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MULTI-ITEM PROBABILISTIC INVENTORY MODELS CONSIDERING DISCOUNT, EXPIRATION, WAREHOUSE CAPACITY, AND CAPITAL CONSTRAINTS

Claresta Tirta Saputera Gunawan[1], Dharma Lesmono[2*], Daniel Salim[3]

[1,2,3]*Center for Mathematics and Society, Department of Mathematics,*
Faculty of Information Technology and Sciences, Universitas Katolik Parahyangan
Jln. Ciumbuleuit No. 94, Bandung, 40141, Indonesia

*Corresponding author's e-mail: * **jdharma@unpar.ac.id***

ABSTRACT

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In general, companies have inventory stored in warehouses to be used or sold in the future. To optimize inventory, companies require a model to determine the appropriate order quantity and optimal reorder timing to minimize total inventory costs. To stay within the set capacity, several factors need to be considered in inventory management, such as a product's shelf life, warehouse capacity, and capital. On the other hand, suppliers may offer discounts, and companies tend to take advantage of them. However, they must consider the warehouse capacity and capital availability. This paper constructs two probabilistic multi-item inventory models, considering discounts, expiration, warehouse capacity, and capital constraint. The first model considers all-unit discounts, while the second deals with incremental discounts. We consider three items to be managed and examine three replenishment policies, namely individual order, joint order, and a combined policy of individual and joint order, to minimize the total inventory cost. Sensitivity analysis is performed on the joint order policy to determine the influence of parameter values' changes on the reorder time and total inventory cost.

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1. INTRODUCTION

In general, companies have inventory stored in warehouses to meet future needs. Inventory is an asset and a burden because procuring less inventory will cause the company to incur more significant costs to store these raw materials. The costs in question include maintenance, rental, or insurance costs. On the other hand, procuring a small amount of inventory will cause ordering costs to increase, so there is a possibility that the production process will stop due to a shortage of raw materials, which can cause a loss of revenue that should be earned. In addition, there is a possibility of loss of consumer confidence and consumers moving to identical products or other companies.

In a company, several factors need to be considered when managing inventory. The first factor is that the goods sold have a limited shelf life or will deteriorate after a certain period. The second factor needs more capacity, both warehouse capacity and capital. Both factors significantly affect the inventory system, especially in determining the optimal order quantity and reorder time to minimize the total inventory cost in a certain period. Some papers such as **[1]** and **[2]** considered only warehouse capacity in their models, but with other features such as stochastic demands, shortages, and permissible delay in payment. Other papers only considered deterioration with additional factors such as in **[3]** with multi-item and partial backlogging feature, **[4]** with advertisement and selling price dependent demand, and **[5]** with nonlinear holding cost, **[6]** with multi-item, multi-warehouse for agri-fresh products and **[7]** with multi-item for medicines in the hospital.

In addition to the two factors mentioned, there is a discount given by the supplier (supplier) if the order exceeds a certain quantity. It makes the company think buying more goods will increase the discount and reduce shipping costs. However, it should be noted that the goods ordered have an expiration factor, and there are limitations on warehouse capacity and capital to procure these goods. Suppose the expiration time of the goods is different, and the warehouse capacity and capital are limited. In that case, the company cannot optimally utilize the supplier's discount. Therefore, an arrangement is needed to determine the number of orders, considering the expiration time and available warehouse capacity. Many papers have included the discount factor in the model along with warehouse or capacity constraint or other factor, such as in **[8]**, **[9]**, **[10]**, **[11]**, **[12]**, **[13]**, **[14]** and **[15]**. For example, in **[8],** a mathematical model was developed to accommodate delays in payments, discounts, capacity constraints, and shortages. A model for growing products with incremental discounts, storage capacity, and budget was considered **[9].** Models with several types of discount factors were developed in those papers along with factors such as partial credit policy and capacity constraint (**[10]**), probabilistic model (**[11]**), and deterioration (**[12]**, **[13]**, **[14]**, **[15]**). Models for multi-item products or models with two or more warehouses are also considered, such as in **[16]**, **[17]**, **[18]** and **[19]**. In **[16]**, strategies for warehouse capacity allocation are discussed, while multiple-warehouse policy and budget constraints are considered in **[17]**. The dual warehouse is one of the factors considered in **[18]** and **[19]**.

Considering these factors, a normally distributed probabilistic inventory model can be used to estimate the optimal inventory in a company. The normal distribution is used because it is suitable for representing many random variables, where the random variables are relatively easy to convert into normally distributed variables. In addition, according to**[20]**, the normal distribution was chosen because it can reflect the probability of inventory shortages concerning inventory levels well.

In the real world, companies manage multiple items with different expiration times and face limited warehouse capacity and capital constraints. This situation often occurs in food and beverage retail, pharmaceuticals, and cosmetic products. Therefore, in this paper, the goods sold by the company are not specifically mentioned. On the other hand, suppliers will provide all-unit discounts or incremental discounts for purchasing larger quantities of goods. From the conditions of these two parties, this paper develops a multi-item probabilistic inventory model by considering discount, expiration, warehouse capacity constraints, and capital factors. Then, a study of ordering policies for three types of items is added, namely individual order, joint order, and a combination of personal and joint order policies. The purpose of developing this model is to determine the optimal reorder time and quantity of goods that minimize the total inventory cost. In [15], the authors developed a multi-item inventory model by considering deterioration, all-units discount, and warehouse capacity. The model in this paper is an extension of the model in [15] by adding budget constraints and considering another discount scheme, viz. incremental discount. Adding budget constraints will make the model more realistic since, in practice, a company operates under a certain or limited budget.

An algorithm to obtain the optimal solution has also been developed, and a sensitivity analysis of the model is also performed.

2. RESEARCH METHODS

The mathematical model in this paper is developed based on the following notations and assumptions.

2.1 Notations

The notations used in the development of this model followed the ones used in [15] with the addition of new notations $A_{i,m}$ and M, about the incremental discount and budget constraint in our model. The notations are:

D_i	:	Total demand for item i in one cycle (unit/year),
$P_{i,m}$	:	Purchase price item i according to m price split (Rp/unit),
S_i	:	The standard deviation of demand for item i in a cycle (Rp/unit/year),
H_i	:	Storage cost for item i in a cycle (Rp/unit/year),
c_{ui}	:	Shortage cost due to out-of-stock for item i (Rp/unit),
J_i	:	The selling price of item i that will expire (Rp/unit),
n	:	Numbers of item types,
T	:	The optimal joint order (year),
Q_i	:	Order size optimal for item i (unit),
Q_{ki}	:	Number of expired items for item i (unit),
α	:	Probabilities stock out (shortages) of supplies,
z_{α_i}	:	Z value for each item i in normal distribution at level α,
$f_{z_{\alpha_i}}$	:	Ordinate function value for item i based on Z value,
$\psi_{z_{\alpha i}}$	:	Value of 1- cumulative distribution function (cdf) of the standard normal distribution,
N_i	:	Shortages number expected of each item i (unit),
ss_i	:	Number of safety stock for item i (unit),
t_i	:	Short cycle planning period in one horizon planning (year),
t_{1i}	:	The length of keeping period for item i until just before it enters expiration date (year),
t_{2i}	:	The length of the shortage period for item i (year),
θ_i	:	Fraction for item i that is in good condition $(0 < \theta_i < 1)$,
$1-\theta_i$	:	Fraction for item i whose condition is expired $(0 < 1-\theta_i < 1)$,
$U_{i,m}$	:	Limit on the quantity of goods ordered in order to change the purchase price with m price break,
$A_{i,m}$	:	Extra purchase cost for each Q_i item that is not purchased for the price of $P_{i,m}$,
L	:	Lead time order (year),
W	:	Total warehouse capacities (unit of volume),
M	:	Total capital (funds) (Rp),
w_i	:	Volume size for item i (unit of volume),
O_b	:	Total purchase cost for during the cycle (Rp),
O_p	:	Total ordering cost for during the cycle (Rp),
O_s	:	Total holding cost for during the cycle (Rp),
O_k	:	Total lost sales (shortage) cost for during the cycle (Rp),
O_{kd}	:	Total expired cost for during the cycle (Rp),
O_T	:	Total inventory cost for during the cycle (Rp).

2.2 Assumptions

The following assumptions used in the model are the same as the ones in [15], except assumption number 6 below, which is used to include incremental discount in our model.

1. All types of items have the same lead time (L).
2. The good fraction value (θ_i) is 90% for all types of items.
3. The existence of expired items has consequences on two cost components, namely the cost of shortages and the cost of expiration:
    a. Consequences on shortage costs: the existence of expired goods causes reduced availability of goods to meet all demand, so there is a demand that cannot be fulfilled.
    b. Consequences on expiry costs: items entering expiry date will be sold at a lower price than purchase price, resulting in a loss equal to the gap between purchase cost and price of sale items (on expired date).
4. All expired items will be sold at the end of period t_{1i} simultaneously so that there are no expired items left during period t_{2i}.
5. All expired items can still be sold to certain parties at J_i price (where $J_i < P_{i,m}$).
6. The extra purchase cost for each item i with $m = 0$ will be 0 ($A_{i,0} = 0$).
7. This assumption means that expired items can still be used (sold) but not for consumption (food).
8. The volume size of each item is known with certainty at the beginning of a planning period.
9. Stockout items as lost sales.
10. The lost sales cost is opportunity cost, where the amount is equal to the profit of each type of product.
11. Lost sales costs due to probabilistic demand and expired items are the same.

2.3 Model Formulation

In real life, the demand for an item is uncertain (probabilistic), so it can cause a shortage. Therefore, companies need to prepare more safety stock to satisfy the needs of their buyers. Furthermore, the problem becomes more complex when handling multi-item goods having different expiration times, along with limited warehouse capacity and limited budget. Another issue arises when the supplier offers an interesting discount scheme for buying more goods. Therefore, a policy is needed to overcome all these problems. Three types of policies can be chosen when purchasing goods, namely individual orders, joint orders, and a combination of individual and joint order policies. The complexity of the situation described above will be tackled by the model we developed.

The multi-item inventory model is a development concept derived from the single-item EOQ inventory model**.** In the multi-item inventory model, more than one type of item is sold. In this model, three types of policies can be chosen when purchasing goods: individual orders, joint orders, and a combination of individual and joint order policies. In this section, the model is discussed first using the joint order policy, where T is the decision variable that affects the total cost of inventory.

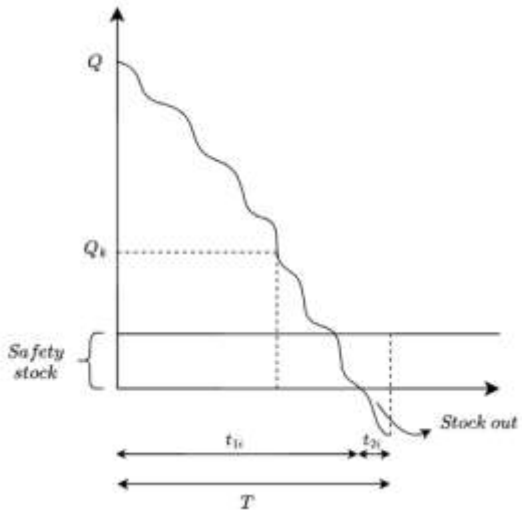

Figure 1. Illustration of inventory in one cycle

In **Figure 1**, T denotes the time between ordering items from one cycle to the next, the highest inventory level at Q_i units for each type of item i, the number of expired items of Q_{ki} for each type of item i that occurs at the end of t_{1i}, and t_{2i} denotes the length of time that shortages occur for each type of item i. Note that the length of T for all items i is the same, so it can be written as $T = \frac{Q}{D}$. Since the policy used is multi-item, the goods ordered come from the same supplier at the same time, then T can be written as T

$$T = \frac{Q_i}{D_i}$$

$$Q_i = T \cdot D_i \tag{1}$$

Next, the length of t_{1i} is determined using the principle of congruence. From **Figure 1**, the following results were obtained.

$$\frac{Q_i}{T} = \frac{Q_i - Q_{ki}}{t_{1i}}$$

$$t_{1i} = \frac{T(Q_i - Q_{ki})}{Q_i}$$

where $Q_{ki} = (1 - \theta_i)Q_i$ and $\theta_i = \frac{Q_i - Q_{ki}}{Q_i}$, so $t_{1i} = T\theta_i$. And we can find the length t_{2i} by using the relationship:

$$T = t_{1i} + t_{2i}$$

$$t_{2i} = (1 - \theta_i)T.$$

Based on the fraction of goods that will expire, it is known at the beginning of the planning period that the average shortage of inventory of expired goods can be expressed in the equation

$$\frac{Q_{ki}}{2} = \frac{Q_i(1 - \theta_i)}{2} = \frac{TD_i(1 - \theta_i)}{2}, \tag{2}$$

where the expiration condition occurs at time t_{2i}, so **Equation (2)** becomes

$$\frac{TD_i(1 - \theta_i)}{2} \times t_{2i} = \frac{T^2 D_i(1 - \theta_i)^2}{2}.$$

The demand probability distribution and the inventory shortage probability distribution are assumed to be normal distributions, so the equation for shortage expectation based on **[21]** is

$$N_i = S_i\sqrt{L}[f(z_\alpha) - z_\alpha \psi(z_\alpha)]$$

Total inventory shortage for one planning horizon caused of demand's nature is as

$$\frac{N_i}{T} = \frac{S_i\sqrt{L}[f(z_\alpha) - z_\alpha \psi(z_\alpha)]}{T}.$$

In multi-item problem, the purchase price of the i-th item is rewritten as follows:

$$P_{i,m} = \begin{cases} P_{i,0}, & \text{jika } U_{i,0} \le Q_i < U_{i,1}; \\ P_{i,1}, & \text{jika } U_{i,1} \le Q_i < U_{i,2}; \\ \vdots & \vdots \\ P_{i,j}, & \text{jika } U_{i,j} \le Q_i < U_{i,j}, \end{cases}$$

where $U_{i,0} < U_{i,1} < \cdots < U_{i,j+1}$ is the number of items at the time of price break and $P_{i,0} > P_{i,1} > \cdots > P_{i,j+1}$ is the price after discount for each unit of goods.

There are five components of the total inventory cost in this case, namely purchase cost (O_b), ordering costs (O_p), holding costs (O_s), lost sales (shortages) cost (O_k), and expiry costs (O_{kd}), which can be arranged in the following equation.

$$O_T = O_b + O_p + O_s + O_k + O_{kd} \tag{3}$$

In this paper, we consider two models. In the first model (Model I), we consider an all-units discount. In contrast, in the second model (Model II), the incremental discount is considered.

2.3.1 Model I and II

There are five cost components for the total cost of inventory for Model I and Model II, that is purchase cost (O_b), ordering cost (O_p), holding cost (O_s), lost sale cost (O_k), and expiry cost (O_{kd}), which each are given in **Table 1**. For Model 1, the formulation for the five cost components comes from **[15]**, while for Model II the formulation of the five components was derived from the modified combination of **[15]** and the basic formula for incremental discount.

Table 1. Five Cost Components for The Total Cost of Inventory

	Model I	Model II
Purchase Cost (O_b)	$\sum_{i=1}^{n} P_{i,m} \cdot D_i$	$\sum_{i=1}^{n} \left(P_{i,m} + \frac{A_{i,m}}{Q_i}\right) \cdot D_i$
Ordering Cost (O_p)	$\frac{C}{T}$	
Holding Cost (O_s)	$\sum_{i=1}^{n} \frac{H_i(TD_i\theta_i(2-\theta_i)+ss_i)}{2}$	
Lost Sales Cost (O_k)	$\sum_{i=1}^{n} \frac{c_{ui}TD_i(1-\theta_i)^2}{2} + \frac{c_{ui}N_i}{T}$	
Expiry Cost (O_{kd})	$\sum_{i=1}^{n} (1-\theta_i)\,(D_i+ss_i)(P_{i,m}-J_i)$	$\sum_{i=1}^{n} (1-\theta_i)\,(D_i+ss_i)\left(P_{i,m}+\frac{A_{i,m}}{Q_i}-J_i\right)$

From **Table 1**, the total inventory cost from **Equation (3)** for Model I can be written as:

$$\begin{aligned} O_T(T) &= O_b + O_p + O_s + O_k + O_{kd} \\ &= \frac{C}{T} + \sum_{i=1}^{n} \Bigg(P_{i,m} \cdot D_i + \frac{H_i(TD_i\theta_i(2-\theta_i)+ss_i)}{2} + \frac{c_{ui}TD_i(1-\theta)^2}{2} + \frac{c_{ui}E[N_i]}{T} \\ &\qquad + (1-\theta_i)(D_i+ss_i) \times \left(P_{i,m} - J_i\right) \Bigg). \end{aligned} \tag{4}$$

with $ss_i = z_{\alpha_i} S_i \sqrt{L_i}$.

The total inventory cost from **Equation (3)** for Model II can be written as:

$$O_T(T) = O_b + O_p + O_s + O_k + O_{kd}$$
$$= \frac{C}{T} + \sum_{i=1}^{n} \left(\left(P_{i,m} + \frac{A_{i,m}}{Q_i} \right) + \frac{H_i(TD_i\theta_i(2-\theta_i) + ss_i)}{2} + \frac{c_{ui}TD_i(1-\theta)^2}{2} + \frac{c_{ui}E[N_i]}{T} + (1-\theta_i)(D_i + ss_i) \times \left(P_{i,m} + \frac{A_{i,m}}{Q_i} - J_i \right) \right), \quad (5)$$

with $ss_i = z_{\alpha_i} S_i \sqrt{L_i}$ and $A_{i,m} = \sum_{e=1}^{i} (U_{i,e-1})(P_{i,e-1} - P_{i,e})$.

The warehouse and capital constraint formula can be written as in **Equation (6)**

$$\sum_{i=1}^{n} w_i Q_i \leq W \quad \text{and} \quad \sum_{i=1}^{n} P_{i,m} Q_i \leq M. \quad (6)$$

Since the decision variable of the model is T, the **Equation (6)** can be written as

$$\sum_{i=1}^{n} w_i D_i T \leq W \quad \text{and} \quad \sum_{i=1}^{n} P_{i,m} D_i T \leq M.$$

For example, suppose the research model under review is Model I is as follows

with constraints

$$\text{Min } O_T(T) = \frac{C}{T} + \sum_{i=1}^{n} \left(P_{i,m} \cdot D_i + \frac{H_i(TD_i\theta_i(2-\theta_i) + ss_i)}{2} + \frac{c_{ui}TD_i(1-\theta)^2}{2} + \frac{c_{ui}E[N_i]}{T} + (1-\theta_i)(D_i + ss_i) \times (P_{i,m} - J_i) \right) \quad (7)$$

with constraints

$$g_1(T) = \sum_{i=1}^{n} w_i D_i T \leq W, \quad (8)$$

$$g_2(T) = \sum_{i=1}^{n} P_{i,m} D_i T \leq M. \quad (9)$$

Based on the Kuhn-Tucker condition **[22]**, the following equations are obtained:

$$\sum_{i=1}^{n} \left(\frac{H_i D_i \theta_i (2-\theta_i)}{2} + \frac{c_{ui} D_i (1-\theta_i)^2}{2} - \frac{c_{ui} E[N_i]}{T^2} + \lambda_1 w_i D_i + \lambda_2 P_{i,m} D_i \right) = \frac{C}{T^2}; \quad (10)$$

$$\lambda_1 \left(\sum_{i=1}^{n} w_i D_i T - W \right) = 0; \quad (11)$$

$$\lambda_2 \left(\sum_{i=1}^{n} P_{i,m} D_i T - M \right) = 0 \quad (12)$$

$$\lambda_1, \lambda_2 \geq 0.$$

Of the four λ's possibilities, then

1. If $\lambda_1, \lambda_2 = 0$, from **Equation (10)** obtained,

$$\frac{C}{T^2} = \sum_{i=1}^{n} \left(\frac{H_i D_i \theta_i (2-\theta_i)}{2} + \frac{c_{ui} D_i (1-\theta_i)^2}{2} - \frac{c_{ui} \mathrm{E}[N_i]}{T^2} \right)$$

$$T_{unconstrained} = \sqrt{\frac{C + \sum_{i=1}^{n} c_{ui} \mathrm{E}[N_i]}{\sum_{i=1}^{n} \frac{1}{2} \left(H_i \left(D_i \theta_i (2-\theta_i) \right) + c_{ui} D_i (1-\theta_i)^2 \right)}} \quad (13)$$

2. If $\lambda_1 = 0$ and $\lambda_2 > 0$, then
   - from **Equation (10)** obtained
$$\lambda_2 = \frac{\frac{C}{T^2} - \sum_{i=1}^{n}\left(\frac{H_i D_i \theta_i (2-\theta_i)}{2} + \frac{c_{ui} D_i (1-\theta_i)^2}{2} - \frac{c_{ui}\mathrm{E}[N_i]}{T^2}\right)}{\sum_{i=1}^{n} P_{i,m} D_i}.$$
   - From **Equation (12)** obtained
$$T_{capital} = \frac{M}{\sum_{i=1}^{n} P_{i,m} D_i}. \tag{14}$$

3. If $\lambda_1 > 0$ and $\lambda_2 = 0$, then
   - from **Equation (10)**, obtained
$$\lambda_1 = \frac{\frac{C}{T^2} - \sum_{i=1}^{n}\left(\frac{H_i D_i \theta_i (2-\theta_i)}{2} + \frac{c_{ui} D_i (1-\theta_i)^2}{2} - \frac{c_{ui}\mathrm{E}[N_i]}{T^2}\right)}{\sum_{i=1}^{n} w_i D_i}.$$
   - From **Equation (11)**, obtained
$$T_{warehouse} = \frac{W}{\sum_{i=1}^{n} w_i D_i}. \tag{15}$$

4. If $\lambda_1, \lambda_2 > 0$, then
   - from **Equation (11)**, obtained
$$T_{warehouse} = \frac{W}{\sum_{i=1}^{n} w_i D_i}, \text{ and}$$
   - from **Equation (12)**, obtained
$$T_{capital} = \frac{M}{\sum_{i=1}^{n} P_{i,m} D_i}.$$

Of the four possibilities, the value of T that can be used to find the reorder time is as in **Equation (13)**, **Equation (14)**, and **Equation (15)**. The optimum T value will be used to find the appropriate Q value for each item and the minimum total inventory cost.

For the case of individual order, the optimum T value obtained will vary because the order for each item is obtained from different suppliers where the order for each type of item does not affect each other. Suppose n is the number of types of goods, then the total inventory cost of the individual order policy is

$$O_{T_{individual}} = \sum_{i=1}^{n} O_{T_i}, \tag{16}$$

where O_{T_i} is the total cost for each type of good i.

For the combined case between individual and joint order, if the goods are ordered simultaneously, then the optimum T value used will be the same, but if the goods are ordered individually, then the optimum T value used is $T_{individual}$. Suppose the company places an order for three types of goods, then there are three alternatives that the company can choose, namely

- The first and second items are ordered simultaneously, while the third item individually. The total inventory cost for the first alternative is

$$O_T = O_{p_{1,2}} + \sum_{i=1}^{2}\left(O_{b_i} + O_{s_i} + O_{k_i} + O_{kd_i}\right) + \left(O_{p_3} + O_{b_3} + O_{s_3} + O_{k_3} + O_{kd_3}\right) \tag{17}$$

  where $O_{p_{1,2}}$ is the one-time cost for the first and second items.
- The second and third items are ordered simultaneously, while the first is ordered individually. The total inventory cost for the first alternative is

$$O_T = O_{p_{2,3}} + \sum_{i=2}^{3} \left(O_{b_i} + O_{s_i} + O_{k_i} + O_{kd_i}\right) + \left(O_{p_1} + O_{b_1} + O_{s_1} + O_{k_1} + O_{kd_1}\right) \quad (18)$$

where $O_{p_{2,3}}$ is the one-time cost for the second and third items.

- The first and third items are ordered simultaneously, while the second item individually. The total inventory cost for the first alternative is

$$O_T = O_{p_{1,3}} + \sum_{i\in 1,3} \left(O_{b_i} + O_{s_i} + O_{k_i} + O_{kd_i}\right) + \left(O_{p_2} + O_{b_2} + O_{s_2} + O_{k_2} + O_{kd_2}\right) \quad (19)$$

where $O_{p_{1,3}}$ is the one-time cost for the first and third items.

2.4 Algorithm for Model I

The algorithm for obtaining the optimal reordering time and quantity of goods ordered for model I with the ordering policy joint order with the aim of minimizing the total cost of inventory is as follows:

1. Calculate the value of $T_{unconstrained}$ by using the formula in **Equation (13)**.
2. Calculate the value of $T_{warehouse}$ by using the formula in **Equation (15)**.
3. Calculate the value of $T_{capital}$ by using **Equation (14)** with each price for all goods $(P_{i,m})$.
4. If the value of $T_{unconstrained}$ is greater than $T_{warehouse}$ and $T_{capital}$, meaning there are no warehouse and capital constraints, then the optimum value of T is between the value of $T_{warehouse}$ or $T_{capital}$.
5. If the value of $T_{warehouse} \leq T_{capital}$, then the optimum value of T is $T_{capital}$, and vice versa.
6. Calculate the value of Q_i for each individual item using the **Equation (1)**.
7. If Q_i is around the m price split, then the price used is $P_{i,m}$.
8. Calculate the total cost of inventory using the **Equation (4)**.

3. RESULTS AND DISCUSSION

3.1 Results

Suppose a company sell three types of items with parameter values shown in **Table 2**.

Table 2. Parameter values

Description	Notation	Item 1	Item 2	Item 3
Total demand expectation	D_i	550	400	800
Volume size	w_i	2	2	2
Total warehouse	W		500	
Total capital	M		2500	
Standard deviation of demand	S_i	57.75	42	72
Lead time	L_i	0.0083012	0.0083012	0.0083012
Shortages expectation	N_i	0.3596	0.3046	0.6028
Fraction good condition	θ_i		0.9	
Probabilities shortages	α_i	5%	6%	7%
Z value in normal distribution	z_{α_i}	1.65	1.55	1.45

Description	Notation	Item 1	Item 2	Item 3
Ordinate value based on Z value	$f_{z_{\alpha_i}}$	0.1023	0.12	0.1394
Partial expectation	$\psi_{z_{\alpha i}}$	0.0206	0.0261	0.0328
Price	$P_{i,m} = \begin{cases} \\ \end{cases}$	12 $\quad Q_1 < 201$ 10.5 $\quad Q_1 \geq 201$	15 $\quad Q_2 < 131$ 13 $\quad Q_2 \geq 131$	8 $\quad Q_3 < 301$ 7 $\quad Q_3 \geq 301$
Expired item price	J_i	10.2	12.75	6.8
Holding cost	H_i	0.12	0.225	0.08
Shortage cost	c_{ui}	3	4	2

The ordering cost of the joint order policy and combined order policy between individual and joint orders is given in **Table 3**.

Table 3. Ordering cost for each policy

Description	Joint Order	Combined	
		Joint Order	Individual Order
Ordering cost	20	16	12

Before entering the numerical simulation for model 1 and model 2, we will find the amount of safety stock for the three types of goods in one cycle using the equation $ss_i = z_{\alpha_i} S_i \sqrt{L_i}$. From **Table 4**, the company can provide safety stock for item 1 of 9 units, item 2 of 6 units, and item 3 of 10 units. This amount of safety stock applies to each ordering policy.

Table 4. The amount of safety stock for each item

Description	Notation	Item 1	Item 2	Item 3
Amount of safety stock	ss_i	9	6	10

If the company places orders simultaneously from the same supplier (joint order), then using **Equation (4)**, the total cost incurred for each selected T_{joint} is shown in **Table 5**.

Table 5. Total cost for joint order policy

Notation	Item 1	Item 2	Item 3
Q_i	72	52	105
$T_{capital}$		0.1316	
O_T		19,488.32	

From **Table 5**, the minimum total cost with joint order policy is when T_{joint} using $T_{capital} = 0.1316$. And the number of items to be ordered at the beginning of each cycle is 72, 52, and 105 units, so the purchase price used for item 1 is 12 (because $Q_1 \leq 201$), item 2 is 15 (because $Q_2 \leq 131$), and item 3 is 8 (because $Q_3 \leq 301$), so the total inventory cost that the company must incur is 19,488.32.

Furthermore, it will be checked whether the order quantities for the first, second, and third items satisfy the constraints in **Equation (8)** and **Equation (9)**. Based on this, ordering using a joint order policy does not exceed the warehouse capacity and capital provided by the company.

If the company places orders items separately (individual orders), then using **Equation (16)** will obtain the reorder time, order quantity for each item, and the total cost. In this case, the total cost obtained is 17,657.53 by ordering items 1 by 238, 2 by 192, and 3 by 250. However, the items ordered exceed the warehouse capacity and capital provided by the company. One of the solutions that can be given is to order the quantity of items obtained from the joint order policy. Therefore, the solution for the individual order policy is shown in **Table 6**.

Table 6. Total cost for individual order policy

Notation	Item 1	Item 2	Item 3
Q_i	72	52	105
T	0.1309	0.13	0.1312
O_{T_i}	6,806.87	6,201.18	6,603.73
O_T		19,611.78	

From **Table 6**, it can be inferred that the company will order items more often in a year. In addition, the total inventory cost that must be incurred is more significant, amounting to 19,611.78. The solution does not provide an optimal value, but it is within the existing warehouse and capital constraints.

Suppose the company places orders for items using a combination policy. In that case, using **Equation (17)**, **Equation (18)**, and **Equation (19)** for each alternative will obtain the reorder time, order quantity for each item, and the total cost. However, the items ordered exceed the warehouse capacity and capital provided by the company. One of the solutions that can be given is to order the quantity of items obtained from the joint order policy. Therefore, the solution for individual order policy is shown in **Table 7**, **Table 8**, and **Table 9**.

Table 7. Total cost for the first alternative

Notation	Item 1	Item 2	Item 3
Q_i	72	52	105
T	0.1305		0.1312
O_{T_i}	12,946.66		6,603.72
O_T	19,550.38		

From **Table 7**, it can be seen that the company needs to order the first and second items every 0.1305 years and the third item every 0.1312 years.

Table 8. Total cost for the second alternative

Notation	Item 1	Item 2	Item 3
Q_i	72	52	105
T	0.1309	0.1308	
O_{T_i}	6,806.87	12,743.46	
O_T	19,550.33		

From **Table 8**, it can be seen that the company needs to order the first item every 0.1309 years and the second and the third items every 0.1308 years.

Table 9. Total cost for the third alternative

Notation	Item 1	Item 3	Item 2
Q_i	72	105	52
T	0.1311		0.13
O_{T_i}	13,349.53		6,201.18
O_T	19,550.71		

Table 9 shows that the company needs to order the first and the third items every 0.1311 years and the second items every 0.13 years. We can also infer that using the combined policy of the three alternatives, the minimum total cost is around 19,550.

3.2 Discussion

Based on the known parameter values, using the all-units discount or incremental discount, using the joint order policy will result in the resulting total cost being cheaper than using the individual order policy and the combined policy. Therefore, in this section, only sensitivity analysis will be conducted on the joint order policy to determine the effect of changing parameter values on the model with a multi-item ordering policy for reorder time T and total inventory cost ($O_{T_{joint}}$).

Table 10. The effect of parameter changes for joint order policy

Parameters (initial value)	% of Changes	T_{joint}	Variation (%) T_{joint}	Q_1	Q_2	Q_3	$O_{T_{joint}}$	Variation (%) $O_{T_{joint}}$
	-50%	0.1316	0	72	52	105	19,412.32	-0.3899
	-20%	0.1316	0	72	52	105	19,427.52	-0.312
	-10%	0.1316	0	72	52	105	19,473.12	-0.078
C	0	0.1316	0	72	52	105	19,488.32	0

Parameters (initial value)	% of Changes	T_{joint}	Variation (%) T_{joint}	Q_1	Q_2	Q_3	$O_{T_{joint}}$	Variation (%) $O_{T_{joint}}$
(20)	+10%	0.1316	0	72	52	105	19,503.52	0.078
	+20%	0.1316	0	72	52	105	19,518.72	0.156
	+50%	0.1316	0	72	52	105	19,564.32	0.39
	-50%	0.0714	-45.75	39	28	57	19,630.73	-0.7307
	-20%	0.1143	-13.15	62	45	91	19,513.04	-0.1268
	-10%	0.1286	-2.28	70	51	102	19,492.10	-0.0307
W	0	0.1316	0	72	52	105	19,488.32	0
(500)	+10%	0.1316	0	72	52	105	19,488.32	0
	+20%	0.1316	0	72	52	105	19,488.32	0
	+50%	0.1316	0	72	52	105	19,488.32	0
	-50%	0.0658	-50	36	26	52	19,658.18	0.8716
	-20%	0.1052	-20.06	57	42	84	19,529.47	0.2111
M	-10%	0.1184	-10.03	65	47	94	19,506.41	0.0928
(2500)	0	0.1316	0	72	52	105	19,488.32	0
	+10%	0.1429	8.59	78	57	114	19,475.72	-0.0647
	+20%	0.1429	8.59	78	57	114	19,475.72	-0.0647
	+50%	0.1429	8.59	78	57	114	19,475.72	-0.0647
	-50%	0.1316	0	72	52	105	20,878.10	-7.13
	-20%	0.1316	0	72	52	105	20,028.43	-2.77
θ	-10%	0.1316	0	72	52	105	19,756.41	-1.37
(0.9)	0	0.1316	0	72	52	105	19,488.32	0
	+10%	0.1316	0	72	52	105	19,225.17	1.35
	+20%	0.1316	0	72	52	105	18,966.95	2.67
	+50%	0.1316	0	72	52	105	18,221.90	6.5
	-50%	0.1316	0	72	52	105	19,479.59	-0.04
	-20%	0.1316	0	72	52	105	19,484.83	-0.0179
O_s	-10%	0.1316	0	72	52	105	19,486.57	-0.0089
(17.47)	0	0.1316	0	72	52	105	19,488.32	0
	+10%	0.1316	0	72	52	105	19,490.07	0.0089
	+20%	0.1316	0	72	52	105	19,491.81	0.0179
	+50%	0.1316	0	72	52	105	19,497.05	0.0448

From **Table 10**, it can be seen that the greater the ordering cost (C) for the joint order policy, the greater the total $(O_{T_{joint}})$ inventory cost that must be incurred, but the T_{joint} reorder time does not change, so the number of the three items does not increase. The more warehouse capacity (W), the longer the ordering time. This happens because the number of three items to be ordered increases and the total inventory cost to be incurred is getting smaller. However, at a certain change in capacity, there will be no change in ordering time and total inventory cost. The more capital (M), the longer the reorder time and the smaller the total inventory cost incurred due to an increase in the number of items ordered.

Furthermore, an increase or decrease in the value of the fraction of good goods θ does not affect the reorder time, but greatly affects the total inventory cost. The higher the fraction of items that are good, the smaller the total inventory cost to be incurred. From **Table 10**, it can also be inferred that variation in the values of θ has the largest effect on the $\boldsymbol{O_{T_{joint}}}$ variation compared to variation on other parameters. The larger the value of θ, the smaller the total inventory cost. But if the value of θ is close to 1 (or 100% of the goods are all good), changes in the value of θ have no effect on the $\boldsymbol{O_{T_{joint}}}$ variation. The greater the storage cost (O_s), the greater the total inventory cost to be incurred, but the reorder time does not change.

3.3 The Effect of Changes in Ordering Costs on Ordering Policy

We will see the effect of changes in ordering cost (C) for the joint order policy on the total inventory cost (O_T). Then the results will be compared with the total inventory cost for the individual order policy, and a combination of individual and joint orders. The joint ordering policy has three alternatives that can be done, but in this section only the first alternative policy will be taken as an example for sensitivity analysis.

Table 11. The impact of changes in ordering costs for goods ordering policy on total inventory costs

Order cost with joint order policy	$O_{T_{joint}}$	$O_{T_{combined}}$	$O_{T_{individual}}$
20	19,488.32		
25	19,526.32	19,550.38	19,611.78
29	19,556.72		

From Table 11, when the joint order message cost is 29, the total inventory cost obtained is greater than the total inventory cost with the joint policy. Using a joint order policy is the best choice because it results in a cheaper total inventory cost. However, when the order cost value is increased, the joint order policy is less attractive for companies because the combined policy provides a cheaper total inventory cost..

4. CONCLUSIONS

In this paper, a multi-item probabilistic inventory model that considers discounts, expiration, warehouse capacity limitations, and capital has been developed. This model was developed to obtain the optimal ordering time for goods and the minimum total inventory cost. Companies can choose from two types of discounts and three alternative ordering policies to minimize total inventory costs. Based on our numerical examples and sensitivity analysis, we conclude that:

a. The model's total inventory cost increases if an item's ordering cost and storage cost are getting higher. Conversely, the total inventory cost decreases when the fraction of a good item is large.

b. The ordering time in the model will be longer when the capital and warehouse capacity is large because the quantity of goods purchased is large. Conversely, the ordering time will be faster when the capital capacity and warehouse are small due to the limited quantity of goods purchased.

c. The joint order policy gives the minimum total cost compared to other policies.

d. In determining the ordering policy for the three types of goods, the ordering cost significantly affects the total inventory cost.

For further research, it can be assumed that the items sold are related (complementary). In addition, it can also use different discount schemes for each item and demand distributions other than the normal distribution.

REFERENCES

[1] J. Sicilia, L. A. San-José, D. Alcaide-López-de-Pablo and B. Abdul-Jalbar, "Optimal policy for multi-item systems with stochastic demands, backlogged shortages and limited storage capacity," *Applied Mathematical Modelling*, vol. 108, pp. 236-257, 2022.

[2] R. Y. Silitonga and E. E. L. B. Sembiring, "A multi-item eoq model considering permissible delay in payment and limited warehouse capacity," *Jurnal Optimasi Sistem Industri*, vol. 21, no. 1, pp. 28-37, 2022.

[3] T. Joviani, D. Lesmono and T. Limansyah, "A multi-item inventory model with various demand functions considering deterioration and partial backlogging," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 17, no. 2, pp. 1069-2080, 2023.

[4] M. A.-A. Khan, A. A. Shaikh, I. Konstantaras, A. K. Bhunia and L. E. Cárdenas-Barrón, "Inventory models for perishable items with advanced payment, linearly time-dependent holding cost and demand dependent on advertisement and selling price," *International Journal of Production Economics*, vol. 230, 107804, 2020.

[5] A. Macías-López, L. E. Cárdenas-Barrón, R. E. Peimbert-García and B. Mandal, "An inventory model for perishable items with price-, stock-, and time-dependent demand rate considering shelf-life and nonlinear holding costs," *Mathematical Problems in Engineering,* vol. 2021, pp. 1-36, 2021.

[6] P. Paam, R. Berretta, R. García-Flores and S. K. Paul, "Multi-warehouse, multi-product inventory control model for agri-fresh products – A case study," *Computers and Electronics in Agriculture,* vol. 194, 106783, 2022.

[7] W. William, T. Ai and W. Lee, "An inventory model to control multi-item medicines with consideration of space capacity in the hospital and joint replenishment," *International Journal of Industrial Engineering and Engineering Management (IJIEEM),* vol. 2, no. 2, pp. 39-48, 2020.

[8] A. Duary, S. Das, M. G. Arif, K. M. Abualnaja, M. A.-A. Khan, M. Zakarya and A. A. Shaikh, "Advance and delay in payments with the price-discount inventory model for deteriorating items under capacity constraint and partially backlogged shortages," *Alexandria Engineering Journal,* vol. 61, no. 2, pp. 1735-1745, 2022.

[9] Y. A. Hidayat, V. N. Riaventin and O. Jayadi, "Economic order quantity model for growing items with incremental quantity discounts, capacitated storage facility, and limited budget," *Jurnal Teknik Industri*, vol. 22 no.1, pp. 1-10, 2020.

[10] M. A.-A. Khan, A. A. Shaikh and L. E. Cárdenas-Barrón, "An inventory model under linked-to-order hybrid partial advance payment, partial credit policy, all-units discount and partial backlogging with capacity constraint," *Omega,* vol. 103, 102418, 2021.

[11] D. Lesmono, T. Limansyah and H. Koswara, "Optimal replenishment policy for multi-item probabilistic inventory model with all-units discount," *Journal of Physics: Conference Series,* vol. 1218, 012036, 2019.

[12] D. Lesmono and T. Limansyah, "A probabilistic inventory model with deterioration factor and continuous discount function," in *AIP Confrence Proceedings*, vol. 2192, 060013, 2019.

[13] D. Lesmono, T. Limansyah and N. Loedy, "A joint return policy for a multi-item perishable inventory model with deterministic demands, return and all-unit discount," *International Journal of Mathematical, Engineering and Management Sciences,* vol. 5, no. 3, pp. 416-431, 2020.

[14] T. Limansyah, D. Lesmono and I. Sandy, "Economic order quantity model with deterioration factor and all-units discount," *Journal of Physics: Confrence Series,* vol. 1490, no. 1, 012052, 2019.

[15] R. Y. Silitonga, L. R. Kristiana and t. A. Parley, "A multi-item probabilistic inventory model that considers expiration factor, all unit discount policy and warehouse capacity constraints," *Jurnal Teknik Industri,* vol. 23, no. 2, pp. 39-147, 2021.

[16] S. Hu, D. Z. Yu and K. Fu, "Online platforms' warehouse capacity allocation strategies for multiple products," *Transportation Research Part E: Logistics and Transportation Review,* vol. 175, 103170, 2023.

[17] Y.-D. Huang, T.-H. Chen, M. Chih, W.-J. Chang and C.-C. Lien, "An adaptive GLNPSO method for inventory replenishment supply chain problem with multiple-warehouse policy and budget consideration," *Engineering Applications of Artificial Intelligence,* vol. 126, 107124, 2023.

[18] R. Motla, A. Kumar, N. Saxena and S. S. Sana, "Inventory optimization in a green environment with two warehouses," *Innovation and Green Development,* vol. 2, no. 4, 100087, 2023.

[19] R. Qiu, Y. Sun and M. Sun, "A robust optimization approach for multi-product inventory management in a dual-channel warehouse under demand uncertainties," *Omega,* vol. 109, 102591, 2022.

[20] J. Korponai, Á. B. Tóth and B. Illés, "Effect of the safety stock on the probability of occurrence of the stock shortage," *Procedia Engineering,* vol. 182, pp. 335-341, 2017.

[21] S. N. Bahagia, Sistem Inventori, Bandung: ITB, 2006.

[22] W. L. Winston, Operations Research: Applications and Algorithms 4th edition, Thomson Learning, Inc, 2004.