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PARAMETER ESTIMATION OF LOGNORMAL AND PARETO TYPE I DISTRIBUTIONS USING FREQUENTIST AND BAYESIAN INFERENCES

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ABSTRACT

Extreme events are events that rarely occur but they cause substantial losses. Insurance companies need to take extreme events into account in risk management because extreme events can have a negative impact on the company's financial health. As a result, insurance companies need an appropriate loss model that matches the empirical data from these extreme events. A distribution that is heavy-tailed and skewed to the right is a good distribution for modeling the magnitude of losses from extreme events. In this paper, two distributions with heavy tails and skew to the right will be used to model the magnitude of losses from extreme events, namely the lognormal distribution and the Pareto distribution type I. The parameters of these distributions are estimated using two inferences, namely the frequentist and Bayesian inferences. In the frequentist inference, two methods are applied, namely the moment method and maximum likelihood. On Bayesian inference, two prior distributions are used, namely uniform and Jeffrey. Test model suitability is carried out by visually comparing the model distribution function with the empirical distribution function, as well as by comparing the Root Mean Square Error (RMSE) value. The visualization results of the distribution function and RMSE values show that in general, the Bayesian inference is better at estimating parameters than the frequentist inference. In the frequentist inference, the maximum likelihood method can provide better estimated values than the moment method. In the Bayesian inference, the two prior distributions show a relatively similar fit to the data and tend to be better than the frequentist inference.

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1. INTRODUCTION

Risk is the possibility of experiencing various kinds of losses, such as financial losses that can be controlled or those that cannot be controlled [1]. The financial loss is caused by a detrimental event. Risk can be transferred from one party to another party through an insurance agreement that follows legal rules and the application of related principles in accordance with the agreement of the parties involved. An insurance agreement between the insurer, namely the insurance company and the insured party, is usually also called a policy. The insured party needs to pay the cost of risk coverage, called a premium, to the insurance company because the insurance company has taken over the risk from the insured party.

Insurance companies cover various risks that tend to be uncertain, especially in cases where the amount of loss is relatively large and the frequency of occurrence is relatively small. This case can be said to be an extreme case. The characteristics of extreme cases can be captured by heavy-tailed distributions because these distributions take into account greater probability values at relatively large loss values compared to light-tailed distributions. Even though relatively large losses tend to occur rarely, the probability value of this event still needs to be observed because it can have a large loss impact on the insurance company if it is not estimated properly. Some parametric distributions with heavy tails are lognormal, Pareto type I, Weibull, and Cauchy distributions. In this paper, the two most frequently used heavy-tailed distributions will be used, namely the lognormal distribution [2] and Pareto type I [3].

Distribution models can be obtained by estimating the parameters of each distribution used. There are two inferences that can be used to estimate parameters, namely the frequentist and Bayesian inferences. The frequentist inference is an inference whose distribution parameters are constant, while the Bayesian inference is an inference whose parameters follow a certain distribution. In research [4], it has been discussed regarding parameter estimation from the Weibull distribution using the maximum likelihood method, variance and covariance estimation, Bayesian inference, Laplace approximation, and Lindley approximation. Several studies related to parameter estimation of the Weibull distribution using those methods have also been conducted by [5]-[19] In this research, other distributions are used, namely the lognormal and Pareto type I distributions using a frequentist inference with moment and maximum likelihood methods, as well as a Bayesian inference with uniform and Jeffrey prior distributions. Results of parameter estimation using various methods will be compared by applying large loss data. The estimated results will be compared with the actual data used to see the distribution and inference that best estimates the parameters for estimating the amount of loss.

This research refers to several previous studies that discussed parameter estimation with the Pareto type I distribution using a frequentist inference and a Bayesian inference. The frequentist inference uses the moment and maximum likelihood methods, while the Bayesian inference uses a uniform prior distribution. Then, the obtained parameter estimates are compared using data simulation. This paper will discuss the Pareto type I distribution using a Bayesian inference with a non-informative prior distribution, namely the Jeffrey prior distribution [20], and discuss parameter estimation with a lognormal distribution using a Bayesian inference where the prior distribution used is a uniform prior distribution. In this paper, development will be carried out using a non-informative prior distribution, namely the Jeffrey prior [21]. By developing the methods used to estimate parameters, this paper will also use large-loss data to analyze methods that can estimate parameters well.

2. RESEARCH METHODS

This research is quantitative research using lognormal distribution modeling and the Pareto I distribution. The parameters of this distribution are estimated using two inferences, namely the frequentist and Bayesian inferences. In the frequentist inference, the moment and maximum likelihood methods are applied. Then, the Bayesian inference uses two prior distributions, namely uniform and Jeffrey. The model suitability test was carried out by visually comparing the model distribution function with the empirical distribution function, as well as by comparing the Root Mean Square Error (RMSE) value.

2.1 Probability Density Function (PDF)

In this research, we utilize two distributions characterized by heavy tails. The first distribution is based on the assumption that the random variable X follows a lognormal distribution, with a scale parameter μ and shape parameter σ. The probability density function for this distribution is given by:

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \tag{1}$$

with $x > 0$; $-\infty < \mu < \infty$; $\sigma > 0$. The second distribution is the Pareto type I distribution, characterized by a shape parameter α and location parameter k. The probability density function for this distribution is expressed as follows:

$$f(x;\mu,\sigma) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}} \tag{2}$$

with $0 < k < x < \infty$; $\alpha > 0$.

2.2 Fisher Information

Fisher information is a way to measure the amount of information provided by the observed random variable X about unknown parameters. If the parameter is at the endpoint of the random variable interval, then the parameter value will be fixed and known so that Fisher information cannot be calculated **[22]**. We suppose that the parameter you want to estimate is θ and the parameter is not at the endpoint of the interval of the random variable.

2.3 Bayesian Inference

The Bayesian inference is an inference that combines information from sample data and information from previous data, thereby producing a conclusion. In the Bayesian inference, there is a prior distribution and a posterior distribution **[23]**. The posterior distribution is a distribution that combines the prior distribution and the likelihood function. In this research, two non-informative prior distributions will be used, namely the Jeffrey prior distribution and the uniform prior distribution.

The first prior distribution applied is Jeffrey's prior, which is derived using the formula

$$\pi(a) \propto \sqrt{I(a)} = \sqrt{-E\left(\frac{\partial^2 \ln f(x;a,b)}{\partial a^2}\right)}, \tag{3}$$

with the assumption that only parameter a is unknown. The second prior distribution used is the uniform distribution with $a = 0$ and $b = 1$, resulting in the following expression for the prior distribution:

$$\pi(a) \propto 1. \tag{4}$$

2.4 *Root Mean Square Error* (RMSE)

Root Mean Square Error (RMSE) in **Equation (5)** is the root of the average squared difference between observed or actual data and predicted data.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n}} \tag{5}$$

n: the number of data, y_i: observation data point i, $\hat{y}_i$: estimation data point i.

The value of RMSE can measure how good the prediction results produced by a model are. With the resulting RMSE value getting smaller, it can be concluded that the model can predict the observed data nicely.

2.5 Parameters Estimation of the Lognormal Distribution with the Frequentist Inference Using Method of Moments

The application of the method of moments yielded **Equation (6)** and **Equation (7)** for parameters estimation in the lognormal distribution:

$$\hat{\mu} = 2\ln\left(\sum_{i=1}^{n} x_i\right) - \frac{\ln\left(\sum_{i=1}^{n} x_i^2\right)}{2} - \frac{3}{2}\ln n \quad (6)$$

$$\hat{\sigma} = \ln\left(\sum_{i=1}^{n} x_i^2\right) - 2\ln\left(\sum_{i=1}^{n} x_i\right) + \ln n \quad (7)$$

n: the number of data, x_i: observation data point i.

2.6 Parameters Estimation of the Lognormal Distribution with the Frequentist Inference Using Maximum Likelihood Method

The parameters of the lognormal distribution were estimated using the maximum likelihood method, resulting in **Equation (8)** and **Equation (9)**:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{2} \quad (8)$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (\ln x_i - \hat{\mu})^2}{n} \quad (9)$$

n: the number of data, x_i: observation data point i.

2.7 Parameters Estimation of the Lognormal Distribution with a Bayesian Inference Using Uniform Prior Distribution

Estimation of the parameters of the lognormal distribution using the Bayesian inference with a uniform prior distribution yielded **Equation (10)** and **Equation (11)**:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{2} \quad (10)$$

$$\hat{\sigma}^2 = \frac{\beta}{n-5} \quad (11)$$

β is defined in **Equation (12)**:

$$\beta = \sum_{i=1}^{n} (\ln x_i)^2 - \frac{\left(\sum_{i=1}^{n} \ln x_i\right)^2}{n} \quad (12)$$

n: the number of data, x_i: observation data point i.

2.8 Parameters Estimation of The Lognormal Distribution with a Bayesian Inference Using Jeffrey Prior Distribution

Estimation of the parameters of the lognormal distribution using the Bayesian inference with Jeffrey's prior distribution yielded **Equation (13)** and **Equation (14)**:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln x_i}{n} \quad (13)$$

$$\hat{\sigma}^2 = \frac{\beta}{n-2} \quad (14)$$

n: the number of data, x_i: observation data point i, and β is defined in **Equation (12)**.

2.9 Parameters Estimation of the Pareto Type I Distribution with the Frequentist Inference Using Method of Moments

The application of the method of moments yielded **Equation (15)** and **Equation (16)** for parameters estimation in the Pareto type I distribution:

$$\hat{\alpha} = \frac{n\bar{X} - x_{min}}{n(\bar{X} - x_{min})} \tag{15}$$

$$\hat{k} = \frac{x_{min}(n\alpha - 1)}{n\alpha} \tag{16}$$

n: the number of data, x_{min}: the minimum value of the observation data, $\bar{X}$: the mean value of the data.

2.10 Parameters Estimation of the Pareto Type I Distribution with the Frequentist Inference Using the Maximum Likelihood Method and Bayesian Inference Using Jeffrey Prior Distribution

The parameters of the Pareto type I distribution were estimated using the maximum likelihood method, resulting in **Equation (17)**:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln x_i \; - \; n \ln k} \tag{17}$$

We assume that the parameter $\hat{k}$ is already known and defined in **Equation (18)**:

$$\hat{k} = \min(x_1, \ldots, x_n) \tag{18}$$

n: the number of data, x_i: observation data point i.

2.11 Parameters Estimation of the Pareto Type I Distribution with a Bayesian Inference Using Uniform Prior Distribution

Estimation of the parameters of the Pareto type I distribution using the Bayesian inference with uniform prior distribution yielded **Equation (19)**:

$$\hat{\alpha} = \frac{n+1}{\sum_{i=1}^{n} \ln x_i \; - \; n \ln \hat{k}} \tag{19}$$

n: the number of data, x_i: observation data point i, and $\hat{k}$ is defined in **Equation (18)**.

3. RESULTS AND DISCUSSION

3.1 Big Data Losses

The results of this research will present two datasets containing large loss data from two extreme events that have occurred previously. The data will later be modeled using two distributions, namely the lognormal distribution and Pareto type I. Meanwhile, the first data presented in **Table 1** is data on large losses from natural disasters caused by wind in the United States in 1977 in millions of US dollars **[24]**.

Table 1. Large Losses Caused by Natural Disasters are Related to Wind

2	2	2	2	2	2	2	2
2	2	2	2	3	3	3	3
4	4	4	5	5	5	5	6
6	6	6	8	8	9	15	17
22	23	24	24	25	27	32	43

Based on the data in **Table 1**, the following is a histogram of the data:

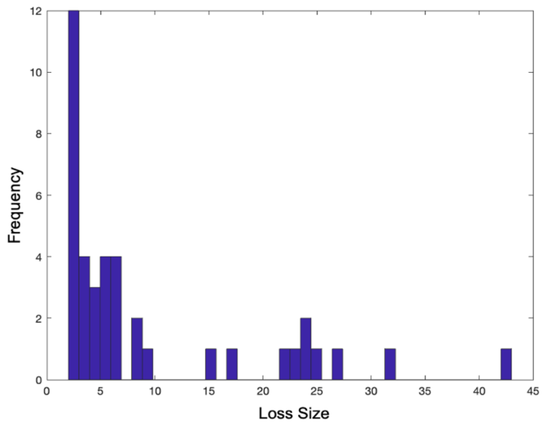


Figure 1. **First Data Histogram of Loss Size**

Based on **Figure 1**, it is known that the histogram is skewed to the right, where a large frequency of events results in small losses and a small frequency of events results in large losses. From this data, the average value of the data stated in $\bar{X}$ is 9.225 and the variance of this data is 102.1744. Then, for the second data, we will present data on the magnitude of losses from fire incidents in Norway in 1975 in units of thousands of Norwegian krone [25].

Table 2. **Large Losses Caused by Fire**

500	500	500	502	515	515	528	530	530	530	540	544
550	550	551	552	557	558	570	572	574	579	583	584
586	593	596	596	600	600	600	605	610	610	613	615
620	622	632	635	635	640	650	650	650	650	672	674
680	700	725	728	736	737	740	748	752	756	756	777
798	800	800	800	826	835	862	885	900	900	910	912
927	940	940	948	957	1000	1002	1009	1013	1020	1024	1033
1038	1041	1104	1108	1137	1143	1180	1243	1248	1252	1280	1285
1291	1293	1298	1300	1305	1327	1387	1455	1475	1479	1485	1491
1515	1519	1587	1700	1708	1820	1822	1848	1906	2110	2251	2362
2497	2690	2760	2794	2886	2924	2953	3289	3860	4016	4300	4397
4585	4810	6855	7371	7771	7834	13000	13484	17237	526000		

Based on the data in **Table 2**, the following histogram of the data is presented:

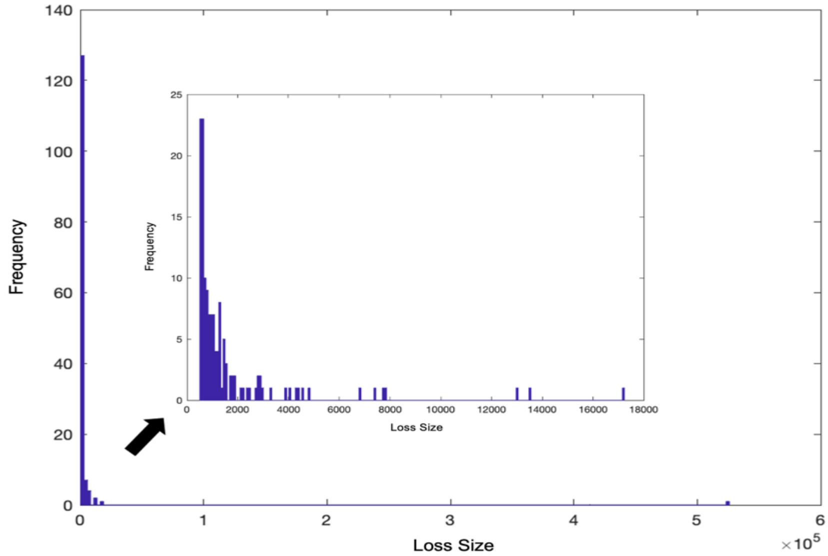


Figure 2. **Second Data Histogram of Loss Size**

Figure 2 shows that the data has characteristics skewed to the right where the frequency of occurrence of small losses has a higher frequency than large losses. From this data, the average of the data stated in $\bar{X}$ is 5,351 and the variance is 1,928,048,222.

3.2 Parameter Estimation Results Using First Data

Based on the data presented in **Table 1**, the parameters were estimated for both distributions through various methods.

Table 3. Parameter Estimation Results from Lognormal and Pareto Type I Distributions with Frequentist and Bayesian Inferences Using First Data

Distribution	Parameter	Frequentist Inference		Bayesian Inference	
		Moment Method	**Maximum *Likelihood* Method**	**Distribution Prior Uniform**	**Distribution Prior Jeffrey**
Lognormal	$\hat{\mu}$	1.8275	1.7174	1.7174	1.7174
	$\hat{\sigma}^2$	0.7887	0.9345	1.0681	0.9837
Pareto Type I	$\hat{\alpha}$	1.2699	0.9763	1.0007	0.9763
	$\hat{k}$	1.9606	2	2	2

Based on the equation calculations that have been carried out through parameter estimates that have been determined in sub-chapters 2.4 to 2.10, it is known that the results of the parameter estimation from the Lognormal and Pareto Distribution type I with the Frequentist and Bayesian Inference using the first data are shown in **Table 3**.

3.3 Parameter Estimation Results Using Second Data

Based on the data presented in **Table 2**, the parameters were estimated for both distributions through various methods.

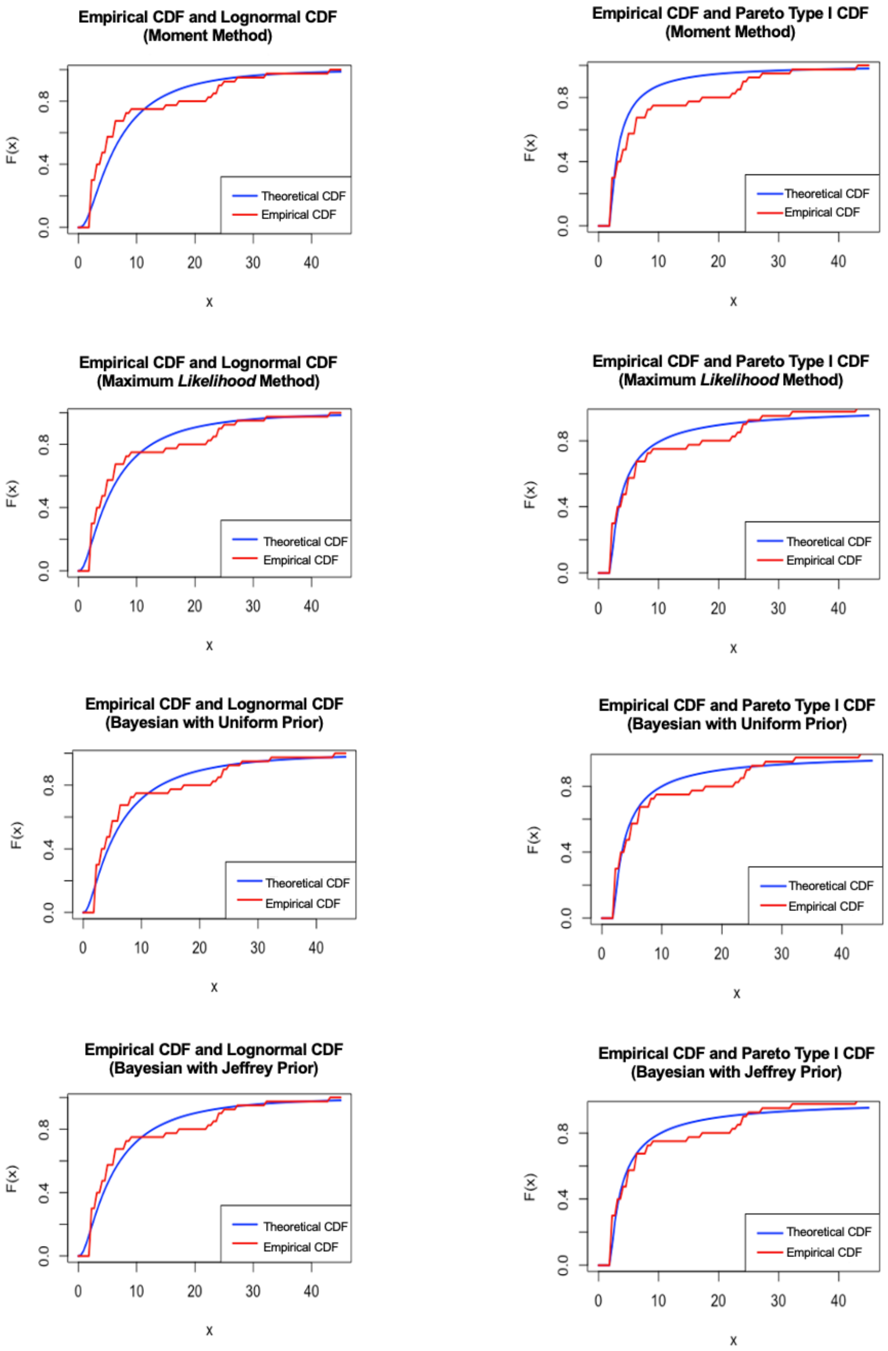
Table 4. Parameter Estimation Results from Lognormal and Pareto Type I Distributions with Frequentist and Bayesian Inferences Using Second Data

Distribution	Parameter	Frequentist Inference		Bayesian Inference	
		Moment Method	**Maximum *Likelihood* Method**	**Distribution Prior Uniform**	**Distribution Prior Jeffrey**
Lognormal	$\hat{\mu}$	6.4731	7.0521	7.0521	7.0521
	$\hat{\sigma}^2$	4.2242	0.8294	0.8597	0.8413
Pareto Type I	$\hat{\alpha}$	1.1023	1.1940	1.2024	1.1940
	$\hat{k}$	496.8057	500	500	500

Based on the equation calculations that have been carried out through the parameter estimates that have been determined in sub-chapters 2.4 to 2.10, it is known that the results of the parameter estimation from the Lognormal and Pareto Distribution type I with the Frequentist and Bayesian Inference using the second data are shown in **Table 4**.

3.4 Model Fit Test Using Distribution Function

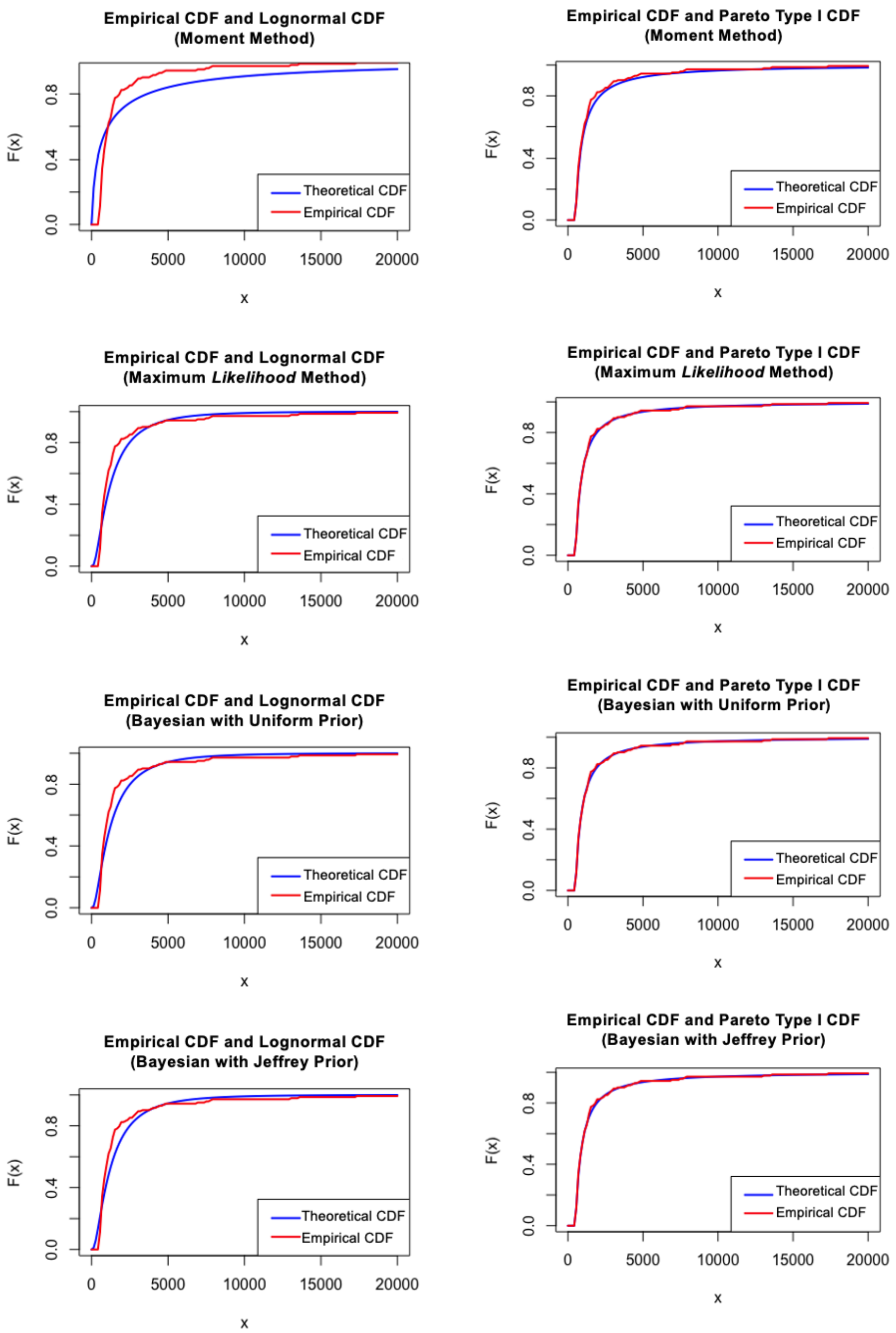
Following this, the visual representation of the model distribution function will be juxtaposed with the empirical data distribution function to enable further analysis.



(a) (b)

Figure 3. This Figure Compares the Distribution Functions for the First Dataset Using (a) the Lognormal Distribution and (b) the Pareto Type I Distribution

From the empirical data and model in **Figure 3**, the model uses lognormal and Pareto type I distributions so that the differences in each model visualization result can be seen. In the visualization results of a model with a lognormal distribution using the maximum likelihood method, it is clear that the maximum likelihood method can obtain results that are closer to empirical data than the moment method. Based on this visualization, the differences between the Bayesian inference with uniform and Jeffrey prior distributions are not clearly visible so it is difficult to determine a better prior distribution, however, the Bayesian inference still shows better results when compared to the frequentist inference with the method of moments. The model with the Pareto distribution type I using a frequentist inference that is closest to empirical data is the model that uses the maximum likelihood method. The model with the Pareto distribution type I using the Bayesian inference does not show a significant difference. The frequentist inference uses the maximum method likelihood, but the Bayesian inference shows better results than the frequentist inference with the method of moments. Therefore, it can be concluded that the frequentist inference with the moment method provides the worst model parameter estimation results.



(a) **(b)**

Figure 4. This Figure Compares the Distribution Functions for the Second Dataset Using (a) the Lognormal Distribution and (b) the Pareto Type I Distribution

Based on the model visualization results in Figure 4, you can see the model visualization results of the lognormal and Pareto type I distributions using the frequentist and Bayesian inferences. In the lognormal distribution with a frequentist inference using the maximum likelihood method, it can be seen that the distribution function of this model can approximate the distribution function of the actual data better than the frequentist inference using the moment method. In the visualization results of the model with a lognormal distribution using the Bayesian inference, it is not clear which prior distribution is better. However, the Bayesian inference with both prior distributions remains better compared to the sequential inference using the method of moments and does not show much difference with the frequentist inference using the maximum likelihood method. In the visualization results of the model with Pareto type I distribution, it is shown that the frequentist inference with the maximum likelihood method is also better when compared to the moment method. This can be seen because the lines of the distribution function of the empirical data and the model coincide with each other in the model with frequentist inference using the maximum likelihood method. The results of model visualization with the Pareto type I distribution using the Bayesian inference show quite small differences from the two prior distributions used, but the Bayesian inference produces better model parameter estimates when compared to the frequentist inference using the moment method.

3.5 Test Model Suitability by Comparing RMSE Values

Based on the previously obtained parameter estimation results, the adequacy of the estimates is evaluated by comparing the values derived from parameter estimation in **Table 3** and **Table 4** with the actual values in **Table 1** and **Table 2**, resulting in the calculation of the RMSE as defined in **Equation 5**.

Table 5. Comparison of RMSE Values from Each Distribution with Frequentist and Bayesian Inferences Using First Data

	Frequentist Inference		Bayesian Inference	
	Moment Method	**Maximum *Likelihood* Method**	**Distribution Prior Uniform**	**Distribution Prior Jeffrey**
Lognormal Distribution	0.0739	0.0567	0.0530	0.0552
Pareto Type I Distribution	0.0722	0.0558	0.0566	0.0558

Based on **Table 5**, the parameter estimation results using the frequentist inference are performed better by the maximum likelihood method, which is indicated by the RMSE value with the lognormal and Pareto type I distribution which is smaller in the maximum likelihood method compared to the moment method. For the Bayesian inference, the estimation results with the uniform prior distribution are better than the Jeffrey prior distribution for the lognormal distribution. The estimation results with the Bayesian inference using the Jeffrey prior distribution are better than the uniform prior distribution when estimating parameters from the Pareto type I distribution. Overall, the Bayesian inference can estimate parameters better than the frequentist inference. So, the lognormal distribution with a Bayesian inference using a uniform prior distribution produces the best model for the first data.

Table 6. Comparison of RMSE Values from Each Distribution with Frequentist and Bayesian Inferences Using Second Data

	Frequentist Inference		Bayesian Inference	
	Moment Method	**Maximum *Likelihood* Method**	**Distribution Prior Uniform**	**Distribution Prior Jeffrey**
Lognormal Distribution	0.1694	0.0921	0.0928	0.0924
Pareto Type I Distribution	0.0301	0.0165	0.0156	0.0165

Based on **Table 6**, the frequentist inference using the maximum likelihood method can estimate parameters better when compared to the moment method. The RMSE value of the parameter estimates from the lognormal distribution with a Bayesian inference using two prior distributions has relatively small differences, but Jeffrey's prior distribution is slightly better. In the Pareto type I distribution using the Bayesian inference, the parameter estimation results were performed better using the uniform prior distribution even though it did not show a relatively large difference with the Jeffrey prior distribution. Parameter estimation with a lognormal distribution for the second data is best carried out by a frequentist inference using the maximum likelihood method even though the RMSE value has a relatively small difference when compared with the RMSE value from the parameter estimation results using the Bayesian inference. Parameter estimation with a Pareto type I distribution is best carried out by Bayesian inference with a uniform prior distribution where the parameter estimates produce a model the best for the second data.

4. CONCLUSIONS

Conclusions that can be drawn from this research include the following: 1) The frequentist inference carried out using the moment and maximum likelihood methods shows that the maximum likelihood method is better than the moment method. The Bayesian inference can estimate parameters better when compared to the frequentist inference using the moment method. The Bayesian inference with uniform and Jeffrey prior distributions has relatively small differences because the prior distribution depends on the data used. 2) The model visualization results show that the results of the model using the Bayesian inference and the frequentist inference using the maximum likelihood method show relatively small differences with empirical data. For the first data, the Bayesian inference for lognormal and Pareto type I distributions does not show much difference from the results obtained by the frequentist inference using the maximum likelihood method. In general, the Bayesian inference produces better parameter estimates compared to the frequentist inference, although there are conditions where the lognormal distribution model with the frequentist inference using the maximum likelihood method is better than the Bayesian inference on the second data. 3) The first large loss data can be modeled best by a lognormal distribution with a Bayesian inference using a uniform prior distribution, while the second large loss data can be best modeled by a Pareto type I distribution with a Bayesian inference using a uniform prior distribution. So, the two large loss data used are best modeled with two different distributions using the same inference and prior distribution.

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