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PERMANENT AND DOMINANT OF MATRIX OVER INTERVAL MIN-PLUS ALGEBRA

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ABSTRACT

A min-plus algebra is a set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$, where $\mathbb{R}$ is the set of all real numbers equipped with two binary operations, namely minimum $(\oplus')$ and addition $(\otimes)$. Every square matrix in min-plus algebra can always be calculated as a permanent and dominant matrix. The min-plus algebra can be extended to an interval min-plus algebra, where the elements are closed intervals denoted $I(\mathbb{R})_{\varepsilon'}$ with two binary operations, minimum $(\overline{\oplus'})$ and addition $(\overline{\otimes})$. Min-plus interval algebra can be defined in a square matrix. This research will discuss the permanent and dominant a matrix over min-plus interval algebra, the relationship between permanent and dominant matrix, and bideterminant matrix over min-plus interval algebra. From the research results obtained, permanent and dominant formulas, it found that the dominant is greater than or equal to the permanent and the bideterminant formulas.

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1. INTRODUCTION

Algebra is the study of mathematical symbols and the rules for manipulating these symbols. There are several structures in algebra, one of which is conventional algebra. Conventional algebra is the set of real numbers with addition $(+)$ and multiplication $(\cdot)$ operations [1]. In conventional algebra, determinant is the value obtained from a square matrix. Determinant is defined as $\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$, S_n is a symmetric group on the set $\{1,2,\cdots,n\}$ [2].

Another structure analogous to conventional algebra is max-plus algebra. Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$, with $\mathbb{R}$ is the set of all real numbers, equipped with maximum $(\oplus)$ and addition $(\otimes)$ operations [3]. Max-plus algebra is a semifield denoted as $(\mathbb{R}_{maks}, \oplus, \otimes)$ with $\varepsilon = -\infty$ is the identity to the maximum operation and $e = 0$ is the identity to the addition operation [4], [5]. The set of matrix of size $m \times n$ whose components are $\mathbb{R}_\varepsilon$ elements is referred to as the set of matrix over max-plus algebra denoted as $\mathbb{R}_\varepsilon^{m\times n}$ [6]. The maximum operation has no inverse in max-plus algebra, so the determinant in max-plus algebra is not defined similarly to conventional algebra. Determinants in max-plus algebra are represented by permanent and dominant approaches [7].

Max-plus algebra is extended to interval max-plus algebra. Interval max-plus algebra is an $I(\mathbb{R})_\varepsilon$ set equipped with maximum $(\overline{\oplus})$ and addition $(\overline{\otimes})$ operations [8]. The matrix set is denoted as $I(\mathbb{R})_\varepsilon^{m\times n}$ [9]. The maximum operation in interval max-plus algebra also has no inverse, so the determinant of interval max-plus algebra is represented by two approaches, permanent and dominant [10], [11].

Min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$, where $\mathbb{R}$ is the set of all real numbers, equipped with minimum operations $(\oplus')$ and addition operation $(\otimes)$. The min-plus algebra is a semiring denoted by $\mathbb{R}_{min} = (\mathbb{R}_{\varepsilon'}, \oplus', \otimes)$ and has a neutral element $\varepsilon' = +\infty$ for the minimum operation, and $e = 0$ for the addition operation [12]. Min-plus algebra can be formed into a set of matrix of size $m \times n$, whose entries are elements of $\mathbb{R}_{\varepsilon'}$ and are denoted as $\mathbb{R}_{\varepsilon'}^{n\times n}$ [13], [14]. Similar to max-plus algebra, in min-plus algebra the determinant is defined with two approaches, permanent i.e. $perm(A) = \bigoplus'_{\sigma\in P_n} \bigotimes_{i=1}^{n} (a_{i\sigma(i)})$ with σ and P_n is the set of all permutations of $1, 2, \dots, n$ and dominant i.e. using the matrix z^A,

$$dom(A) = \begin{cases} lowest\ exponent\ in \det(z^A), if \det(z^A) \neq 0 \\ \varepsilon', if \det(z^A) = 0 \end{cases}.$$

The relationship between permanent and dominant is $dom(A) \geq perm(A)$. The bideterminant matrix is another approach to calculate the permanent [15].

Interval min-plus algebra is an extension of min-plus algebra, where the elements are closed intervals. The set of interval min-plus algebra, $I(\mathbb{R})_{\varepsilon'}$, is equipped with two binary operations, namely minimum $(\overline{\oplus'})$ and addition $(\overline{\otimes})$. Then the interval min-plus algebra is denoted as $I(\mathbb{R})_{min} = (I(\mathbb{R})_{\varepsilon'}, \overline{\oplus'}, \overline{\otimes})$ [16]. The matrix with the notation $I(\mathbb{R})_{\varepsilon'}^{m\times n}$ is a matrix over the interval min-plus algebra with size $m \times n$. If $m = n$, the set of square matrix is obtained, namely $I(\mathbb{R})_{\varepsilon'}^{n\times n}$ [17]. In this article, we will discuss the concepts of permanent and dominant matrices in interval min-plus algebra. We will explore the relationship between the permanent and dominant matrix within this algebraic structure, and additionally, we will examine bideterminant matrix under interval min-plus algebra.

2. RESEARCH METHODS

The research method used in writing this article is a literature study using references to books, journals, or writings on interval min-plus algebra, a matrix over interval min-plus algebra, and systems of linear equations over interval min-plus algebra. In addition, it also uses references that discuss min-plus algebra and its determinants.

In this study, three steps were taken, which are described below.

1. Determine the definition of a permanent and dominant matrix over interval min-plus algebra.
2. Determine the relationship between permanent and dominant matrix.
3. Determine the definition of bideterminant matrix over interval min-plus algebra.

3. RESULTS AND DISCUSSION

Determinant is the value obtained from a square matrix. In interval min-plus algebra, the minimum operation does not have an inverse, so the determinant in interval min-plus algebra is not defined similarly to conventional algebra. Permanent and dominant approaches represent determinants in interval min-plus algebra. This section describes the definition of permanent and dominant matrix over interval min-plus algebra, the relationship between permanent and dominant matrix, and the definition of bideterminant matrix over interval min-plus algebra taken from [15].

3.1 Permanent and Dominant over Interval Min-Plus Algebras

Suppose $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$, $A \approx [\underline{A}, \overline{A}]$ with $\underline{A} = \underline{a}_{ij}, \overline{A} = \overline{a}_{ij} \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Since $A \approx [\underline{A}, \overline{A}]$, then $\underline{a}_{ij} \leq \overline{a}_{ij}$ for every $i, j \in \{1, 2, \dots, n\}$. This results in $perm(\underline{A}) = \bigoplus'_{\sigma\in P_n} \bigotimes_{i=1}^{n} \left(\underline{a}_{i\sigma(i)}\right) \leq \bigoplus'_{\sigma\in P_n} \bigotimes_{i=1}^{n} \left(\overline{a}_{i\sigma(i)}\right) = perm(\overline{A})$ with σ and $P_n$is the set of all permutations of $1, 2, \dots, n$. Permanent can be defined as:

Definition 1. *Given $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}, A \approx [\underline{A}, \overline{A}]$. Permanent of A is defined as $perm(A) = [perm(\underline{A}), perm(\overline{A})]$.*

To formulate the dominant definition, matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ will be formed into a matrix z^A where z is a variable. The matrix z^A is a matrix of size $n \times n$ with entries $z^{[\underline{a}_{ij},\overline{a}_{ij}]} = [z^{\underline{a}_{ij}}, z^{\overline{a}_{ij}}]$.

Definition 2. *Given a matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$, the dominant of matrix A is defined as $dom(A) = \left[min\left(dom(\underline{A}), dom(\overline{A})\right), dom(\overline{A})\right]$ with $dom(\underline{A}) = \begin{cases} lowest\ exponent\ in\ det(z^{\underline{A}}), if\ det\ (z^{\underline{A}}) \neq 0 \\ \varepsilon', if\ det\ (z^{\underline{A}}) = 0 \end{cases}$ and $dom(\overline{A}) = \begin{cases} lowest\ exponent\ in\ det\left(z^{\overline{A}}\right), if\ det\ (z^{\overline{A}}) \neq 0 \\ \varepsilon', if\ det\ (z^{\overline{A}}) = 0 \end{cases}$.*

Based on definition and lemma exponential function in min-plus algebra [15], the variable z is replaced with e^s. Therefore, the definitions of matrix e^{sA} and $dom(A)$ are obtained as follows:

Definition 3. *Given a matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$, the matrix entry e^{sA} is $e^{[s\underline{a}_{ij}, s\overline{a}_{ij}]} = [e^{s\underline{a}_{ij}}, e^{s\overline{a}_{ij}}]$ where $\underline{a}_{ij} \in \mathbb{R}_{\varepsilon'}$ is an element of $\underline{A}$ and $\overline{a}_{ij} \in \mathbb{R}_{\varepsilon'}$ is an element of $\overline{A}$.*

Definition 4. *Let matrix $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$, the dominant of matrix A is defined as $dom(A) = \left[min\left(dom(\underline{A}), dom(\overline{A})\right), dom(\overline{A})\right]$ with $dom(\underline{A}) = \begin{cases} \lim\limits_{s\to\infty} s^{-1} ln\ |det(e^{s\underline{A}})|, if\ det\ (e^{s\underline{A}}) \neq 0 \\ \varepsilon', if\ det\ (z^{\underline{A}}) = 0 \end{cases}$ and $dom(\overline{A}) = \begin{cases} \lim\limits_{s\to\infty} s^{-1} ln\ |det\left(e^{s\overline{A}}\right)|, if\ det\ (e^{s\overline{A}}) \neq 0 \\ \varepsilon', if\ det\ (z^{\overline{A}}) = 0 \end{cases}$.*

With the exponential function, $|\det(e^{s\underline{A}})| \asymp e^{s\ dom(\underline{A})}$ and $|\det(e^{s\overline{A}})| \asymp e^{s\ dom(\overline{A})}$. The following example is given to calculate the permanent and dominant matrix.

Example 1. Given a matrix $A \in I(\mathbb{R})_{\varepsilon'}$ of size 3×3

$$A = \begin{pmatrix} [1,2] & [1,3] & [0,3] \\ [0,1] & [-1,3] & [1,2] \\ [-1,1] & [0,2] & [-2,-1] \end{pmatrix}$$

The interval matrix A can be written as interval matrix $[\underline{A}, \overline{A}]$ i.e.

$$\begin{aligned} A &\approx [\underline{A}, \overline{A}] \\ &= \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & -1 \end{pmatrix}\right] \end{aligned}$$

Therefore, the permanent of matrix $\underline{A}$ is calculated as follows

$$\begin{aligned} perm(\underline{A}) &= \oplus'_{\sigma\in P_n} \otimes_{i=1}^{n} (\underline{a}_{i\sigma(i)}) \\ &= (\underline{a}_{11} \otimes \underline{a}_{22} \otimes \underline{a}_{33}) \oplus' (\underline{a}_{11} \otimes \underline{a}_{23} \otimes \underline{a}_{32}) \oplus' (\underline{a}_{12} \otimes \underline{a}_{21} \otimes \underline{a}_{33}) \oplus' (\underline{a}_{12} \otimes \underline{a}_{23} \\ &\quad \otimes \underline{a}_{31}) \oplus' (\underline{a}_{13} \otimes \underline{a}_{21} \otimes \underline{a}_{32}) \oplus' (\underline{a}_{13} \otimes \underline{a}_{22} \otimes \underline{a}_{31}) \\ &= (1 \otimes -1 \otimes -2) \oplus' (1 \otimes 1 \otimes 0) \oplus' (1 \otimes 0 \otimes -2) \oplus' (1 \otimes 1 \otimes -1) \oplus' (0 \otimes 0 \\ &\quad \otimes 0) \oplus' (0 \otimes -1 \otimes -1) \\ &= -2 \oplus' 2 \oplus' -1 \oplus' 1 \oplus' 0 \oplus' -2 \\ &= -2 \end{aligned}$$

Next, the permanent matrix $\overline{A}$ is calculated which is written as

$$\begin{aligned} perm(\overline{A}) &= \oplus'_{\sigma\in P_n} \otimes_{i=1}^{n} (\overline{a}_{i\sigma(i)}) \\ &= (\overline{a}_{11} \otimes \overline{a}_{22} \otimes \overline{a}_{33}) \oplus' (\overline{a}_{11} \otimes \overline{a}_{23} \otimes \overline{a}_{32}) \oplus' (\overline{a}_{12} \otimes \overline{a}_{21} \otimes \overline{a}_{33}) \oplus' (\overline{a}_{12} \otimes \overline{a}_{23} \\ &\quad \otimes \overline{a}_{31}) \oplus' (\overline{a}_{13} \otimes \overline{a}_{21} \otimes \overline{a}_{32}) \oplus' (\overline{a}_{13} \otimes \overline{a}_{22} \otimes \overline{a}_{31}) \\ &= (2 \otimes 3 \otimes -1) \oplus' (2 \otimes 2 \otimes 2) \oplus' (3 \otimes 1 \otimes -1) \oplus' (3 \otimes 2 \otimes 1) \oplus' (3 \otimes 1 \otimes 2) \oplus' (3 \\ &\quad \otimes 3 \otimes 1) \\ &= 4 \oplus' 6 \oplus' 3 \oplus' 6 \oplus' 6 \oplus' 7 \\ &= 3 \end{aligned}$$

We get $perm(A) = [-2,3]$. Then determine the dominant matrix $\underline{A}$

$$\begin{aligned} \det(e^{s\underline{A}}) &= e^{s(\underline{a}_{11}+\underline{a}_{22}+\underline{a}_{33})} - e^{s(\underline{a}_{11}+\underline{a}_{23}+\underline{a}_{32})} - e^{s(\underline{a}_{12}+\underline{a}_{21}+\underline{a}_{33})} + e^{s(\underline{a}_{12}+\underline{a}_{23}+\underline{a}_{31})} + e^{s(\underline{a}_{13}+\underline{a}_{21}+\underline{a}_{32})} \\ &\quad - e^{s(\underline{a}_{13}+\underline{a}_{22}+\underline{a}_{31})} \\ &= e^{s(1+(-1)+(-2))} - e^{s(1+1+0)} - e^{s(1+0+(-2))} + e^{s(1+1+(-1))} + e^{s(0+0+0)} - e^{s(0+(-1)+(-1))} \\ &= -e^{2s} - e^{-1s} + e^{1s} + e^{0s} \\ &\neq 0 \end{aligned}$$

Since $\det(e^{s\underline{A}}) \neq 0$, $dom(\underline{A}) = -1$. Next, the dominant matrix $\overline{A}$ is calculated, namely

$$\begin{aligned} \det\left(e^{s\overline{A}}\right) &= e^{s(\overline{a}_{11}+\overline{a}_{22}+\overline{a}_{33})} - e^{s(\overline{a}_{11}+\overline{a}_{23}+\overline{a}_{32})} - e^{s(\overline{a}_{12}+\overline{a}_{21}+\overline{a}_{33})} + e^{s(\overline{a}_{12}+\overline{a}_{23}+\overline{a}_{31})} + e^{s(\overline{a}_{13}+\overline{a}_{21}+\overline{a}_{32})} \\ &\quad - e^{s(\overline{a}_{13}+\overline{a}_{22}+\overline{a}_{31})} \\ &= e^{s(2+3+(-1))} - e^{s(2+2+2)} - e^{s(3+1+(-1))} + e^{s(3+2+1)} + e^{s(3+1+2)} - e^{s(3+3+1)} \\ &= e^{4s} - e^{3s} + e^{6s} - e^{7s} \\ &\neq 0 \end{aligned}$$

Since $\det\left(e^{s\overline{A}}\right) \neq 0$, $dom(\overline{A}) = 3$. We get $dom(A) = [-1,3]$.

3.2 The Relationship between Permanent and Dominant over Min-Plus Interval Algebra

Since the permanent value is the minimum of the diagonal value for all permutations of columns in matrix A, the following theorem is obtained:

Theorem 1. *If $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$, then $dom(A) \geq perm(A)$.*

Proof. Note that $dom(A) = \left[\min\left(dom(\underline{A}), dom(\overline{A})\right), dom(\overline{A})\right]$ and $perm(A) = [perm(\underline{A}), perm(\overline{A})]$. Based on the permanent and dominant relations in min-plus algebra, if $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ then $dom(A) \geq perm(A)$. Hence, in $A \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$ it follows that $dom(\underline{A}) \geq perm(\underline{A})$ and $dom(\overline{A}) \geq perm(\overline{A})$. By definition there are two possibilities, namely:

a. $dom(\underline{A}) < dom(\underline{A})$

$$dom(A) = [dom(\underline{A}), dom(\overline{A})] \geq [perm(\underline{A}), perm(\overline{A})] = perm(A)$$

b. $dom(\underline{A}) > dom(\underline{A})$

Based on the statement

$$dom(\overline{A}) \geq perm(\overline{A}) \geq perm(\underline{A})$$
$$dom(\overline{A}) \geq perm(\underline{A})$$

obtained

$$dom(A) = [dom(\overline{A}), dom(\overline{A})] \geq [perm(\underline{A}), perm(\overline{A})] = perm(A)$$

therefore, it is obtained:

$$dom(A) = \left[\min\left(dom(\underline{A}), dom(\overline{A})\right), dom(\overline{A})\right]$$
$$\geq \left[perm(\underline{A}), perm(\overline{A})\right] = perm(A).$$

3.3 Bideterminant Matrix over Min-Plus Interval Algebra

The following defines the bideterminant of matrix A as another approach to calculate the permanent as follows:

Definition 5. *Given $A = (a_{ij}) \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$. Suppose that $w_{\underline{A}}(\sigma) = \underline{a}_{1\sigma(1)} \otimes \underline{a}_{2\sigma(2)} \otimes \ldots \otimes \underline{a}_{n\sigma(n)}$ and $w_{\overline{A}}(\sigma) = \overline{a}_{1\sigma(1)} \otimes \overline{a}_{2\sigma(2)} \otimes \ldots \otimes \overline{a}_{n\sigma(n)}$, P_n^e and P_n^0 are the set of even and odd permutations of {1,2,.} respectively. The bideterminant of matrix A is $bidet(A) = \begin{pmatrix}[\Delta_1'(\underline{A}), \Delta_1'(\overline{A})]\\ [\Delta_2'(\underline{A}), \Delta_2'(\overline{A})]\end{pmatrix}$ with $\Delta_1'(\underline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\underline{A}}(\sigma), \Delta_1'(\overline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\overline{A}}(\sigma), \Delta_2'(\underline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\underline{A}}(\sigma)$, and $\Delta_2'(\overline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\overline{A}}(\sigma)$.*

According to the definition of bideterminant and permanent, the following theorem is obtained:

Theorem 2. *Given $A = (a_{ij}) \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$. If the bideterminant of matrix A is $bidet(A) = \begin{pmatrix}[\Delta_1'(\underline{A}), \Delta_1'(\overline{A})]\\ [\Delta_2'(\underline{A}), \Delta_2'(\overline{A})]\end{pmatrix}$ with $\Delta_1'(\underline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\underline{A}}(\sigma), \Delta_1'(\overline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\overline{A}}(\sigma), \Delta_2'(\underline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\underline{A}}(\sigma)$, and $\Delta_2'(\overline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\overline{A}}(\sigma)$, P_n^e and P_n^0 are the set of even and odd permutations of {1,2,.} respectively, then $perm(A) = [perm(\underline{A}), perm(\overline{A})]$ with $perm(\underline{A}) = \Delta_1(\underline{A}) \oplus' \Delta_2(\underline{A})$ and $perm(\overline{A}) = \Delta_1(\overline{A}) \oplus' \Delta_2(\overline{A})$.*

Proof. Given $A = (a_{ij}) \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$ and $\underline{A} = (\underline{a}_{ij})$, $\overline{A} = (\overline{a}_{ij}) \in \mathbb{R}_{\varepsilon'}^{n\times n}$.

$$bidet(A) = \begin{pmatrix}[\Delta_1'(\underline{A}), \Delta_1'(\overline{A})]\\ [\Delta_2'(\underline{A}), \Delta_2'(\overline{A})]\end{pmatrix}$$
$$\approx \left[\begin{pmatrix}\Delta_1'(\underline{A})\\ \Delta_2'(\underline{A})\end{pmatrix}, \begin{pmatrix}\Delta_1'(\overline{A})\\ \Delta_2'(\overline{A})\end{pmatrix}\right] = [bidet(\underline{A}), bidet(\overline{A})]$$

Thus $bidet(\underline{A}) = \begin{pmatrix}\Delta_1'(\underline{A})\\ \Delta_2'(\underline{A})\end{pmatrix}$ and $bidet(\overline{A}) = \begin{pmatrix}\Delta_1'(\overline{A})\\ \Delta_2'(\overline{A})\end{pmatrix}$. Therefore, the permanent matriks A is $perm(A) = [perm(\underline{A}), perm(\overline{A})]$ with $perm(\underline{A}) = \Delta_1(\underline{A}) \oplus' \Delta_2(\underline{A})$ and $perm(\overline{A}) = \Delta_1(\overline{A}) \oplus' \Delta_2(\overline{A})$.

The following example for calculating the permanent matrix is given using **Theorem 2**.

Example 2. Given an example of bideterminant with matrix $A \in I(\mathbb{R})_{\varepsilon'}$ of size 3×3

$$A = \begin{pmatrix}[1,2] & [1,3] & [0,3]\\ [0,1] & [-1,3] & [1,2]\\ [-1,1] & [0,2] & [-2,-1]\end{pmatrix}$$

The interval matrix A can be written as interval matrix $[\underline{A}, \overline{A}]$ i.e.

$$A \approx [\underline{A}, \overline{A}]$$
$$= \left[\begin{pmatrix}1 & 1 & 0\\ 0 & -1 & 1\\ -1 & 0 & -2\end{pmatrix}, \begin{pmatrix}2 & 3 & 3\\ 1 & 3 & 2\\ 1 & 2 & -1\end{pmatrix}\right]$$

Based on the matrix A, $w_{\underline{A}} = \{-2,2,-1,1,0,-2\}$ and $w_{\overline{A}} = \{4,6,3,6,6,7\}$ are obtained. Furthermore, the values of $\Delta_1'(\underline{A}), \Delta_1'(\overline{A}), \Delta_2'(\underline{A})$, and $\Delta_2'(\overline{A})$ are calculated, namely

$$\Delta_1'(\underline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\underline{A}}(\sigma) = \min(-2,1,0) = -2,$$
$$\Delta_1'(\overline{A}) = \bigoplus'_{\sigma\in P_n^e} w_{\overline{A}}(\sigma) = \min(4,6,6) = 4,$$
$$\Delta_2'(\underline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\underline{A}}(\sigma) = \min(2,-1,-2) = -2, \text{ and}$$
$$\Delta_2'(\overline{A}) = \bigoplus'_{\sigma\in P_n^0} w_{\overline{A}}(\sigma) = \min(6,3,7) = 3$$

Thus obtained the bideterminant of matrix A is

$$bidet(A) = \begin{pmatrix}[\Delta_1'(\underline{A}), \Delta_1'(\overline{A})]\\ [\Delta_2'(\underline{A}), \Delta_2'(\overline{A})]\end{pmatrix}$$

$$= \begin{pmatrix} [-2,4] \\ [-2,3] \end{pmatrix}$$

Using **Theorem 2**, we get

$$perm(\underline{A}) = \Delta_1(\underline{A}) \oplus' \Delta_2(\underline{A}) = \min(-2,-2) = -2$$
$$perm(\overline{A}) = \Delta_1(\overline{A}) \oplus' \Delta_2(\overline{A}) = \min(4,3) = 3$$
$$perm(A) = [-2,3].$$

4. CONCLUSIONS

According to the results and discussion, permanent and dominant formulas are obtained, dominant is always greater or equal to permanent, and bideterminant formulas are obtained.

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