \right\}$. An element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in T_2(\mathbb{Z}_2)$ is not a regular element. The explanation is as follows. Here for $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $ad - bc = 0$ and only entries b that are nonzero elements. Based on Proposition 1 point 3, an element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is said to be a regular element if there exists $\begin{bmatrix} 0 & 0 \\ 1^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in ring $T_2(\mathbb{Z}_2)$. We know that element $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin T_2(\mathbb{Z}_2)$. Therefore an element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not a regular element. For clarity, the results of the multiplication between elements in ring $T_2(\mathbb{Z}_2)$ will be provided below, and it will be shown that there is no element $T_2(\mathbb{Z}_2)$of $T_2(\mathbb{Z}_2)$ such that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in T_2(\mathbb{Z}_2)$ is a regular.

$$T_2(\mathbb{Z}_2)=\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\0&1\end{bmatrix}\right\}$$

Table 1. Multiplication of Elements in $T_2(\mathbb{Z}_2)$

$\cdot$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$
$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$
$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$
$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$
$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$
$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$
$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$
$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$
$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Based on the multiplication results among elements in ring $T_2(\mathbb{Z}_2)$, it is concluded that there exists no element in ring $T_2(\mathbb{Z}_2)$ such that element $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ is regular. Thus, element $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ is not a regular element. Consequently, ring $T_2(\mathbb{Z}_2)$ is not a regular ring.

3.2 The Subrings of $M_2(\mathbb{Z}_P)$

The aim of this research is to find the characteristics of strongly r-clean of the subrings of $M_2(\mathbb{Z}_P)$. The subrings are as follows.

Table 2. The Subrings of $M_2(\mathbb{Z}_P)$

No.	The subring of $M_2(\mathbb{Z}_P)$
1.	$\left(\left\{\begin{bmatrix}a&0\\0&0\end{bmatrix}: a\in\mathbb{Z}_P\right\},+,\cdot\right)$
2.	$\left(\left\{\begin{bmatrix}0&b\\0&0\end{bmatrix}: b\in\mathbb{Z}_P\right\},+,\cdot\right)$
3.	$\left(\left\{\begin{bmatrix}0&0\\c&0\end{bmatrix}: c\in\mathbb{Z}_P\right\},+,\cdot\right)$
4.	$\left(\left\{\begin{bmatrix}0&0\\0&d\end{bmatrix}: d\in\mathbb{Z}_P\right\},+,\cdot\right)$
5.	$\left(\left\{\begin{bmatrix}a&b\\0&0\end{bmatrix}: a,b\in\mathbb{Z}_P\right\},+,\cdot\right)$
6.	$\left(\left\{\begin{bmatrix}a&0\\c&0\end{bmatrix}: a,c\in\mathbb{Z}_P\right\},+,\cdot\right)$
7.	$\left(\left\{\begin{bmatrix}a&0\\0&d\end{bmatrix}: a,d\in\mathbb{Z}_P\right\},+,\cdot\right)$
8.	$\left(\left\{\begin{bmatrix}0&b\\0&d\end{bmatrix}: b,d\in\mathbb{Z}_P\right\},+,\cdot\right)$
9.	$\left(\left\{\begin{bmatrix}0&0\\c&d\end{bmatrix}: c,d\in\mathbb{Z}_P\right\},+,\cdot\right)$
10.	$\left(\left\{\begin{bmatrix}a&b\\0&d\end{bmatrix}: a,b,d\in\mathbb{Z}_P\right\},+,\cdot\right)$
11.	$\left(\left\{\begin{bmatrix}a&0\\c&d\end{bmatrix}: a,c,d\in\mathbb{Z}_P\right\},+,\cdot\right)$

No.	The subring of $M_2(\mathbb{Z}_P)$
12.	$\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} : 0 \in \mathbb{Z}_P\right\}, +, \cdot\right)$
13.	$\left(\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$

Based on the sufficient and necessary theorem of subrings ([18], [22]), we can check the set on the second column of Table 2.

Example 3. Let $\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$ be a subset of $M_2(\mathbb{Z}_P)$. We will show $\left(\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}, +, \cdot\right)$ is a subring of $(M_2(\mathbb{Z}_P), +, \cdot)$.

i. The set of $\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$ is non empty set, since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$.

ii. Let $\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \in \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$,

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 +_P (-a_2) & 0 \\ 0 & 0 \end{bmatrix}.$$

Since, $\mathbb{Z}_P$ is a ring and $a_2 \in \mathbb{Z}_P$ then there exist $-a_2 \in \mathbb{Z}_P$. Thus, $a_1 +_P (-a_2) \in \mathbb{Z}_P$ and $\begin{bmatrix} a_1 +_P (-a_2) & 0 \\ 0 & 0 \end{bmatrix} \in \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$.

iii. Let $\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \in \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$.

$$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 \cdot_P a_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\mathbb{Z}_P$ is a ring and $a_1, a_2 \in \mathbb{Z}_P$ then $a_1 \cdot_P a_2 \in \mathbb{Z}_P$ and $\begin{bmatrix} a_1 \cdot_P a_2 & 0 \\ 0 & 0 \end{bmatrix} \in \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}$.

From (i), (ii), and (iii), we have $\left(\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}, +, \cdot\right)$ is a subring of $(M_2(\mathbb{Z}_P), +, \cdot)$.

3.3 The Cleanness Properties of the Subrings of $M_2(\mathbb{Z}_P)$

For the first, we present the subrings of $M_2(\mathbb{Z}_P)$ are strongly r-clean rings. Before discussing the subrings of $M_2(\mathbb{Z}_P)$, we need to study the following lemma.

Lemma 1. *Given a ring $(R, +, \cdot)$. If $a \in R$ is a regular element, then a is a strongly r-clean element.*

Proof. Since R is a ring, then there exists zero element $0 \in R$ and $0 \cdot 0 = 0^2 = 0$. For any regular element $a \in R$, we have

$$a = a + 0,$$

where $a \in Reg(R)$ and $0 \in Id(R)$ and $a0 = 0a$. Thus, a is strongly r-clean element.■

Consequence of the Lemma 1, regular rings are strongly r-clean rings.

Next, we will show the subrings of $M_2(\mathbb{Z}_P)$ are strongly r-clean rings in this proposition that follows.

Proposition 2. *Let $(\mathbb{Z}_P, +_P, \cdot_P)$ and $(M_2(\mathbb{Z}_P), +, \cdot)$ be rings i.e.*

$$M_2(\mathbb{Z}_P) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_P\right\},$$

then ring $(M_2(\mathbb{Z}_P), +, \cdot)$ is a strongly r-clean ring.

Proof. Based on Proposition 1, for any element $A \in M_2(\mathbb{Z}_P)$ we can find $B \in M_2(\mathbb{Z}_P)$ such that $ABA = A$. Then every element of $M_2(\mathbb{Z}_P)$ is a regular element. According to Lemma 1, every element of $M_2(\mathbb{Z}_P)$ is a strongly r-clean element. Thus, ring $(M_2(\mathbb{Z}_P), +, \cdot)$ is a strongly r-clean ring. ■

Based on Proposition 2 and Lemma 1, we can conclude that if all elements of a subring of $M_2(R)$ is regular, then the subring is a strongly r-clean ring. Therefore, the following proposition is obtained.

Proposition 3. The subrings of $M_2(\mathbb{Z}_P)$ that are regular rings, i.e., the subrings of $M_2(\mathbb{Z}_P)$ in **Table 2**, in numbers 1, 4, 7, 12, and 13 are strongly r-clean rings.

Furthermore, we will identify the subrings of $M_2(\mathbb{Z}_P)$ that has a unity element.

Proposition 4. *Let $(\mathbb{Z}_P, +_P, \cdot_P)$ and $(T_2(\mathbb{Z}_P), +, \cdot)$ be rings, i.e.*

$$T_2(\mathbb{Z}_P) = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{Z}_P\right\},$$

then ring $(T_2(\mathbb{Z}_P), +, \cdot)$ is a strongly r-clean ring.

ok of set $T_2(\mathbb{Z}_P)\setminus\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ from ring $T_2(\mathbb{Z}_P)$, there exists an element $X \in T_2(\mathbb{Z}_P)$ such that $TXT = T$. Thus, every element of set $T_2(\mathbb{Z}_P)\setminus\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ is a regular element. Consequently, based on **Lemma 1**, every element of set $T_2(\mathbb{Z}_P)\setminus\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ is a strongly r-clean element. It is clear that the subring $T_2(\mathbb{Z}_P)$ is not a regular ring because there exist elements in $T_2(\mathbb{Z}_P)$ that are not a regular element ($\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$). Nevertheless, we want to show that even though the subring $T_2(\mathbb{Z}_P)$ is not a regular ring but subring $T_2(\mathbb{Z}_P)$ is a strongly r-clean ring. Now, we want to prove that elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ are strongly r-clean elements. The elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ can be represented as the sum of

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix} \in Reg(T_2(\mathbb{Z}_P))$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in Id(T_2(\mathbb{Z}_P))$, and $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$. These facts show that the elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ are strongly r-clean elements. Since every element of $T_2(\mathbb{Z}_P)$ is a strongly r-clean element, thus $T_2(\mathbb{Z}_P), +, \cdot)$ is a strongly r-clean ring.■

According to **Proposition 4**, there is a subring of $M_2(\mathbb{Z}_P)$ which is not a regular ring, but it has a unity element; thus, the subrings are a strongly r-clean ring. As a result, we have the following proposition.

Proposition 5. The subrings of $\boldsymbol{M_2}(\mathbb{Z}_{\boldsymbol{P}})$ that are not regular rings, but have a unity element, i.e., the subrings of $\boldsymbol{M_2}(\mathbb{Z}_{\boldsymbol{P}})$ in **Table 2**, in numbers 10 and 11, are strongly r-clean rings.

We know the fact that every strongly r-clean is an r-clean, but the converse is not always true. On the next, we present the subrings of $M_2(\mathbb{Z}_P)$ are r-clean rings but not strongly r-clean rings.

Proposition 6. *Let $(\mathbb{Z}_P, +_P, \cdot_P)$ and $(A_2(\mathbb{Z}_P), +, \cdot)$ be a ring, i.e.*

$$A_2(\mathbb{Z}_P) = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_P\right\}$$

Then ring $(A_2(\mathbb{Z}_P), +, \cdot)$ is an r-clean ring but not a strongly r-clean ring.

Proof. Following **Proposition 1**, every element of $A_2(\mathbb{Z}_P)\setminus\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ is a regular element. Then, the elements of $A_2(\mathbb{Z}_P)\setminus\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ are strongly r-clean elements. Next, we check the elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ in $A_2(\mathbb{Z}_P)$ are strongly r-clean elements or not. Idempotent elements in $A_2(\mathbb{Z}_P)$ is $Id(A_2(\mathbb{Z}_P)) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix} : e \neq 0, e \in Id(\mathbb{Z}_P)\right\}$. The elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ in $A_2(\mathbb{Z}_P)$ can be represented as the sum of regular and idempotent elements as follows.

i. $$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -e & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} -e & b \\ 0 & 0 \end{bmatrix} \in Reg(A_2(\mathbb{Z}_P))$ and $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in Id(A_2(\mathbb{Z}_P))$. Here,

$$\begin{bmatrix} -e & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -e & b \\ 0 & 0 \end{bmatrix}$$

ii. $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -e & -e+b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix}$,

where $\begin{bmatrix} -e & -e+b \\ 0 & 0 \end{bmatrix} \in Reg(A_2(\mathbb{Z}_P))$ and $\begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix} \in Id(A_2(\mathbb{Z}_P))$. Here,

$$\begin{bmatrix} -e & -e+b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} e & e \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -e & -e+b \\ 0 & 0 \end{bmatrix}.$$

iii. $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -(1-e) & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} -(1-e) & b \\ 0 & 0 \end{bmatrix} \in Reg(A_2(\mathbb{Z}_P))$ and $\begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix} \in Id(A_2(\mathbb{Z}_P))$. Here,

$$\begin{bmatrix} -(1-e) & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1-e & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -(1-e) & b \\ 0 & 0 \end{bmatrix}.$$

iv. $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -(1-e) & -(1-e)+b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1-e & 1-e \\ 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} -(1-e) & -(1-e)+b \\ 0 & 0 \end{bmatrix} \in Reg(A_2(\mathbb{Z}_P))$ and $\begin{bmatrix} 1-e & 1-e \\ 0 & 0 \end{bmatrix} \in Id(A_2(\mathbb{Z}_P))$. Here,

$$\begin{bmatrix} -(1-e) & -(1-e)+b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1-e & 1-e \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1-e & 1-e \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -(1-e) & -(1-e)+b \\ 0 & 0 \end{bmatrix}$$

Hence, the elements of $\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \neq 0, b \in \mathbb{Z}_P\right\}$ in $A_2(\mathbb{Z}_P)$ are r-clean elements but not a strongly r-clean elements. ■

Analog to **Proposition 6**, the subrings of $M_2(\mathbb{Z}_P)$ in **Table 2**, numbers 5, 6, 8, and 9 are r-clean rings but not strongly r-clean rings. Next, we identify the subrings of $M_2(\mathbb{Z}_P)$ that are not strongly r-clean rings or r-clean rings either in the following proposition.

Proposition 7. *Let $(\mathbb{Z}_P, +_P, \cdot_P)$ and $(B_2(\mathbb{Z}_P), +, \cdot)$ be a ring, i.e.,*

$$B_2(\mathbb{Z}_P) = \left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{Z}_P\right\}.$$

Then ring $(B(\mathbb{Z}_P), +, \cdot)$ is not a strongly r-clean ring or r-clean either.

Proof. The multiplication result of the elements of $B_2(\mathbb{Z}_P)$ are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $Reg\big(B_2(\mathbb{Z}_P)\big) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ and $Id\big(B_2(\mathbb{Z}_P)\big) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$. From the regular and idempotent elements contained in ring $B_2(\mathbb{Z}_P)$, we have that every non-zero element in ring $B_2(\mathbb{Z}_P)$ cannot be expressed as the sum of a regular and an idempotent element. Clearly, $(B(\mathbb{Z}_P), +, \cdot)$ is not a strongly r-clean ring or r-clean either. ■

Based on the **Proposition 7**, we also have the following useful facts.

Proposition 8. For $A, B \in M_2(\mathbb{Z}_P)$, if AB is not equal to a zero element, then the subring is clearly not a strongly r-clean ring or r-clean ring either. Thus, the subrings of $M_2(\mathbb{Z}_P)$ in **Table 2**, number 2 and 3 are not a strongly r-clean ring or r-clean ring either.

Hence, we have a necessary and sufficient condition for subrings of $\boldsymbol{M_2}(\mathbb{Z}_{\boldsymbol{P}})$ is strongly $\boldsymbol{r}$-clean ring.

Theorem 1. Let $M_2(\mathbb{Z}_P)$ is a ring and $\boldsymbol{A}$ is a subring of $M_2(\mathbb{Z}_P)$. Subring $\boldsymbol{A}$ is a strongly r-clean ring if only if $\boldsymbol{A}$ is a regular ring or $\boldsymbol{A}$ has a unity element.

Proof. $\Longrightarrow$ If A is a regular ring or A has a unity element then A is a strongly r-clean element.
According to **Proposition 3** and **Proposition 5**, clearly, if A is a regular ring or A has a unity element then A is a strongly r-clean element.
$\Longleftarrow$ If A is a strongly r-clean element then A is a regular ring or A has a unity element. Based on the observations, the results of the subrings of $M_2(\mathbb{Z}_P)$ are obtained as strongly r-clean rings, r-clean rings, or neither. These results are presented in **Table 3**.

Table 3. **The Cleanness of the Subrings of $M_2(\mathbb{Z}_P)$**

No.	The subring of $M_2(\mathbb{Z}_P)$	Regular ring	Strongly *r*-clean ring	*r*-clean ring
1.	$\left(\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z}_P\right\}, +, \cdot\right)$	✓	✓	✓
2.	$\left(\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	-
3.	$\left(\left\{\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : c \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	-
4.	$\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} : d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	✓	✓	✓
5.	$\left(\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	✓
6.	$\left(\left\{\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} : a, c \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	✓
7.	$\left(\left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	✓	✓	✓
8.	$\left(\left\{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} : b, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	✓
9.	$\left(\left\{\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} : c, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	-	✓
10.	$\left(\left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	✓	✓
11.	$\left(\left\{\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} : a, c, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	-	✓	✓
12.	$\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} : 0 \in \mathbb{Z}_P\right\}, +, \cdot\right)$	✓	✓	✓
13.	$\left(\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_P\right\}, +, \cdot\right)$	✓	✓	✓

Based on the observations in **Table 3**, subrings number 1, 4, 7, 10, 11, 12, and 13 are strongly r-clean rings, and these subrings are regular rings or rings with unity element. It is proven that if subring A is a strongly r-clean ring, then A is a regular ring or a ring with a unity element. ■

4. CONCLUSIONS

Through this study, we have the characteristics of the cleanness properties of the subring of $M_2(\mathbb{Z}_P)$. If the subring of $M_2(\mathbb{Z}_P)$ is a regular ring, then the subring is a strongly r-clean ring. If the subring of $M_2(\mathbb{Z}_P)$ is not a regular ring but has an unity element, then the subring is also a strongly r-clean ring. However, if all the multiplication result of the elements in the subring of $M_2(\mathbb{Z}_P)$ is a zero element, then the subring is not a strongly r-clean ring or r-clean ring either.

AKNOWLEDGEMENT

This paper was presented in First Joint IIIMT (Indonesia-India-Iran-Malay-Türkiye) – Algebra Forum on December 16-17, 2n023. The authors would like to express their heartfelt gratitude to all the individuals who have contributed to the success of this paper. Our sincere appreciation goes out to our colleagues and peers who have provided valuable feedback and insightful suggestions throughout the research process.

REFERENCES

[1] W. K. Nicholson, “Lifting Idempotents and Exchange Rings,” *Transaction of the America Mathematical Society*, vol. 299, pp. 269–278, 1977.

[2] H. Zhang and V. Camillo, “On Clean Rings,” *Communication in Algebra*, vol. 44, pp. 2475–2481, 2016.

[3] S. Kar and D. Das, “Clean Semiring,” *Beitr Algebra Geom*, vol. 64, pp. 197–207, 2023.

[4] W. K. Nicholson, “Strongly Clean Rings and Fitting’s Lemma,” *Communication in Algebra*, vol. 27, no. 8, pp. 3583–3592, 1999.

[5] J. Ster, “Example of Strongly Clean Rings,” *Communication in Algebra*, vol. 47, no. 11, pp. 4684–4696, 2019.

[6] N. Ashrafi and E. Nasibi, “Rings in Which Elements are the Sum of an Idempotent and a Regular Element,” *Bulletin of the Iranian Mathematical Society*, vol. 39, no. 3, pp. 579–588, 2013.

[7] N. Ashrafi and E. Nasibi, “On r-Clean Rings,” *Math Subject Classification*, vol. 3, pp. 579–588, 2013.

[8] N. Ashrafi and E. Nasibi, “r-Clean Group Rings,” *Iranian Journal Science Technology* , vol. 41, pp. 65–68, 2017.

[9] D. A. Yuwaningsih, I. E. Wijaya, and B. Surodjo, “On r-Clean Ideals,” *Palestine Journal of Mathematics*, vol. 12, no. 2, pp. 217–224, 2023.

[10] G. Sharma and A. B. Singh, “Strongly r-Clean Rings,” *International Journal of Mathematics*, vol. 13, no. 2, pp. 207–214, 2018.

[11] B. Li and L. Feng, “f-Clean Rings and Rings Having Many Full Elements,” *Journal of the Korean Mathematical Society*, vol. 47, no. 2, pp. 247–261, 2010.

[12] A. Andari, “The Relationships Between Clean Rings, r-Clean Rings, and f-Clean Rings,” in *AIP Conference Proceedings*, 2018.

[13] S. Jamshidvand, H. H. S. Javadi, and N. V. Javeheri, “Generalized f-Clean Rings,” *Journal of Linear and Topological Algebra*, vol. 3, no. 1, pp. 55–60, 2014.

[14] S. Sahebi, V. Rahmani, and Hamid. H. S. Javadi, “On a Generalization of f-Clean Rings,” *J Comput Theor Nanosci*, vol. 12, pp. 3936–3939, 2015.

[15] S. Purkait, T. K. Dutta, and S. Kar, “On m-Clean and Strongly m-Clean Rings,” *Commun Algebra*, vol. 48, no. 1, 2020.

[16] S. Purkait, “On Strongly m-Clean Rings and m-Semiperfect Rings,” *Commun Algebra*, vol. 48, no. 10, 2020.

[17] S. Wahyuni, I. E. Wijayanti, D. A. Yuwaningsih, and A. D. Hartanto, *Teori Ring dan Modul*. Daerah Istimewa Yoygakarta, Indonesia: Gadjah Mada University Press, 2016.

[18] P. B. Garrett, *Abstract Algebra*. United States of America: Chapman & Hall/CRC, 2007.

[19] J. von Neumann, “On Regular Rings,” *Proceedings of the National Academy of Sciences*, vol. 22, pp. 700–713, 1936.

[20] S. Wahyuni, I. E. Wijayanti, A. Munandar, and N. Hijriati, *Teori Representasi Grup Hingga*. Yogyakarta: Gadjah Mada University Press, 2018.