BAREKENG: Journal of Mathematics and Its Applications
December 2024 Volume 18 Issue 4 Page 2543–2554
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss4pp2543-2554

HOW MANY SUBSETS WHICH THEIR OPERATIONS WITH A FIXED SUBSET CONTAIN AT LEAST ONE ELEMENT OF A GIVEN COLLECTION?

Muhammad Nurul Huda[1*], Veni Rizki Lestari[2]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada*
Jl. Geografi, Sekip Utara BLS 21, Yogyakarta, 55821, Indonesia

[2]*Ion Operation Unit Graha, Gamping, Sleman, Yogyakarta, Indonesia*

*Corresponding author's e-mail: *muhammadnurulhuda2000@mail.ugm.ac.id*

Article History:

Received: 2nd, May 2024
Revised: 1st, July 2024
Accepted: 6th,August 2024
Published:14th, October 2024

Keywords:

Binary Set Operations;
Bonferroni's Inequality;
Corrádi's Lemma;
Hypergraph;
Inclusion-Exclusion Principle;

ABSTRACT

We pose the following problem related to binary set operations on finite sets. Given a finite set F. Let a binary set operation $*$ and $\mathcal{L}$ be a non-empty collection of non-empty subsets of F. For a fixed subset A of F, where $|A| \geq \max_{X \in \mathcal{L}} |X|$, how many subsets of F which their operation $*$ with A contains at least one element of $\mathcal{L}$?. In this paper, we give the solution of this problem, especially for the subsets of size $|A|$, using the inclusion-exclusion principle, Corrádi's lemma, and Bonferroni's inequality. In this context, the problem is related to determining the degree of nodes in certain graphs, such as graphs constructed with the adjacency rule depends on $*$ and the node set is a hypergraph.

How to cite this article:

M.N. Huda and V.R. Lestari, "HOW MANY SUBSETS WHICH THEIR OPERATIONS WITH A FIXED SUBSET CONTAIN AT LEAST ONE ELEMENT OF A GIVEN COLLECTION?," *BAREKENG: J. Math. & App.*, vol. 18, iss. 4, pp. 2543-2554, December, 2024.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article • Open Access

1. INTRODUCTION

In set theory, one of the tools for determining the size of the set unions of finite sets is the inclusion-exclusion principle. For any finite collection of finite sets $\{X_i\}_{i\in I}$,

$$\left|\bigcup_{i\in I} X_i\right| = \sum_{\emptyset\neq J\subseteq I} (-1)^{|J|-1} \left|\bigcap_{j\in J} X_j\right|$$

It can be challenging to perform calculations involving a large number of distinct sets. In such cases, it might be sufficient to determine their bounds. Bonferroni's inequality is a commonly used tool in combinatorics and probability theory to establish the upper bounds (see [1]). For any positive integer m, the classical Bonferroni's inequality, with a measure μ, is stated as follows:

$$\mu\left(\bigcup_{i\in I} X_i\right) \leq \sum_{J\subseteq I;\, 0<|J|\leq q} (-1)^{|J|-1} \mu\left(\bigcap_{j\in J} X_j\right)$$

if q is odd, and we get the lower bound for the otherwise. In terms of probability measure ($\mu = \mathbf{P}$), let $\{X_i\}_{i\in I}$ be a collection of events. When $q = 1$, the inequality $\mathbf{P}(\bigcup_{i\in I} X_i) \leq \sum_{i\in I} \mathbf{P}(X_i)$ is known as the union bound or also as Boole's inequality for finite events. We call Bonferroni's inequality with measure 'Cardinality of set' especially for $q = 1$ as simplex Bonferroni's inequality. Another useful tool is Corrádi's lemma (see [2]), which provides the lower bound and is often optimal, especially in cases like a projective plane (see [3]). For the finite collection $\{X_i\}_{i\in I}$, the lemma asserts as follows, if $|X_i| \geq k$ and $|X_i \cap X_j| \leq \lambda$ for all i, j where $i \neq j$, then

$$\left|\bigcup_{i\in I} X_i\right| \geq \frac{|I|k^2}{k + (|I| - 1)\lambda}.$$

A couple of years ago, Li, Broersma, and Wang in [4] proved the generalized type of Corrádi's lemma, which they applied to study the growth rates of the Erdős-Gyarfas function. However, it involves somewhat detailed assumptions and can lead to more complicated results.

For brevity, we denote $\binom{X}{k}$ as the collection of all subsets having k elements of the set X. Let G be a *graph*, denoted by $G = (V_G, E_G)$, with V_G and E_G as the node set and the edge set of G, respectively. The edge of G is represented by a subset of $\binom{V_G}{2}$. The degree of node v, denoted by $d_G(v)$, represents the number of nodes in G which are adjacent to v. The generalization of a graph G is called a *hypergraph*, which was introduced by Claude Berge (see [5]). A hypergraph is a pair (P, Q) where P is a set of nodes and Q is a collection of hyperedges (another name for edges in hypergraph) satisfying $\emptyset \neq R \subseteq 2^P$, for all $R \in Q$, such that $\bigcup_{R\in Q} R \subseteq P$. In other words, an ordinary graph and a hypergraph both differ in the form of their edges, where the edge of an ordinary graph connects exactly two nodes, but the edge of a hypergraph can connect more than one node. There has been extensive study on hypergraphs, for example [6]–[10].

Let G be any graph. The non-uniform subset graph associated with G was introduced in [11]. It is a graph whose node set is $\binom{V_G}{k}$ and any two distinct nodes are joined by an edge if and only if their intersection contains at least one edge of G. Additionally, the author of [11] has studied various graph properties, including clique number, girth, bipartiteness, and hamiltonicity, or the non-uniform subset graph. On the other hand, there is the token graph of G which was introduced by Fabila-Monroy et al. (see [12]). It is a graph whose node set is $\binom{V_G}{k}$ and any two distinct nodes are joined by an edge if and only if their symmetric difference is an edge of G. The authors of [12] have studied various graph properties of the token graph, including chromatic number and hamiltonicity, and their research has been continued by other authors, for example [13]–[20]. It is clear that non-uniform subset graphs and token graphs are not hypergraphs themselves, but they can be associated with hypergraphs.

We pose the following problem related to binary set operations on finite sets.

Problem 1. Given a finite set F. Let $*$ be a binary set operation and $\mathcal{L}$ be a non-empty collection of non-empty subsets of F. For a fixed subset A of F, where $|A| \geq \max_{X \in \mathcal{L}} |X|$, how many subsets which their operation $*$ with A contains at least one element of $\mathcal{L}$?.

In general, when we are given two systems $\mathcal{G}$ and $\mathcal{H}$, such that $\mathcal{H}$ is depending on $\mathcal{G}$. There is also defined a relation $\sim$ between elements of $\mathcal{G}$. For a fixed one element in $\mathcal{G}$, determine the number of other elements in $\mathcal{G}$ which the relation $\sim$ with the fixed one is well-defined in $\mathcal{H}$. For instance, determining the degree of every node on Cayley graphs. For a given finite group G' and $S \subset G'$, where $S = S^{-1}$, a Cayley graph of G' is a graph where the node set is G', and two distinct nodes x and y are adjacent if and only if $xy^{-1} \in S$. In this context, the relation $\sim$ is defined as x multiplies with the inverse of other elements in G'. However, this problem is easily solved, as it has been established that $d_{\text{Cayley group of } G'}(x) = |S|$ for every $x \in G'$. The research on Cayley graphs can be found, for example [21], [22]. Other examples are coprime graphs and non-coprime graphs. Those are introduced by Ma et. al. (2014) (see [23]) and Mansoori et al (2016) (see [24]), respectively. The coprime graph of the group G' is a graph in which the node set is G' and two distinct nodes are adjacent if and only if their orders is coprime (relatively prime). The non-coprime graph of the group G' is a graph in which the node set is $G' - \{i\}$, where i is the identity element of G', and two distinct nodes are adjacent if and only if their orders are not coprime. These graphs make a connection between elements by the relation of their coprimality whether included in $\{(a,b): a,b \in \mathbb{N}, \gcd(a,b) = 1\}$ or $\mathbb{N}^2 - \{(a,b): a,b \in \mathbb{N}, \gcd(a,b) = 1\}$. The study of these graphs is quite lots see for example [23]–[25], and there was research that explains the connection between them (see [26]). Specifically, related to the systems consisting of sets, the problem is similar to determining the degree of every node in certain graphs, such as graphs constructed with the adjacency rule depends on $\sim$ and the node set is a graph or even hypergraph since the elements of $\mathcal{L}$ are not necessarily the same size. Examples of these graphs are uniform subset graphs, non-uniform subset graphs, and token graphs.

In determining the problem's solution, we are using the principle of inclusion-exclusion, Corradi's lemma, and simplex Bonferroni's inequality, specifically placing the last two in the following moment.

$$\text{Corradi's lemma} \leq \textbf{Size of union of some finite sets} \leq \text{Simplex Bonferroni's inequality}.$$

The upper and lower bounds can be gained using Bonferroni's inequality by tinkering values of q, but it does not quite match in this context since in the discussion, some calculations include binomial terms that do not hold for $q > 1$.

The purpose of this paper is to solve **Problem 1** which focused on binary set operations such as union, intersection, difference, and symmetric difference, by using three supporting materials such as the inclusion-exclusion principle, Corrádi's lemma, and simplex Bonferroni's inequality. The solutions will be applicable to determine the degree of every node in certain graphs. For example, graphs constructed with the adjacency rule depend on the binary set operation and the node set is a graph or even hypergraph.

2. RESEARCH METHODS

This research uses a literature review methodology, drawing materials from various books and articles published in specific journals. The discussion groove in this research is solving the posed problem by employing the inclusion-exclusion principle, Corrádi's lemma, and simplex Bonferroni's inequality.

3. RESULTS AND DISCUSSION

Recall that in this paper, we just solve **Problem 1**, particularly for the subsets that will be searched to have the same size as the fixed subset. Before directly solving the problem, we re-state the assertion of **Problem 1** into a more specific assertion as follows.

Problem 2. Given a finite set F of size $f > 0$ and some integers $k \geq m \geq n > l \geq 0$ where $f > k$. Let $*$ be a binary set operation and

$$\mathcal{L} := \{C \subseteq F : n \leq |C| \leq m, |C \cap C'| \leq l\}.$$

For a fixed $A \in \binom{F}{k}$, how many $B \in \binom{F}{k} - \{A\}$ such that $B * A$ contains at least one element of $\mathcal{L}$?

The question on **Problem 2** means that the subset B should satisfy $2^{B*A} \cap \mathcal{L} \neq \emptyset$. For brevity, we denote $[B * A]_{\mathcal{L}}$ which represents the size of $2^{B*A} \cap \mathcal{L}$, $N_*(A)$ represents the set of subsets B such that $[B * A]_{\mathcal{L}} \neq 0$, and $\#N_*(A)$ represents the size of $N_*(A)$. It is pretty clear that the set of all subset B is included in $\binom{F}{k}$, so we obtain $\#N_*(A) \leq \binom{f}{k}$. For every $1 \leq i \leq t$, let $S_i \subset F$ be any subsets of size less than or equal to k, we denote $C_k(S_i)$ be the collection of $D \in \binom{F}{k}$ containing S_i. The size of $C_k(S_i)$ must be $\binom{f-|S_i|}{k-|S_i|}$ for every $1 \leq i \leq t$. It is easy to verify that $\bigcap_{1\leq i\leq t} C_k(S_i) = C_k(\bigcup_{1\leq i\leq t} S_i)$ and it has the size $\binom{f-|\bigcup_{1\leq i\leq t} S_i|}{k-|\bigcup_{1\leq i\leq t} S_i|}$. To get $N_*(A)$, we can see how the interaction between A and each element of $\mathcal{L}$, whether A contains some elements of $\mathcal{L}$ or not. This interaction lets us divide the solution to the problem into two cases i.e., $[A]_{\mathcal{L}} = 0$ or $[A]_{\mathcal{L}} \neq 0$. Now, first, consider the fact that $\mathcal{L}$ is non-empty. If $* = \cup$, then no matter what A is, we must have $N_*(A) \neq \emptyset$. It also holds for $* = \triangle$, since $\mathcal{L}$ is non-empty, then there exists $C \in \mathcal{L}$ such that $n \leq |C| \leq k$. If $C \cap A \neq \emptyset$, then there exists $B \in \binom{F}{k} - \{A\}$ and $B \supset C - A$ such that $B \triangle A \supset C$, so $[B \triangle A]_{\mathcal{L}} \neq 0$. If $C \cap A = \emptyset$, then there exists $B \in \binom{F}{k} - \{A\}$ and $B \supset C$ such that $B \triangle A \supset C$, so $[B \triangle A]_{\mathcal{L}} \neq 0$. In other words, $N_{\triangle}(A) \neq \emptyset$ for any fixed A. Nevertheless, if $* = \cap$, then $N_{\cap}(A)$ depends on $[A]_{\mathcal{L}}$, and also what the precise value of $\#N_{\triangle}(A)$ is also depending on $[A]_{\mathcal{L}}$. This is the reason why we are necessary to consider the cases $[A]_{\mathcal{L}} = 0$ or $[A]_{\mathcal{L}} \neq 0$.

Theorem 1.

$$\#N_{\cap}(A) = \begin{cases} -1 + \sum_{p=1}^{[A]_{\mathcal{L}}} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{\min\left\{k,\, p \max_{C\in 2^A\cap\mathcal{L}} |C|\right\}} \binom{f-r}{k-r} \gamma_{p,r} & , [A]_{\mathcal{L}} \neq 0 \\ 0, & , [A]_{\mathcal{L}} = 0. \end{cases}$$

where $\gamma_{p,r}$ represents the number of subcollections $\{C_i\}_{1\leq i\leq p}$ of $2^A \cap \mathcal{L}$ satisfy $\left|\bigcup_{1\leq i\leq p} C_i\right| = r$.

Proof. If $[A]_{\mathcal{L}} = 0$, then all subsets B satisfy $[B \cap A]_{\mathcal{L}} = 0$. So, $N_{\cap}(A) = \emptyset$. Now, let $[A]_{\mathcal{L}} \neq 0$. We determine $\#N_{\cap}(A)$ using the same method as in [11] (proof of Proposition 3.4). Consider $N_{\cap}(A)$ must consist of subsets containing the same element from $\mathcal{L}$ i.e.

$$N_{\cap}(A) = \left\{B \in \binom{X}{k} - \{A\} : \exists C \in 2^A \cap \mathcal{L}, C \subseteq B\right\}.$$

Therefore,

$$N_{\cap}(A) = \left(\bigcup_{C\in 2^A\cap\mathcal{L}} C_k(C)\right) - \{A\}.$$

By the inclusion-exclusion principle, we have

$$\#N_{\cap}(A) = -1 + \sum_{\emptyset\neq J\subseteq\{1,2,\ldots,[A]_{\mathcal{L}}\}} (-1)^{|J|-1} \left|\bigcap_{j\in J} C_k(C_j)\right| = -1 + \sum_{\emptyset\neq J\subseteq\{1,2,\ldots,[A]_{\mathcal{L}}\}} (-1)^{|J|-1} \binom{f - \left|\bigcup_{j\in J} C\right|}{k - \left|\bigcup_{j\in J} C_j\right|}.$$

Recall that $|C_i| \geq n$ and $|C_i \cap C_j| \leq l$ for all i, j and $i \neq j$. For every $1 \leq p \leq [A]_{\mathcal{L}}$, by Corrádi's lemma and simplex Bonferroni's inequality, we have

$$\frac{pn^2}{n+(p-1)l} \leq \left| \bigcup_{j\in I;\ |I|=p} C_j \right| \leq p \max_{C\in 2^A\cap\mathcal{L}} |C|.$$

If there exists $\{C_j' : j \in I, |I| = p\}$ such that $k < \left|\bigcup_{j\in I;\ |I|=p} C_j'\right| \leq p \max_{C\in 2^A\cap\mathcal{L}} |C|$ satisfying

$$\begin{pmatrix} f - \left| \bigcup_{j\in I;\ |I|=p} C_j' \right| \\ k - \left| \bigcup_{j\in I;\ |I|=p} C_j' \right| \end{pmatrix} = 0,$$

then we have

$$\frac{pn^2}{n+(p-1)l} \leq \left| \bigcup_{j\in I;\ |I|=p} C_j \right| \leq \min\left\{k, p \max_{C\in 2^A\cap\mathcal{L}} |C|\right\} \tag{1}$$

Therefore,

$$\begin{matrix}\text{The number of all elements of } N_\cap(A) \\ \text{containing } p \text{ elements of } 2^A \cap \mathcal{L}\end{matrix} = \sum_{r=\frac{pn^2}{n+(p-1)l}}^{\min\left\{k,p \max_{C\in 2^A\cap\mathcal{L}}|C|\right\}} \binom{f-r}{k-r} \gamma_{p,r} \tag{2}$$

where $\gamma_{p,r}$ represents the number of subcollections $\{C_i\}_{1\leq i\leq p}$ of $2^A \cap \mathcal{L}$ satisfy $\left|\bigcup_{1\leq i\leq p} C_i\right| = r$. By the inclusion-exclusion principle running for all p, we obtain

$$\#N_\cap(A) = -1 + \sum_{p=1}^{[A]_\mathcal{L}} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{\min\left\{k,p \max_{C\in 2^A\cap\mathcal{L}}|C|\right\}} \binom{f-r}{k-r} \gamma_{p,r}.$$

■

The following **Theorem 2** and **Theorem 4** show the bounds of $\#N_\triangle(A)$.

Theorem 2. *If* $[A]_\mathcal{L} = 0$, *then*

$$\#N_\triangle(A) \leq \sum_{p=1}^{|\mathcal{L}|} (-1)^{p-1} \sum_{r=\frac{p\left(\min_{C\in\mathcal{L}}|C-A|\right)^2}{\min_{C\in\mathcal{L}}|C-A|+(p-1)l}}^{\min\left\{k,p \max_{C\in\mathcal{L}}|C-A|\right\}} \binom{f-r}{k-r} \phi_{p,r},$$

where $\phi_{p,r}$ *represents the number of subcollections* $\{C_i\}_{1\leq i\leq p}$ *of* $\mathcal{L}$ *satisfy* $\left|\bigcup_{1\leq i\leq p}(C_i - A)\right| = r$.

Proof. Consider that for all $C \in \mathcal{L}$, these satisfy $C \not\subset A$ since $[A]_\mathcal{L} = 0$, but they are certainly satisfied either $A \cap C = \emptyset$ or $A \cap C \neq \emptyset$. If $A \cap C \neq \emptyset$, then $C - A \neq \emptyset$. Also, if $A \cap C = \emptyset$, then $C - A = C$. In other words, $N_\triangle(A)$ consists of $B \in \binom{F}{k}$ satisfies $B \supseteq C - A$, equivalently

$$N_\triangle(A) \subseteq \bigcup_{C\in\mathcal{L}} C_k(C - A).$$

Using the inclusion-exclusion principle, we have

$$\#N_{\triangle}(A) \leq \sum_{\emptyset \neq J \subseteq \{1,2,\dots,|\mathcal{L}|\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} C_k(C_j - A) \right| = \sum_{\emptyset \neq J \subseteq \{1,2,\dots,|\mathcal{L}|\}} (-1)^{|J|-1} \binom{f - \left| \bigcup_{j \in J} (C_j - A) \right|}{k - \left| \bigcup_{j \in J} (C - A) \right|}.$$

Let $m_* := \min\{|C - A| : C \in \mathcal{L}\}$ and $m^* := \max\{|C - A| : C \in \mathcal{L}\}$. Consider that $|C_i - A| \geq m_*$ and

$$\left|(C_i - A) \cap (C_j - A)\right| = \left|(C_i \cap A^c) \cap (C_j \cap A^c)\right| = |(C_i \cap C) \cap A^c| \leq |C_i \cap C_j| \leq l$$

for all i, j and $i \neq j$. For every $1 \leq p \leq |\mathcal{L}|$, using Corrádi's lemma and simplex Bonferroni's inequality, we have

$$\frac{pm_*^2}{m_* + (p-1)l} \leq \left| \bigcup_{j \in I;\, |I| = p} (C_j - A) \right| \leq pm^*.$$

If there exists $\{C_j' - A : j \in I, |I| = p\}$ such that $k < \left| \bigcup_{j \in I;\, |I|=p} (C_j' - A) \right| \leq pm^*$ satisfying

$$\binom{f - \left| \bigcup_{j \in I;\, |I|=p} (C_j' - A) \right|}{k - \left| \bigcup_{j \in I;\, |I|=p} (C_j' - A) \right|} = 0,$$

then we have

$$\frac{pm_*^2}{m_* + (p-1)l} \leq \left| \bigcup_{j \in I;\, |I| = p} (C_j - A) \right| \leq \min\{k, pm^*\}.$$

Therefore,

$$\begin{matrix}\text{The number of all elements of } N_{\triangle}(A) \\ \text{containing } p \text{ elements of } 2^A \cap \mathcal{L}\end{matrix} = \sum_{r = \frac{pm_*^2}{m_* + (p-1)l}}^{\min\{k, pm^*\}} \binom{f - r}{k - r} \phi_{p,r},$$

where $\phi_{p,r}$ represents the number of subcollections $\{C_i\}_{1 \leq i \leq p}$ of $\mathcal{L}$ satisfy $\left| \bigcup_{1 \leq i \leq p} (C_i - A) \right| = r$. By the inclusion-exclusion principle running for all p, we obtain

$$\#N_{\triangle}(A) \leq \sum_{p=1}^{|\mathcal{L}|} (-1)^{p-1} \sum_{r = \frac{pm_*^2}{m_* + (p-1)l}}^{\min\{k, pm^*\}} \binom{f - r}{k - r} \phi_{p,r}.$$

■

Proposition 3. *There is at least one subset* $B \in \binom{F}{k}$ *disjoints with all* $C \in 2^A \cap \mathcal{L}$ *if and only if* $k \leq f - |\bigcup_{C \in 2^A \cap \mathcal{L}} C|$.

Proof. It is trivial to know that if $k \leq f - |\bigcup_{C\in 2^A \cap \mathcal{L}} C|$, then $\binom{f - |\bigcup_{C\in 2^A\cap\mathcal{L}} C|}{k} > 0$. So, there is at least one subset $B \in \binom{F}{k}$ disjoining with all elements of $2^A \cap \mathcal{L}$. Conversely, let

$$Y = \left\{B \in \binom{F}{k} : B \cap C = \emptyset, \forall C \in 2^A \cap \mathcal{L}\right\} \neq \emptyset.$$

Take any element $B \in Y$. We have $B \cap C_1 = \emptyset$, $B \cap C_2 = \emptyset$, ..., $B \cap C_{[A]_\mathcal{L}} = \emptyset$. Consider the following.

$$\emptyset = \emptyset \cup \emptyset \cup \ldots \cup \emptyset = (B \cap C_1) \cup (B \cap C_2) \cup \ldots \cup \left(B \cap C_{[A]_\mathcal{L}}\right) = B \cap \left(\bigcup_{C\in 2^A\cap\mathcal{L}} C\right).$$

In other words, B disjoint with the union of all elements of $2^A \cap \mathcal{L}$. Therefore, $\binom{f - |\bigcup_{C\in 2^A\cap\mathcal{L}} C|}{k} > 0$ or equivalently, $k \leq f - |\bigcup_{C\in 2^A\cap\mathcal{L}} C|$.

■

Theorem 4. *If* $[A]_\mathcal{L} \neq 0$, *then*

$$\#N_\triangle(A) \geq \sum_{p=1}^{[A]_\mathcal{L}} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{p \max_{C\in 2^A\cap\mathcal{L}} |C|} \binom{f-r}{k} \gamma_{p,r},$$

where $\gamma_{p,r}$ *represents the number of subcollections* $\{C_i\}_{1\leq i\leq p}$ *of* $2^A \cap \mathcal{L}$ *satisfy* $\left|\bigcup_{1\leq i\leq p} C_i\right| = r$.

Proof. Since $[A]_\mathcal{L} \neq \emptyset$, then let $2^A \cap \mathcal{L} = \left\{C_1, C_2, \ldots, C_{[A]_\mathcal{L}}\right\}$. The set $B \in \binom{F}{k}$ which disjoints with at least one element of $2^A \cap \mathcal{L}$, say $C \in 2^A \cap \mathcal{L}$, satisfies $B \triangle A \supseteq C$. It follows that $[B \triangle A]_\mathcal{L} \neq 0$. So, $B \in N_\triangle(A)$. Therefore,

$$N_\triangle(A) \supseteq \left(\bigcup_{p=1}^{[A]_\mathcal{L}} \left\{B \in \binom{F}{k} : B \cap C_p = \emptyset\right\}\right)$$

or

$$\#N_\triangle(A) \geq \left|\bigcup_{p=1}^{[A]_\mathcal{L}} \left\{B \in \binom{F}{k} : B \cap C_p = \emptyset\right\}\right|.$$

By using the inclusion-exclusion principle,

$$\#N_\triangle(A) \geq \left(\sum_{\emptyset\neq J\subseteq I} (-1)^{|J|-1} \left|\bigcap_{j\in J} \left\{B \in \binom{F}{k} : B \cap C_j = \emptyset\right\}\right|\right)$$

$$= \left(\sum_{\emptyset\neq J\subseteq I} (-1)^{|J|-1} \left|\left\{B \in \binom{F}{k} : B \cap \left(\bigcup_{j\in J} C_j\right) = \emptyset\right\}\right|\right)$$

$$\#N_\triangle(A) \geq \left(\sum_{\emptyset\neq J\subseteq I} (-1)^{|J|-1} \binom{f - \left|\bigcup_{j\in J} C_j\right|}{k}\right)$$

Based on **Equation (1)** and **Equation (2)**, we obtain

$$\#N_\triangle(A) \geq \left(\sum_{p=1}^{[A]_\mathcal{L}} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{p \max_{C\in 2^A\cap\mathcal{L}} |C|} \binom{f-r}{k} \gamma_{p,r}\right).$$

■

The following **Theorem 5** and **Theorem 6** show the size of $N_\cup(A)$.

Theorem 5. $$\#N_\cup(A) = \begin{cases} \left|\left\{B \in \binom{F}{k} : \exists A_0 \subseteq A, [A_0 \cup B]_\mathcal{L} \neq 0\right\}\right| & ,[A]_\mathcal{L} = 0 \\ \binom{f}{k} - 1 & ,[A]_\mathcal{L} \neq 0. \end{cases}$$

Proof. It is pretty clear for the case $[A]_\mathcal{L} = 0$. If $[A]_\mathcal{L} \neq 0$, then every subset $B \in \binom{F}{k} - \{A\}$ satisfies $[B \cup A]_\mathcal{L} \neq 0$ since $2^{B\cup A} \cap \mathcal{L} \supseteq 2^A \cap \mathcal{L} \neq \emptyset$. Therefore, $\#N_\cup(A) = \binom{f}{k} - 1$.

■

Theorem 6. $\#N_\cup(A) \geq |N_\cap(A) \cup N_\triangle(A)|$.

Proof. Take any $B \in N_\cap(A) \cup N_\triangle(A)$. If $B \in N_\cap(A)$, then there exists $C_1 \in \mathcal{L}$ such that $C_1 \subseteq B \cap A \subseteq B \cup A$. Similarly, if $B \in N_\triangle(A)$, then there exists $C_2 \in \mathcal{L}$ such that $C_2 \subseteq B \triangle A \subseteq B \cup A$. Therefore, there exists $C = C_1 \cup C_2$ such that $C \subseteq B \cup A$. In other words, $B \in N_\cup(A)$.

■

Theorem 4 does not always hold equality i.e., $\#N_\cup(A) = |N_\cap(A) \cup N_\triangle(A)|$. It because not all elements of $N_\cup(A)$ are contained in $N_\cap(A) \cup N_\triangle(A)$. Perhaps there exists $B \in N_\cup(A)$ which all $C \in 2^A \cap \mathcal{L}$ satisfy $C \cap (B \cap A) \neq \emptyset$ and $C \cap (B \triangle A) \neq \emptyset$. It follows that $C \nsubseteq (B \cap A)$ and $C \nsubseteq (B \triangle A)$. Therefore, $B \notin \left(N_\cap(A) \cup N_\triangle(A)\right)$. For instance, given a hypergraph $(H, \Re)$ where

$$H = \binom{\{n: 1 \leq n \leq 11\}}{5} \ \wedge \ \Re = \{\{2,4,6\}\}.$$

For a fixed node $u = \{1,2,3,4,5\}$. There exists $v = \{4,5,6,7,8\}$ such that $\{2,4,6\} \cap (v \cap u) = \{4\} \neq \emptyset$ and $\{2,4,6\} \cap (v \triangle u) = \{2,6\} \neq \emptyset$. Therefore, we have $\#N_\cup(u) > |N_\cap(u) \cup N_\triangle(u)|$.

Theorem 7. *If* $[A]_\mathcal{L} = 0$, *then*

$$\#N_-(A) = \sum_{p=1}^{|\mathcal{L}^+|} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{min\{k,pm\}} \binom{f-r}{k-r} \psi_{p,r},$$

where $\mathcal{L}^+ \coloneqq \{C \in \mathcal{L} : C \cap A = \emptyset\}$ *and* $\psi_{p,r}$ *represents the number of subcollections* $\{C_i\}_{1\leq i\leq p}$ *of* $\mathcal{L}^+$ *satisfy* $\left|\bigcup_{1\leq i\leq p} C_i\right| = r$.

Proof. Consider that for any $B \in \binom{F}{k}$, we have the fact that for some $C \in \mathcal{L}$, $B - A \supset C$ if and only if $B \supset C$ with $C \cap A = \emptyset$. Therefore,

$$\#N_-(A) = \left|\left\{B \in \binom{F}{k} : \exists C \in \mathcal{L}, B \supset C, C \cap A = \emptyset\right\}\right| = \left|\bigcup_{C\in\mathcal{L}^+} C_k(C)\right|,$$

where $\mathcal{L}^+ \coloneqq \{C \in \mathcal{L} : C \cap A = \emptyset\}$. By using the similar technique of proving **Theorem 1** and **Theorem 2**, we obtain

$$\#N_-(A) = \sum_{p=1}^{|\mathcal{L}^+|} (-1)^{p-1} \sum_{r=\frac{pn^2}{n+(p-1)l}}^{\min\left\{k,p \max_{C\in\mathcal{L}^+}|C|\right\}} \binom{f-r}{k-r} \psi_{p,r},$$

where $\psi_{p,r}$ represents the number of subcollections $\{C_i\}_{1\leq i\leq p}$ of $\mathcal{L}^+$ satisfy $\left|\bigcup_{1\leq i\leq p} C_i\right| = r$.

■

The results of **Theorem 1** and **Theorem 7** are precise and accurate even though the formulas are sort of complicated to use (as $f \to \infty$) because one has to consider every subcollection of $2^A \cap \mathcal{L}$ to get the values of $\gamma_{p,r}$ and $\psi_{p,r}$. Nevertheless, we believe that the values of $\#N_\cap(A)$ and $\#N_-(A)$ are bounded as follows,

$$\frac{1}{\frac{1}{[A]_{\mathcal{L}}}+\binom{f-2n+l}{k-2n+l}}<1+\#N_{\cap}(A)<[A]_{\mathcal{L}}\frac{(f-\alpha)^{k-\alpha}}{(k-\alpha)!},$$

and if $[A]_{\mathcal{L}}=0$, then

$$\frac{|\mathcal{L}|\binom{f-m}{k-m}^2}{\binom{f-m}{k-m}+(|\mathcal{L}|-1)\binom{f-\frac{2n^2}{n+l}}{k-\frac{2n^2}{n+l}}}\le \#N_{-}(A)\le |\mathcal{L}|\frac{(f-n)^{k-n}}{(k-n)!}.$$

Example 1. An information technology company saves about five different important data i.e., $d_1, d_2, \ldots, d_5$. The relationship between data is shown in the following figure.

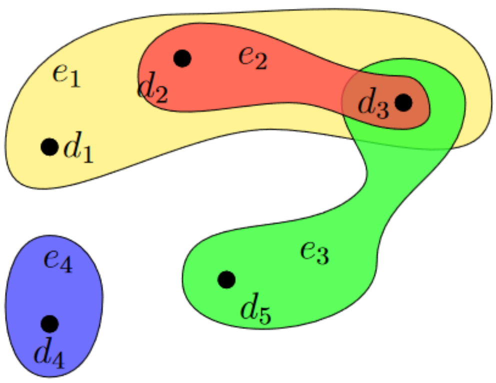

Figure 1. Data Relationship

In **Figure 1**, a group of data that lie in the same region is connected data. Furthermore, the data will be distributed into exactly 10 different secret boxes where each box has 3 different data. The chairman wants the employers to analyze how many relations for every box based on the following rules.

"Two distinct boxes, u and v, are related if and only if $v * u$ contains at least one connected data",

where $*$ be a binary set operation.

Now, define a hypergraph $(H, \mathcal{E})$ where $H = \{d_1, d_2, d_3, d_4, d_5\}$ and

$$\mathcal{E} = \{e_1 = \{d_1, d_2, d_3\}, e_2 = \{d_2, d_3\}, e_3 = \{d_3, d_5\}, e_4 = \{d_4\}\}.$$

The data relationship is interpreted as a hypergraph $(H, \mathcal{E})$. Define a graph $G = (V_G, E_G)$ in which the set of nodes is $V_G = \binom{H}{3}$. Let the nodes are

$$\begin{aligned}
u_1 &= \{d_1, d_2, d_3\}, & u_2 &= \{d_1, d_2, d_4\},\\
u_3 &= \{d_1, d_2, d_5\}, & u_4 &= \{d_1, d_3, d_4\},\\
u_5 &= \{d_1, d_3, d_5\}, & u_6 &= \{d_1, d_4, d_5\},\\
u_7 &= \{d_2, d_3, d_4\}, & u_8 &= \{d_2, d_3, d_5\},\\
u_9 &= \{d_2, d_4, d_5\}, & u_{10} &= \{d_3, d_4, d_5\}.
\end{aligned}$$

We can represent the relation between boxes as a graph G. The chairman asked what is the degree of every node in G based on certain adjacency rules? Consider the following rules.

RULE-1. "Two distinct boxes, u and v, are related if and only if $v \cap u$ contains at least one connected data". The rule means that whether u and v share the same connected data. For every secret box u, it relates to

$$d_G(u)=\begin{cases}-1+\sum_{p=1}^{[u]_\mathcal{E}}(-1)^{p-1}\sum_{r=\frac{p}{2p-1}}^{\min\{3,p\max_{e\in 2^u\cap\mathcal{E}}|e|\}}\binom{5-r}{3-r}\gamma_{p,r}\,, & [u]_\mathcal{E}\neq 0\\ 0, & [u]_\mathcal{E}=0\end{cases}$$

other data. By some calculations using **Theorem 1**, we have the following table.

Table 1. Degree of Every Node of G.

Node	Degree
u_1	2
u_2	5
u_3	0
u_4	5
u_5	2
u_6	5
u_7	7
u_8	4
u_9	5
u_{10}	7

RULE-2. "Two distinct boxes, u and v, are related if and only if $v \triangle u$ contains at least one connected data". The rule u and v means that each of their different data collectively contains connected data. For every secret box u, it relates to

$$d_G(u)\begin{cases}\geq \sum_{p=1}^{[u]_\mathcal{E}}(-1)^{p-1}\sum_{r=\frac{p}{2p-1}}^{p\max_{C\in 2^u\cap\mathcal{E}}|C|}\binom{5-r}{3}\gamma_{p,r} & ,[u]_\mathcal{E}\neq 0\\ \leq \sum_{p=1}^{|\mathcal{E}|}(-1)^{p-1}\sum_{r=\frac{p\left(\min_{e\in\mathcal{E}}|e-u|\right)^2}{\min_{e\in\mathcal{E}}|e-u|+(p-1)l}}^{\min\{3,p\max_{e\in\mathcal{E}}|e-u|\}}\binom{5-r}{3-r}\phi_{p,r} & ,[u]_\mathcal{E}=0\end{cases}$$

other data. By some calculations using **Theorem 2** and **Theorem 4**, we have the following table of the degree of every node of G.

Table 2. The degree of every node of G.

Node	Degree
u_1	≥ 1
u_2	≥ 4
u_3	≤ 9
u_4	≥ 4
u_5	≥ 1
u_6	≥ 4
u_7	≥ 5
u_8	≥ 1
u_9	≥ 4
u_{10}	≥ 5

RULE-3. "Two distinct boxes, u and v, are related if and only if $v \cup u$ contains at least one connected data". The rule means that whether u and v collectively contain some connected data. For every secret box u, it relates to

$$d_G(u)=\begin{cases}|\{v\in V_G:\exists u_0\subseteq u,[u_0\cup v]_\mathcal{E}\neq 0\}| & ,[u]_\mathcal{E}=0\\ 9 & ,[u]_\mathcal{E}\neq 0\end{cases}$$

other data. By some calculations using **Theorem 5**, we have the following.

Table 3. The degree of every node of G.

Node	Degree
u_1	9
u_2	9
u_3	9
u_4	9
u_5	9
u_6	9
u_7	9
u_8	9
u_9	9
u_{10}	9

By **Theorem 7**, we obtain particularly for box u_3 that $d_G(u_3) = 6$.

Hereafter, in addition to the knowledge of non-uniform subset graphs and token graphs, one can create new graphs using the remaining operations such as difference and union. Let U be a graph constructed from any hypergraph and $*$ represents a binary set operation. In this construction, the node set comprises all subsets of k elements from the node set of the hypergraph, and the edge set consists of all pairs of nodes A and B where there exists a hyperedge C satisfies $C \subseteq A * B$ or $C \subseteq B * A$. The degree of every node of those graphs is partially determined by the solutions of **Problem 2**. Especially, for the token graph of a hypergraph, if each hyperedge of the hypergraph has an odd number of elements, then the token graph has no edges because the symmetric difference of any two nodes has an even number of elements. Therefore, the degree of every node is zero. In the case of the token graph of a graph G, the degree of every node A is strictly less than $\#N_{\triangle}(A)$ because there are some pairs of nodes whose symmetric difference has more than two elements.

4. CONCLUSIONS

In this paper, the newly posed problem related to binary set operations on finite sets is defined and subsequently solved for certain operations. These are solved using the principle of inclusion-exclusion, Corrádi's lemma, and Bonferroni's inequality. Interestingly, this problem corresponds to a problem in graph theory, specifically determining the degree of every node of certain graphs.

REFERENCES

[1] K. Dohmen, "Improved Bonferroni Inequalities via Union-Closed Set Systems," *J. Comb. Theory, Ser. A*, vol. 92, pp. 61–67, 2000, doi: 10.1006/jcta.1999.3030.

[2] K. Corrádi, "Problem at Schweitzer competition," *Mat. Lapok.*, vol. 20, pp. 159–162, 1969.

[3] S. Jukna, *Extremal Combinatorics With Applications in Computer Science*. Berlin: Springer-Verlag Berlin Heidelberg, 2011.

[4] X. Li, H. Broersma, and L. Wang, "Grow rates of the bipartite Erdős-Gyarfas function," *arXiv: 2111.00879 [math.CO]*, 2021.

[5] C. Berge, *Graphs and Hypergraphs*. Amsterdam: North-Holland Publishing Company, 1973.

[6] V. Blinovsky and C. Greenhill, "Asymptotic enumeration of sparse uniform hypergraphs with given degrees," *Eur. J. Comb.*, vol. 51, pp. 287–296, 2016, doi: 10.1016/j.ejc.2015.06.004.

[7] X. Ouvrard, "Hypergraphs: an introduction and review," *arXiv: 2002.05014 [cs.DM]*, 2020.

[8] M. M. Jadhav and K. F. Pawar, "On edge product number of hypergraph," *Palest. J. Math.*, vol. 11, no. 4, pp. 183–194, 2022.

[9] V. Blinovsky and C. Greenhill, "Asymptotic enumeration of sparse uniform linear hypergraphs with given degrees," *Electron. J. Comb.*, vol. 23, no. 3, pp. 1–17, 2016, doi: 10.37236/5512.

[10] M. Axenovich, D. Bradač, L. Gishboliner, D. Mubayi, and L. Weber, "Large cliques or cocliques in hypergraphs with forbidden order-size pairs," *Comb. Probab. Comput.*, pp. 286–299, 2023, doi: 10.1017/S0963548323000433.

[11] M. N. Huda, "Graf subset tak seragam yang berasosiasi dengan sebarang graf," Universitas Gadjah Mada, 2024.

[12] R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood, "Token graphs," *Graphs Comb.*, vol. 28, pp. 365–380, 2012, doi: 10.1007/s00373-011-1055-9.

[13] J. Zhang, J.-X. Zhou, Y.-T. Li, and Y. S. Kwon, "The automorphisms of 2-token graphs," *Appl. Math. Comput.*, vol. 446, 2023, doi: 10.1016/j.amc.2023.127872.

[14] Y. Susanti, "On the 2-token graphs of some disjoint union of graph," *Malaysian J. Math. Sci.*, vol. 17, no. 4, pp. 719–730, 2023, doi: 10.47836/mjms.17.4.12.

[15] A. Lew, "Garland's method for token graphs," *arXiv: 2305.02406 [math.CO]*, 2023.

[16] J. Leaños and C. Ndjatchi, "The edge-connectivity of token graphs," *Graphs Comb.*, vol. 37, pp. 1013–1023, 2021, doi: 10.1007/s00373-021-02301-0.

[17] L. E. Adame, L. M. Rivera, and A. L. Trujillo-Negrete, “Hamiltonicity of token graphs of some join graphs,” *Symmetry (Basel).*, vol. 13, no. 6, 2021, doi: 10.3390/sym13061076.
[18] W. Carballosa, R. Fabila-Monroy, J. Leanos, and L. M. Rivera, “Regularity and planarity of token graphs,” *Discuss. Math. Graph Theory*, vol. 37, pp. 573–586, 2017, [Online]. Available: http://dx.doi.org/10.7151/dmgt.1959.
[19] C. Dalfó, F. Duque, R. Fabila-monroy, and M. A. Fiol, “On the Laplacian spectra of token graphs ☆,” *Linear Algebra Appl.*, vol. 625, pp. 322–348, 2021, doi: 10.1016/j.laa.2021.05.005.
[20] S. Barik and P. Verma, “Spectral properties of token graphs,” *Linear Algebra Appl.*, vol. 687, pp. 181–206, 2024, doi: 10.1016/j.laa.2024.02.004.
[21] C. Bagiński and P. Grzeszczuk, “On the generic family of Cayley graphs of a finite group,” *J. Comb. Theory, Ser. A*, vol. 184, p. 105495, 2021, doi: 10.1016/j.jcta.2021.105495.
[22] Y. Susanti and A. Erfanian, “Prime square order Cayley graph of cyclic groups,” *Asian-European J. Math.*, vol. 17, no. 2, 2024, doi: 10.1142/S1793557124500037.
[23] X. Ma, H. Wei, and L. Yang, “The coprime graph of a group,” *Int. J. Gr. Theory*, vol. 3, no. 3, pp. 13–23, 2014.
[24] F. Mansoori, A. Erfanian, and B. Tolue, “Non-coprime graph of a finite group,” 2016, [Online]. Available: https://doi.org/10.1063/1.4954605.
[25] M. N. Huda and S. Ali, “Non-coprime graph energy of dihedral and symmetric group,” 2024.
[26] M. N. Huda, “A note on hamiltonicity conditions of the coprime and non-coprime graphs of a finite group,” *J. Mat. UNAND*, vol. 13, no. 3, pp. 157–162, 2024.