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DYNAMICAL BEHAVIOR IN THE COMPETITIVE MODEL INCORPORATING THE FEAR EFFECT OF PREY DUE TO ALLELOPATHY WITH SHARED BIOTIC RESOURCES

Mifta Kharisma Dewi[1], Dian Savitri[2*], Abadi[3]

[1,2,3]Departement of Mathematics, Faculty of Mathematics and Sciences, Universitas Negeri Surabaya Jl. Ketintang, Gayungan Sub-District, Surabaya, 60231, Indonesia

*Corresponding author's e-mail: * **diansavitri@unesa.ac.id***

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ABSTRACT

This research develops a mathematical model of a natural phenomenon, namely sea snails that can release toxins (allelopathy) so that non-toxic sea snails become afraid. In addition, toxic and non-toxic sea snails share biotic resources. Based on the existing phenomenon, the model of fear effect caused by allelopathy in the competitive interaction model with shared biotic resources will be studied. In this system, three equilibrium points are obtained: extinction point of prey, extinction point of predator, and coexisting point under certain conditions. Analysis of local stability at equilibrium points by linearization shows that all equilibrium points are asymptotically stable with certain conditions. Numerical simulations at the equilibrium point show the same results as the analysis results. Then, numerical continuity was carried out by selecting variation of the fear effect parameter for $k = 0.8, k = 0.9, k = 4, k = 5.5$. Numerical continuity results show that changes in these parameters affect the population of toxic and non-toxic species, marked by the emergence of Transcritical bifurcations, Bifurcation occurs at $k = 1$, the first Saddle-Node bifurcation at $k = 0.880185$, and the second Saddle-Node bifurcation at $k = 5.279992$.

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1. INTRODUCTION

An ecological system formed from the reciprocal relationship between living things and their environment is called an ecosystem [1]. The process of ecosystem interaction can occur in all other living things, such as the interaction of animals with plants and the surrounding environment [2]. Interactions in ecosystems arise not only on land but also in waters such as the sea; one example is the interaction between sea snails and their environment.

Sea snails are members of the *Gastropoda class*, one of the largest classes in the *Mollusca* phylum [3]. Sea snails secrete muricin, a chemical compound in some sea slug species that causes allelopathy by releasing toxins. Allelopathy is the process of secondary metabolites produced by plants, algae, bacteria, fungi, and organisms that affect the growth and development of agricultural and biological systems [4]. A unique phenomenon of sea snail species is when the release of toxins (allelopathy) inhibits another sea snail's development or physiological functions. Species that mainly produce it are *Dicathais orbita* and *Conus Spp* [5].

Toxic and non-toxic sea snails compete for biotic environmental resources such as algae food. Competition between two species is modeled through the Lotka-Volterra model, and its development has been intensively investigated in recent decades, such as Safuan and Musa 2016 who studied the Lotka-Volterra competition model by considering the intraguild phenomenon where two species share the same environmental carrying capacity [6].

Toxin release (allelopathy) was first added to the Lotka - Volterra model by Maynard-Smith to account for the harmful impact that one species has on another. Many scientists considered situations where only one species releases toxin, such as Chen et al. proposed a system for toxin release from a single species. The analysis found in Chen's (2013) model that extinction is not affected at low levels of toxin release, meaning that toxic species cannot wipe out non-toxic species [7]. But in reality, non-toxic species can go extinct even if only affected by lower concentrations of toxins. Non-toxic species also experience fear of other toxic species when the toxic species releases toxins [8].

Non-toxic species that have a fear effect experience a loss; namely, more time and energy spent with fear reduces time and energy to find food so that the growth of non-toxic species is inhibited. The fear effect also forces temporary habitat abandonment so that reproduction rates may decline in the long term [9]. In 2016, Wang considered the fear effect for the first time based on the classical theory of the two-species Lotka-Volterra predator-prey model [10]. Several studies discussed predator-prey models with fear effects, including considering the effect of fear on prey [11]. Other research that considers fear's effect on prey is accompanied by additional food [12]. This is supported by the research of Srivastava (2022), which considers the effect of fear on the competition model between two species [13]. The fear effect has been studied extensively, but the fear effect is less considered in competitive systems. However, there is strong evidence that fear arises in pure competitive systems without predation effects or when predation effects are negligible [14].

In some of the research that has been described previously, there is a model that combines the effects of fear and the release of toxins (allelopathy), namely the research of Chen et al. (2023), who studied the Lotka-Volterra competition model with the fear parameter affecting the allelopathic planktonic system by adding the term fear effect [8]. Based on this explanation, the authors are interested in studying and modifying the interaction model in the Lotka-Volterra competition studied by several previous researchers by considering the fear effect on non-toxic species due to releasing toxins with shared biotic resources.

2. RESEARCH METHODS

2.1 Basic Competition Model

Competition is an interaction between two species trying to obtain the same resources, such as food, shelter, or other environmental factors. There are two types of competition in ecology: intraspecies and interspecies competition [15]. Intraspecies competition is competition from the same species, while interspecies competition is competition from different species. The Lotka-Volterra competition model is a mathematical model that describes the population dynamics of two species competing for the same resources in an ecosystem [16]. The Lotka-Volterra competition model was first introduced by Alfred J. Lotka in 1925. In 1926, a prominent Italian mathematician named Vito Volterra developed the mathematical study.

The competition model assumes that the model grows exponentially, which is usually called the exponential growth model, where the environment does not limit growth, so the species population increases infinitely. It is also assumed that the population decreases due to intraspecies competition (the same species) that competes for nutrients or food, sunlight, and maintaining its place of residence. Besides intraspecies competition, the population also decreases due to the presence of different species. Suppose $u = u(t)$ represents the population of species one at time t and $v = v(t)$ represents the population of species two at time t, the Lotka-Volterra competition model can be expressed as follows:

$$\begin{aligned}\frac{du}{dt} &= a_1u - b_1u^2 - c_1uv,\\ \frac{dv}{dt} &= a_2u - b_2v^2 - c_2uv.\end{aligned} \tag{1}$$

where $\frac{du}{dt}$ is the rate of change of population of first species with respect to time t and $\frac{dv}{dt}$ is the rate of change of population of second species with respect to time t. All parameters are positive real. Parameters a_1 and a_2 are the intrinsic growth of species, b_1 and b_2 are the intraspecies competition rate, c_1 and c_2 are the interspecies competition rate.

2.2 The Fear Effect

The fear effects are behavioral and stress-related physiological changes resulting from the presence of a noxious species [17]. One example is the fear effect in non-toxic species due to toxic species releasing toxic compounds. Non-toxic sea snail species fear sea snails that can release toxins.

The fear effect affects the growth rate of the frightened species assumed as species two, where the growth of species two is not constant but depends on the density of the species, causing the fear assumed as species one. A higher density of species one increases the fear of species two, the fear effect is denoted by the parameter k. The denominator $1 + kx$ is used to represent the level of fear that decreases as the number of species experiencing fear (species two) increases. The fear effect function, according to [10], has the following equation

$$f(k, x) = \frac{1}{1 + kx}. \tag{2}$$

where the variable x is species one and the parameter k is the fear effect coefficient.

2.3 The Competitive Model with Shared Biotic Resources

The Lotka-Volterra competitive interaction model with shared biotic resources has been investigated by previous research without considering the effects of fear and the release of toxins (allelopathy) [6]. The results of Safuan and Musa (2016) showed that interspecies competition affected the system solution, and parameter variations also affected the system solution. The following is the competitive interaction model with shared biotic resources.

$$\begin{aligned}\frac{dX}{dt} &= r_1X\left(1 - \frac{X}{pZ}\right) - aXY,\\ \frac{dY}{dt} &= r_2X\left(1 - \frac{Y}{qZ}\right) - bXY,\\ \frac{dZ}{dt} &= cZ - dXZ - eYZ.\end{aligned} \tag{3}$$

The variables X, Y and Z denote the populations of first species, second species, and biotic resourcZ. Species one and two grow logistically with intrinsic growth rates given by r_1 and r_2 respectively and are limited by the availability of biotic resources Z in proportions pZ and qZ respectively. It is assumed that the environmental carrying capacity evolves linearly with growth rate c, and the resource utilization rates by species one and two are d d and e, respectively. Parameters a and b are the interspecies competition rates of species one and two.

3. RESULTS AND DISCUSSION

3.1 The Model Formulation

The Lotka-Volterra competitive interaction model assumes, based on previous research, that the population growth rate of toxic sea snails (species one) follows logistic growth with shared biotic resources. The population of toxic sea snails decreases due to interspecies competition (from different species) between toxic (allelopathic) and non-toxic (nonallelopathic) sea snails. Furthermore, the population growth of non-toxic sea snails (species two) also follows logistic growth with shared biotic resources. This growth decreases due to the fear effect of the release of toxins from poisonous sea snails; it is also assumed that the population of non-toxic sea snails decreases due to interspecies competition. In addition, non-toxic sea snails may become extinct due to the release of allelopathic sea snail toxins. Based on **Equation (1)**, **Equation (2)**, **Equation (3)**, and the above assumptions, the construction of a model on Lotka-Volterra competition interaction that considers the fear effect on non-toxic species due to the presence of allelopathy with shared biotic resources as is follows

$$\begin{aligned} \frac{dX}{dt} &= r_1 X\left(1-\frac{X}{pZ}\right) - aXY, \\ \frac{dY}{dt} &= \frac{r_2 Y}{1+\eta X}\left(1-\frac{Y}{qZ}\right) - bXY - \xi XY^2, \\ \frac{dZ}{dt} &= cZ - dXZ - eYZ. \end{aligned} \tag{4}$$

The assumed variables and model parameters used to model competition that considers the fear effect on non-toxic species due to allelopathy (toxin release) with shared biotic resources are presented in Table 1.

Table 1. The Variables and Parameters of a Fear Effect on Non-Toxic Species Due to the Release of Toxins with Share Biotic Resources

Symbol	Definition	Type	Unit
X	Population of toxic species	Variable	$tail$
Y	Population of nontoxic species	Variable	$tail$
Z	Biotic resources	Variable	$tail$
r_1	Toxic species growth rate without being influenced by the environment	Parameter	$\frac{1}{day}$
p	Environmental carrying capacity of toxic species populations	Parameter	$tail$
a	Rate of interspecies competition	Parameter	$\frac{1}{day}$
r_2	Nontoxic species growth rate without being influenced by the environment	Parameter	$\frac{1}{day}$
η	The fear effect	Parameter	-
q	Environmental carrying capacity of nontoxic species populations	Parameter	$tail$
b	Rate of interspecies competition	Parameter	$\frac{1}{day}$
ξ	Rate of toxic release	Parameter	$\frac{1}{day}$
c	Growth of biotic resources	Parameter	$\frac{1}{day}$
d	Rate of uptake of the resource by toxic species	Parameter	$\frac{1}{day}$
e	Rate of uptake of the resource by nontoxic species	Parameter	$\frac{1}{day}$

Equation (4) can be written in dimensionless form, referring to **[18]** with $x = \frac{dX}{r_1}$, $y = \frac{eY}{r_1}$, $z = \frac{dpZ}{r_1}$, $\tau = r_1 t$, so that **Equation (4)** becomes

$$\frac{dx}{d\tau} = x\left(1 - \frac{x}{z}\right) - \alpha xy,$$
$$\frac{dy}{d\tau} = \frac{ry}{1+kx}\left(1 - \frac{\mu y}{z}\right) - \beta XY - mXY^2, \tag{5}$$
$$\frac{dz}{d\tau} = \gamma z - xz - yz.$$

with

$$\alpha = \frac{a}{e}, \quad \beta = \frac{b}{d}, \quad \gamma = \frac{c}{r_1}, \quad r = \frac{r_2}{r_1}, \quad \mu = \frac{dp}{eq}, \ k = \frac{\eta r_1}{d}, \quad m = \frac{\xi r_1}{ed},$$

3.2 Model Analysis

The next step is to find the equilibrium point of **Equation (5)**. The equilibrium points in the model competition that considers the fear effect on non-toxic species due to allelopathy (toxin release) with shared biotic resources is obtained if $\frac{dx}{d\tau} = 0, \frac{dy}{d\tau} = 0, \frac{dz}{d\tau} = 0$ [19]. So that the equilibrium points are obtained:

1. The extinction equilibium point of first species $E_1 = (0, \gamma, \mu\gamma)$
2. The extinction equilibrium point of second species $E_2 = (\gamma, 0, \gamma)$
3. The coexist equilibrium point $E_3 = (x^*, y^*, z^*)$, with

$$x^* = \frac{z^* - z^*\alpha\gamma}{1 - z^*\alpha}, y^* = \frac{\gamma - z^*}{1 - z^*\alpha},$$

And z^* is obtained by solving the below equation

$$Az^{*4} + Bz^{*3} + Cz^{*2} + Dz^* + E = 0, \tag{6}$$

with

$$A = -\alpha^3\beta\gamma^2k - \alpha^2\gamma^2km - \alpha^3\beta\gamma + 2\alpha^2\beta\gamma k + \alpha^3 r - \alpha^2\gamma m + 2\alpha\gamma km + \alpha^2\beta - \alpha\beta k + \alpha m - km,$$

$$B = \alpha^2\gamma^3km + \alpha^2\beta\gamma^2k + \alpha^2\gamma^2m - 2\alpha\gamma^2km + 2\alpha^2\beta\gamma - \alpha^2\mu r - 2\alpha\beta\gamma k - 3\alpha^2 r + \gamma km - 2\alpha\beta + \beta k - m,$$

$$C = \alpha^2\gamma\mu r - \alpha\gamma^2 m - \alpha\beta\gamma + 2\alpha\mu r + 3\alpha r + \gamma m + \beta,$$

$$D = -2\alpha\gamma\mu r - \mu r - r,$$

$$E = \gamma\mu r.$$

Equation (6) can be determined by using the Ferarri method, the following are the roots of the solution [20]

$$z_1 = -\frac{B}{4A} + \frac{\sqrt{p+2u} + \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2}, \quad z_2 = -\frac{B}{4A} + \frac{\sqrt{p+2u} - \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2},$$

$$z_3 = -\frac{B}{4A} + \frac{-\sqrt{p+2u} + \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2}, \text{ and } z_4 = -\frac{B}{4A} + \frac{-\sqrt{p+2u} - \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2}.$$

with

$$u = -\frac{5}{6}p + \sqrt[3]{-\frac{\left(-\frac{p^3}{108}+\frac{pr}{3}-\frac{q^3}{8}\right)}{2} \pm \sqrt{\frac{\left(-\frac{p^3}{108}+\frac{pr}{3}-\frac{q^3}{8}\right)^2}{4} + \frac{\left(-\frac{p^2}{12}-r\right)}{27}}} - \frac{-\frac{p^2}{12}-r}{3\sqrt[3]{-\frac{\left(-\frac{p^3}{108}+\frac{pr}{3}-\frac{q^3}{8}\right)}{2} \pm \sqrt{\frac{\left(-\frac{p^3}{108}+\frac{pr}{3}-\frac{q^3}{8}\right)^2}{4} + \frac{\left(-\frac{p^2}{12}-r\right)}{27}}}},$$

$$p = \frac{C}{A} - \frac{3B^2}{8A^2},$$

$$q = \frac{D}{A} - \frac{BC}{2A^2} + \frac{B^3}{8A^3},$$

$$r = \frac{E}{A} - \frac{BD}{4A^2} + \frac{B^2C}{16A^3} - \frac{3B^4}{256A^4},$$

so that the equilibrium point is obtained as follows

$$E_3 = (x_1, y_1, z_1) = \left(\frac{z_1 - z_1\alpha\gamma}{1 - z_1\alpha}, \frac{\gamma - z_1}{1 - z_1\alpha}, -\frac{B}{4A} + \frac{\sqrt{p+2u} + \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2} \right),$$

$$E_4 = (x_2, y_2, z_2) = \left(\frac{z_2 - z_2\alpha\gamma}{1 - z_1\alpha}, \frac{\gamma - z_2}{1 - z_2\alpha}, -\frac{B}{4A} + \frac{\sqrt{p+2u} - \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2} \right),$$

$$E_5 = (x_3, y_3, z_3) = \left(\frac{z_3 - z_3\alpha\gamma}{1 - z_3\alpha}, \frac{\gamma - z_3}{1 - z_3\alpha}, -\frac{B}{4A} + \frac{-\sqrt{p+2u} + \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2} \right),$$

$$E_6 = (x_4, y_4, z_4) = \left(\frac{z_4 - z_4\alpha\gamma}{1 - z_4\alpha}, \frac{\gamma - z_4}{1 - z_4\alpha}, -\frac{B}{4A} + \frac{-\sqrt{p+2u} - \sqrt{-(3p+2u+\frac{2q}{\sqrt{p+2u}})}}{2} \right).$$

The local stability of the three equilibrium points can be determined by the eigenvalues generated from the linearization process of the system of **Equation (5)** around the equilibrium point **[21].** The Jacobian matrix resulting from the linearization process of the system of **Equation (5)** is as follows

$$J(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{dx} & \frac{\partial f_1}{dy} & \frac{\partial f_1}{dz} \\ \frac{\partial f_2}{dx} & \frac{\partial f_2}{dy} & \frac{\partial f_2}{dz} \\ \frac{\partial f_3}{dx} & \frac{\partial f_3}{dy} & \frac{\partial f_3}{dz} \end{pmatrix} \tag{7}$$

where $\frac{dx}{d\tau} = f_1(x,y,z), \frac{dy}{d\tau} = f_2(x,y,z)$, and $\frac{dz}{d\tau} = f_3(x,y,z)$. The matrix elements of $J(x,y,z)$ is

$$\frac{\partial f_1}{dx} = 1 - \frac{2x}{z} - \alpha y,$$

$$\frac{\partial f_2}{dx} = -\alpha x,$$

$$\frac{\partial f_3}{dx} = \frac{x^2}{z^2},$$

$$\frac{\partial f_1}{dy} = -\frac{rky}{(1+kx)^2}\left(1 - \frac{\mu y}{z}\right) - \beta y - my^2,$$

$$\frac{\partial f_2}{dy} = -\frac{r}{1+kx}\left(1 - \frac{\mu y}{z}\right) - \frac{r\mu y}{z(1+kx)} - \beta x - 2mxy,$$

$$\frac{\partial f_3}{dy} = \frac{ry^2\mu}{z^2(1+kx)},$$

$$\frac{\partial f_1}{dz} = -z,$$

$$\frac{\partial f_2}{dz} = -z,$$

$$\frac{\partial f_3}{dz} = \gamma - x - y.$$

Theorem 1. *The equilibrium point* $E_1 = (0, \gamma, \mu\gamma)$ *is asymptotically stable if* $\gamma > \frac{1}{\alpha}$.

Proof. The Jacobian matrix in **Equation (7)** at is as follows:

$$J(E_1) = \begin{pmatrix} 1 - \alpha\gamma & 0 & 0 \\ -\beta\gamma - m\gamma^2 & -r & \frac{r}{\mu} \\ -\mu\gamma & -\mu\gamma & 0 \end{pmatrix}$$

The eigenvalues of the above Jacobian matrix at E_1 are

$$\lambda_1 = -\alpha\gamma + 1, \ \lambda_2 = -\frac{r}{2} + \frac{\sqrt{r^2 - 4r\gamma}}{2}, \text{ and } \lambda_3 = -\frac{r}{2} - \frac{\sqrt{r^2 - 4r\gamma}}{2}$$

Stable when $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 < 0$. Conditions that must be met when $\lambda_1 < 0$ if:

$$-\alpha\gamma + 1 < 0,$$
$$\Leftrightarrow \gamma > \frac{1}{\alpha}.$$

a) The equilibrium point $E_1 = (0, \gamma, \mu\gamma)$ is stable spiral if it satisfies the conditions $\lambda_1 < 0, \lambda_2 < 0$, $\lambda_3 < 0$, and if the discriminant of the quadratic equation is $r^2 - 4r\gamma < 0$ or $\gamma > \frac{r}{4}$.

b) The equilibrium point $E_1 = (0, \gamma, \mu\gamma)$ is stable node if it satisfies the conditions $\lambda_1 < 0, \lambda_2 < 0$, $\lambda_3 < 0$, and if the discriminant of the quadratic equation is $r^2 - 4r\gamma \geq 0$ or $\gamma \leq \frac{r}{4}$.

So, the equilibrium point $E_1 = (0, \gamma, \mu\gamma)$ is asymptotically stable if $\gamma > \frac{1}{\alpha}$.

Theorem 2. *The equilibrium point $E_2 = (\gamma, 0, \gamma)$ is asymptotically stable if $k > \frac{r - \beta\gamma}{\beta\gamma^2}$.*

Proof. The Jacobian matrix in **Equation (7)** at is as follows:

$$J(E_2) = \begin{pmatrix} 1 & -\alpha\gamma & 1 \\ 0 & \dfrac{r}{1 + k\gamma} - \beta\gamma & 0 \\ -\gamma & -\gamma & 0 \end{pmatrix}$$

The eigenvalues of the above Jacobian matrix at E_1 are

$$\lambda_1 = \frac{r}{1+k\gamma} - \beta\gamma,\ \ \lambda_2 = -\frac{1}{2} + \frac{\sqrt{1-4\gamma}}{2},\ \text{and}\ \lambda_3 = -\frac{1}{2} - \frac{\sqrt{1-4\gamma}}{2}$$

Stable when $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 < 0$. Conditions that must be met when $\lambda_1 < 0$ if:

$$\frac{r}{1 + k\gamma} - \beta\gamma < 0,$$
$$\Leftrightarrow k > \frac{r - \beta\gamma}{\beta\gamma^2}.$$

a) The equilibrium point $E_2 = (\gamma, 0, \gamma)$ is stable spiral if it satisfies the conditions $\lambda_1 < 0, \lambda_2 < 0$, $\lambda_3 < 0$, and if the discriminant of the quadratic equation is $r^2 - 4r\gamma < 0$ or $\gamma > \frac{1}{4}$.

b) The equilibrium point $E_2 = (\gamma, 0, \gamma)$ is stable node if it satisfies the conditions $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$, and if the discriminant of the quadratic equation is $r^2 - 4r\gamma \geq 0$ or $\gamma \leq \frac{1}{4}$.

So, the equilibrium point $E_2 = (\gamma, 0, \gamma)$ is asymptotically stable if $k > \frac{r - \beta\gamma}{\beta\gamma^2}$.

Theorem 3. *The equilibrium point $E_3 = (x^*, y^*, z^*)$ is asymptotically stable if $b_1 > 0, b_3 > 0$ and $b_1 b_2 > b_3$*

Proof. The Jacobian matrix in **Equation (7)** at is as follows:

$$J(E_3) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}. \tag{8}$$

where

$h_{11} = 1 - \frac{2x^*}{z^*} - \alpha y^*,$

$h_{12} = -\alpha x^*,$

$h_{13} = \frac{x^{*2}}{z^{*2}},$

$h_{21} = -\frac{rky^*}{(1+kx^*)^2}\left(1 - \frac{\mu y^*}{z^*}\right) - \beta y^* - m y^{*2},$

$h_{22} = -\frac{r}{1+kx^*}\left(1 - \frac{\mu y^*}{z^*}\right) - \frac{r\mu y^*}{z^*(1+kx^*)} - \beta x^* - 2m x^* y^*,$

$h_{23} = \frac{r y^{*2}\mu}{z^{*2}(1+kx^*)},$

$h_{31} = -z^*,$

$h_{32} = -z^*,$

$h_{33} = \gamma - x^* - y^*.$

The corresponding characteristic equation of (8) is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

with

$$b_1 = -(h_{11} + h_{22} + h_{33}),$$
$$b_2 = h_{11}h_{22} + h_{11}h_{33} + h_{22}h_{33} - h_{13}h_{31} - h_{23}h_{32} - h_{12}h_{21},$$
$$b_3 = -h_{11}h_{22}h_{33} - h_{12}h_{23}h_{31} - h_{13}h_{21}h_{32} + h_{13}h_{31}h_{22} + h_{11}h_{23}h_{32} + h_{12}h_{21}h_{33}.$$

By Routh-Hurwitz criteria, equilibrium point $E_3 = (x^*, y^*, z^*)$ is asymptotically stable if $b_1 > 0, b_3 > 0$ and $b_1 b_2 > b_3$.

3.3 Model Simulation

Numerical simulations were performed to validate the results of the analysis method [22]. Numerical simulations were carried out with variations in several parameters to determine how changes in these parameters affect the population of toxic and non-toxic species. The following parameter values are required to perform numerical simulations.

Tabel 2. System Parameter Value

Parameters	Value	References
α	0.5	Safuan & Musa [6]
r	1	Safuan & Musa [6]
μ	0.2	Safuan & Musa [6]
β	0.5	Safuan & Musa [6]
γ	1	Safuan & Musa [6]
k	0.8	Assumptions
m	1	Assumptions

In this article, the parameter k or the fear effect parameter of the nontoxic species due to the toxic species is varied to determine the changes in the stability of some equilibrium points. Numerical simulation is performed by setting different values of parameter k ($k = 0.8, k = 0.9, k = 2, k = 4$). The phase portraits of numerical simulation results based on the parameter values in Table 2 with different values of parameter k are illustrated in Figure 1, Figure 2, Figure 3, and Figure 4.

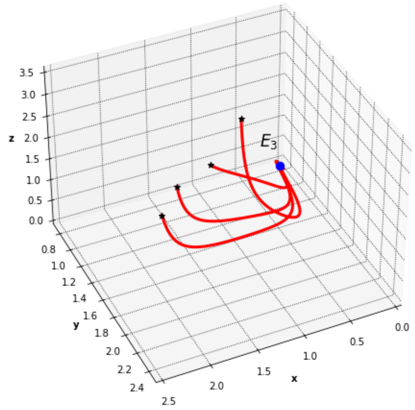

Figure 1. Phase portrait at $k = 0.8$

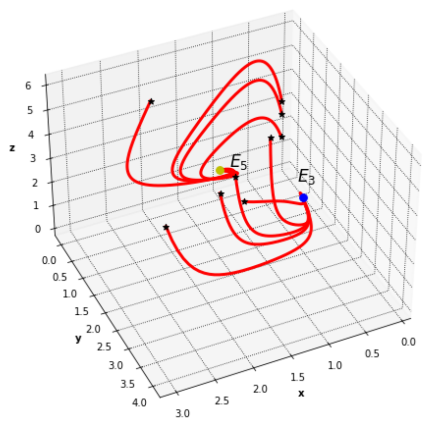

Figure 2. Phase portrait at $k = 0.9$

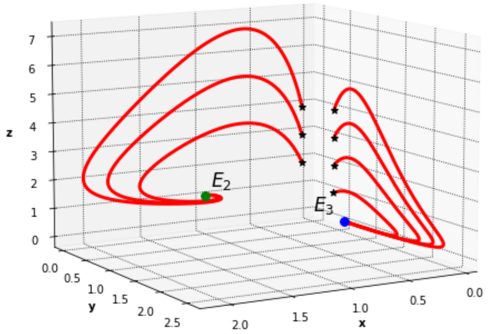

Figure 3. Phase portrait at $k = 4$

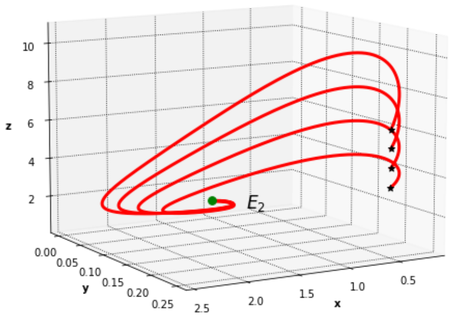

Figure 4. Phase portrait at $k = 5.5$

Based on **Figure 1**, the phase portrait fear effect parameter at $k = 0.8$ so that 3 equilibrium points exist namely E_1, E_2 and E_3. Figure 1 indicates stability towards the equilibrium point $E_3 = (0.117, 0.882, 0.211)$ and the equilibrium points E_1 and E_2 are unstable. The stability at point E_3 shown in the phase portrait means that the populations of toxic, non-toxic, and biotic resource species remain or all three can coexist.

Simulation of the model by increasing the parameter values in **Table 2**, by increasing the parameter values to $k = 0.9$ so that 5 equilibrium points exist namely E_1, E_2, E_3, E_4 and E_5. **Figure 2** is phase portraits that show stability at two points, called bistable. In **Figure 2**, the phase portraits towards the equilibrium points $E_3 = (0.117, 0.882, 0.211)$ and $E_5 = (0.894, 0.1055, 0.9443)$ are spiral stable and the equilibrium points E_1, E_2 and E_4 are unstable. The equilibrium point is $E_5 = (0.894, 0.1055, 0.9443)$, which was originally unstable becomes stable this shows that **Theorem 3** is satisfied

Furthermore, by increasing the fear effect parameter at $k = 4$, so that 4 equilibrium points exist namely E_1, E_2, E_3 and E_4. **Figure 3** is phase portraits that show stability at two points, so they are called bistable. In Figure 5, the phase portraits towards the equilibrium points $E_2 = (1, 0, 1)$ and $E_3 = (0.117, 0.882, 0.211)$ are spiral stable and the equilibrium points E_1 and E_4 are unstable. When the fear effect is given as $k = 0.9$ there is only one stable equilibrium point, namely E_3 and E_5. Then the value of the fear effect parameter is increased to $k = 4$ there are two stable equilibrium points, namely E_2 and E_3. The equilibrium point $E_2 = (1, 0, 1)$ which was originally unstable becomes stable this shows that **Theorem 2** is satisfied and the equilibrium point $E_3 = (0.117, 0.882, 0.211)$ which was originally unstable becomes stable this shows that **Theorem 3** is satisfied. This indicates a bifurcation or change in stability. The stability at point $E_2 = (1, 0, 1)$ shown in the phase portrait means that populations of toxic species and biotic resources remain but populations of non-toxic species are extinct.

For the final simulation by increasing the fear effect parameter $k = 5.5$, so that 2 equilibrium points exist namely E_1 and E_2. Based on **Figure 4**, the phase portrait at $k = 5.5$ shows stability towards the equilibrium point is $E_2 = (1, 0, 1)$, and the equilibrium point E_1 remains unstable. The result of numerical simulation shows that changes in the equilibrium point due to variation the parameter of the fear effect, when $k = 0.8$ there is only one stable point, namely E_3, when the value of k is increased to $k = 0.9$ it becomes bistable, namely at points E_3 and E_5. The value of the parameter k is increased again to $k = 4$ where it is still bistable at points E_2 and E_3, then increased again to $k = 5.5$ shows only one stable point at E_2.

Numerical continuation is performed on **Equation (4)** by running the value of k, the fear effect parameter. The results of numerical continuation of the parameter k cause changes in the stability of the equilibrium point which is illustrated by a bifurcation diagram, as shown in Figure 5.

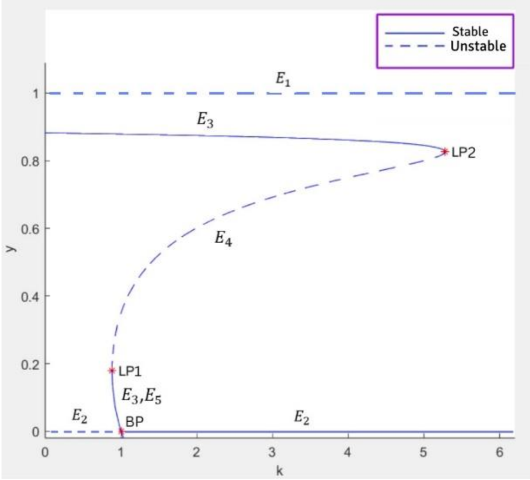

Figure 5. Bifurcation diagram

Continuation begins when the value of $k = 0.8$ indicates that the equilibrium point E_3 is stable, E_1 and E_2 are unstable. Then k is moved forward and LP1 (Limit Point) occurs at $k = 0.880185$. This LP1 shows a Saddle Node bifurcation where two equilibrium points intersect each other, namely E_3 and E_3 and disappear when k > 0.880185 accompanied by the emergence of another stable equilibrium point, namely E_5. Saddle Node bifurcation illustrates that when 0.880185 < k < 1 the system has two stable equilibrium points or called bistable, namely at E_3 and E_5.

The nature of stability in this system changes when passing BP (Branch Point), which is when $k = 1$. Changes in stability that occur in BP (Branch Point) type bifurcations include Transcritical bifurcations characterized by the intersection of the two branches of the equilibrium point. At equilibrium points E_3 and E_5 there is a change in stability after bifurcation where E_3 and E_5 become unstable. When $k < 1$ equilibrium point E_2 is unstable, while when $k > 1$ equilibrium point E_2 is stable. In addition, the previously unstable point E_3 becomes stable.

After that, parameter k moves forward and LP2 (Limit Point) occurs $k = 5.279992$. This LP2 shows the Saddle Node bifurcation that when 1 < k < 5.279992 has two stable equilibrium points or called bistable, namely E_2 and E_3. So it can be seen when bistable if 1 < k < 5.279992, also can see in **Figure 3** for numerical results are in accordance with the analytical results. After passing the Saddle Node bifurcation with a value of k > 5.27999, there are only two equilibrium points that exist, namely, E_1 and E_2 with only E_2 being stable.

4. CONCLUSIONS

1. Based on the assumptions set in this study, a mathematical model of the fear effect caused by allelopathy on competitive interaction model with shared biotic resources is formed, namely in System (4).
2. The system has three equilibrium points namely $E_1 = (0, \gamma, \mu\gamma)$, $E_2 = (\gamma, 0, \gamma)$, and $E_3 = (x^*, y^*, z^*)$. Stability analysis is performed at three equilibrium points, the equilibrium points E_1, E_2, and E_3 are asymptotically stable with certain conditions.
3. Numerical simulations were carried out on parameter values from research related to the fear effect model due to the release of toxins (allelopathy). Numerical simulations at equilibrium points show the same results as the analytical results. Numerical simulations with variations in several parameters show that the parameter k has an effect on changes in the population of toxic and non-toxic species, where the originally unstable equilibrium point becomes stable or otherwise. When the value of the fear effect parameter is low, $k = 0.8$ produces one stable equilibrium point. When the value of the fear effect parameter is increased to $k = 0.9$ and $k = 4$ results in bistable equilibrium points. After the parameter is increased to k = 5.5 there is one stable point. This shows that an increase in the fear effect results in a change in the stability of the three populations not to exist or cannot coexist. Bifurcation arises due to changes in the stability of the equilibrium point, which is expressed in the numerical continuity of variations in the parameters of the fear effect. Transcritical bifurcation occurs at $k = 1$, the first Saddle-Node bifurcation at $k = 0.880185$, and the second Saddle-Node bifurcation at $k = 5.279992$.

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