BAREKENG: Journal of Mathematics and Its Applications
March 2025 Volume 19 Issue 1 Page 0259–0270
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol19iss1pp0259-0270

A STUDY ON THE STRUCTURE OF MATRICES RELATED TO THE $VECH^*$, $VECP^*$, AND VEC OPERATORS

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ABSTRACT

The vec operator is an essential tool in matrix algebra that transforms a matrix into a column vector based on specific rules. This paper introduces two new operators, namely $vech^*$ and $vecp^*$, which take the main diagonal and supra-diagonal elements of the matrix, respectively. In this paper, we obtain the general form of the matrix ${B_n}^{*(p)}$, which transform $vech^*(A)$ to $vecp^*(A)$, with A as a matrix of size $n \times n$. In addition, we also develop the general forms of matrices ${D_n}^{*(h)}$ and ${D_n}^{*(p)}$, which transform $vech^*(A)$ into $vec(A)$ and $vecp^*(A)$ into $vec(A)$, with A as a symmetric matrix of size $n \times n$. This study also explores the properties and relationships between these matrices and their relevance to duplication and commutation matrices, providing deeper insights into the structure and operations of matrices.

Article History:

Received: 28th May 2024
Revised: 21st November 2024
Accepted: 21st November 2024
Published: 13th January 2025

Keywords:

vec;
$vech^*$;
$vecp^*$;
Duplication Matrix;
Commutation Matrix.

How to cite this article:

N. Hidayah, Yanita and A. Nazra., "A STUDY ON THE STRUCTURE OF MATRICES RELATED TO THE $VECH^*$, $VECP^*$, AND VEC OPERATORS," *BAREKENG: J. Math. & App.*, vol. 19, iss. 1, pp. 0259-0270, March, 2025.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

The vec operator transforms a matrix into a column vector by stacking the first column to the last column of a matrix vertically [1]. Similarly, [2] introduces the vech operator for symmetric matrices, which functions like the vec operator but excludes entries above the main diagonal. . In this case, the $vech$ operator transforms a symmetric matrix into a column vector by stacking only the relevant elements from each column, omitting those above the main diagonal. Both vec and $vech$ operators are widely utilized in multivariate statistics for deriving key results [2]. Furthermore, [3] extends the $vech$ operator, initially designed for symmetric matrices, to general square matrices by incorporating supra-diagonal elements. The vec operator also plays a significant role in analyzing structured families of symmetric stochastic matrices(see [4] and [5]), while [6] applies it to individual blocks of arbitrary matrices, providing a straightforward formula.

The vec operator related to the Kronecker product and the vec-permutation matrix can be seen in [7]. The matrix that transforms the vec operator with its transpose is the commutation matrix. The concept of the commutation matrix to the commutation tensor and the use of the commutation tensor to achieve the unification of the two formulae of the linear preserver of the matrix rank is extended by [8]. Another relationship between the vec operator related to the Kronecker product and the commutation matrix can be seen at [9], [10], and [11]. Then, there are three definitions of commutation matrix which are represented by [12] in different ways and it is proven that these three definitions are equivalent. Proof of the equivalent uses the properties in the Kronecker product on the matrix. Kronecker product is used in many branches of mathematics [13] and physics [14]. Rakotonirina [15], gives formulas of Kronecker commutation matrices (KCMs) in terms of some matrices of particle physics. By using the block matrix vec, and Ojeda [16], provides necessary and sufficient conditions for the factorization of a matrix into the Kronecker product of two matrices so that the basic algorithm is obtained to decide whether a matrix has square roots for the Kronecker product. Furthermore, More [17], derives a new approach to describing balanced and unbalanced partitioned block matrices and discusses the properties of the block operator vec and the block Kronecker product.

In addition to permutation and commutation matrices, the vec operator is also associated with the duplication matrix. The duplication matrix changes the $vech$ to vec. The duplication matrix is useful in various cases, such as in control theory [18] to show that the order of the external positive system can be reduced to $\frac{n(n+1)}{2}$ (LTI system analysis via conversion) and in statistics [19] to transform the multivariate to univariate system. There are several other matrix operators defined, such as $vecp$ [20], $vecd$ [21] and its applications [22], $veck$ and its applications [23]. Hidayah, et.al [24], introduces the $vech^*$and $vecp^*$ operators and constructs a matrix that transforms $vech^*$to $vecp^*$ for $n = 2, 3, 4, 5,$ and 6, and finds several properties.

This paper will construct matrices that transform $vech^*$ to $vecp^*$, $vech^*$ to vec, and $vecp^*$ to vec in general. Section I describes the development of the vec operator and its application through research that previous researchers have done. Section II gives some definitions and theorems which will be used. Section III present some interesting results about the $vech^*$ and $vecp^*$ operators.

2. RESEARCH METHODS

This section present definitions, properties, and theorems related to $\boldsymbol{vec}$, $\boldsymbol{vech}$, $\boldsymbol{vech}^*$, and $\boldsymbol{vecp}^*$.

Definition 1. [1] *Let $A = (a_{ij})$ be an $m \times n$ matrix and A_j as its jth column, then $vec(A)$ is the $mn \times 1$ vector gives by*

$$vec(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}.$$

For example, if A is the 2 ×3 matrix given by $A = \begin{pmatrix} 2 & 0 & 5 \\ 8 & 1 & 3 \end{pmatrix}$ then $vec(A)$ is the 6×1 vector given by $vec(A) = (2, 8, 0, 1, 5, 3)^T$.

Theorem 1. [1] *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be any two vectors, whereas A dan B are two matrices of the same size. Then*

a. $vec(\boldsymbol{a}) = vec(\boldsymbol{a}^T) = \boldsymbol{a}$,

b. $vec(\boldsymbol{ab}^T) = \boldsymbol{b} \otimes \boldsymbol{a}$,

c. $vec(\alpha A + \beta B) = \alpha vec(A) + \beta vec(B)$, *where* α *and* β *are scalars.*

Theorem 2. [1] *Let* $A = (a_{ij})$, $B = (b_{ij})$, *and* $C = (c_{ij})$ *be matrices of sizes* $m \times n$, $n \times p$, *and* $p \times q$, *respectively. Then* $vec(ABC) = (C^T \otimes A)vec(B)$.

Theorem 3. [1] *Let* $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, *and* $D = (d_{ij})$ *be matrices of sizes* $m \times n$, $n \times p$, $p \times q$, *and* $q \times m$, *respectively. Then* $tr(ABCD) = (vec(A^T))^T(D^T \otimes B)vec(C)$.

Definition 2. [3] *Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix. Then* $vech(A)$ *is the* $\frac{(n+1)}{2} \times 1$ *vector that is obtained from* $vec(A)$ *by eliminating all supra-diagonal elements of* A.

For example, for $n = 3$, $vech(A) = (a_{11}, a_{21}, a_{31}, a_{22}, a_{32}, a_{33})^T$.

Definition 3. [1] *Let* H_{ij} *be an* $m \times n$ *matrix that has its only nonzero element, a one, in the* (i, j)*th position. The* $mn \times mn$ *matrix, denoted by* K_{mn} *is given by*

$$K_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}(H_{ij} \otimes H_{ij}{}^T).$$

The matrix H_{ij} *can be conveniently expressed in terms of columns from the identity matrices* I_m *and* I_n. *If* $\boldsymbol{e}_{i,m}$ *is the* i*-th column of* I_m *and* $\boldsymbol{e}_{j,n}$ *is the* j*th column of* I_n, *then* $H_{ij} = \boldsymbol{e}_{i,m}\boldsymbol{e}_{j,n}{}^T$.

Theorem 4. [12] *Let* $K_{m,n}$ *be a commutation matrix.* Then the following statements are equivalent:

a. **Definition 3**

b. Let I_n be the identity matrix, and $\boldsymbol{e}_{i,m}$ is an m-dimensional column vector that has 1 in the i^{th} position and 0's elsewhere; that is:

$$\boldsymbol{e}_{i,m} = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T \text{ and } I_n \otimes \boldsymbol{e}_{i,m}{}^T = a_{ij}\boldsymbol{e}_{i,m}{}^T,\ a_{ij} \in I_n.$$

The commutation matrix, denoted by $K_{m,n}$ is given by:

$$K_{m,n} = \begin{pmatrix} I_n \otimes \boldsymbol{e}_{1,m}{}^T \\ I_n \otimes \boldsymbol{e}_{2,m}{}^T \\ \vdots \\ I_n \otimes \boldsymbol{e}_{m,m}{}^T \end{pmatrix}$$

c. A permutation matrix P is called a commutation matrix of a matrix, $m \times n$, if it satisfies the following conditions:

i. $P = [A_{ij}]$ is an $m \times n$ block matrix, with each blok A_{ij} being $n \times m$ matrix.

ii. For each $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $A_{ij} = (a_{st}{}^{(i,j)})$ is a (0,1) matrix with unique 1 which lies at the position (j, i).

We denote this commutation matrix by $K_{m,n}$, *thus a commutation matrix is of size* $mn \times mn$.

Theorem 5. [1] *Let* A *be an* $m \times n$ *matrix and* B *be a* $p \times q$. *Then* $vec(A \otimes B) = (I_n \otimes K_{mn} \otimes I_m)(vec(A) \otimes vec(B))$.

Definition 4. [19] *Let* $A = (a_{ij})$ *be an* $n \times n$ *symmetric matrix. The duplication matrix* D_n *is a matrix that transforms* $vech(A)$ *to* $vec(A)$ *of size* $n^2 \times \frac{n(n+1)}{2}$. *The general form of* D_n *is as follows:*

$$D_n = \begin{cases} \begin{pmatrix} O_{(i-1)\times(n-i+1)} \\ I_{(n-i+1)} \end{pmatrix}, & \text{for } i = j, \\ \left(O_{n\times(n-j+1)}\right), & \text{for } i < j, \\ \begin{pmatrix} O_{(j-1)\times(n-j+1)} \\ E^{(n-j+1)}_{1,(i-j+1)} \end{pmatrix}, & \text{for } i > j, \end{cases}$$

for each $i, j = 1,2, \dots, n$ where $E^n_{1,j} = \boldsymbol{e}_{i,n} \otimes \boldsymbol{e}^T_{j,n}$ with $\boldsymbol{e}_{i,n}$ is the ith unit vector of size $n \times 1$ and $\boldsymbol{e}_{j,n}$ is the jth unit vector of size $n \times 1$.

Definition 5. [24] *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $vech^*(A)$ is the $\frac{n(n+1)}{2} \times 1$ vector that is obtained from $vec(A)$ by eliminating all below elements of A, i.e.:*

$vech^*(A) = (a_{11}, a_{12}, a_{22}, a_{13}, a_{23}, a_{33}, \dots, a_{1n}, a_{2n}, \dots, a_{nn})^T.$

Definition 6. [24] *Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $vecp^*(A)$ is the $\frac{1}{2}n(n+1) \times 1$ vector that stacks the main diagonal elements and then the supra-diagonal elements in order of the first column to the last column of A, i.e.:*

$$vecp^*(A) = \left(a_{11}, a_{22}, a_{33}, \dots, a_{nn}, a_{12}, a_{13}, a_{23}, \dots, a_{1n}, \dots, a_{n-1,n}\right)^T.$$

For example, suppose given a matrix A of size 4×4 as follows:

$$A = \begin{pmatrix} 2 & 0 & 1 & 8 \\ 1 & 8 & 3 & 2 \\ 5 & 8 & 3 & 6 \\ 1 & 9 & 4 & 7 \end{pmatrix},$$

then $vech^*(A) = (2, 0, 8, 1, 3, 3, 8, 2, 6, 7)^T$ and $vecp^*(A) = (2, 8, 3, 7, 0, 1, 3, 8, 2, 6)^T$.

3. RESULTS AND DISCUSSION

Let A be an $n \times n$ matrix. Based on **Definition 5** and **Definition 6**, the operators $vech^*(A)$ and $vecp^*(A)$ have the same elements, only the position of the arrangement of the elements is different. Therefore, each element of $vech^*(A)$ can be associated with exactly one element of $vecp^*(A)$. Consequently, there is a transformation matrix which is symbolized as ${B_n}^{*(p)} = (b_{ij})$ with entries namely 1 and 0, where 1 represents the change in the position of the arrangement of elements from $vech^*(A)$ to $vecp^*(A)$ or $b_{ij} = 1$ represents the change from the jth row of $vech^*(A)$ to the ith row of $vecp * (A)$ such that

$$ {B_n}^{*(p)} vech^*(A) = vecp^*(A) \tag{1}$$

The form of ${B_n}^{*(p)}$ for $n = 2,3,4,5,6$ has been obtained in the article **[24]**. We need several symbols, i.e.:

a. $\boldsymbol{e}_{1,n}$ is a row matrix containing one element 1 in the first column,

b. $O_{m\times n}$ is a zero-matrix consisting of m-row and n-column,

c. $F_n = \left(O_{(n-1)\times 1}, I_{n-1}\right)$.

We obtain the form of ${B_n}^{*(p)}$ of size $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ which satisfies **Equation (1)** for any n as follows:

$$ {B_2}^{*(p)} = \begin{pmatrix} \boldsymbol{e}^T_{1,2} & O_{1\times 1} \\ O_{1\times 2} & \boldsymbol{e}^T_{1,1} \\ O_{1\times 1} & \boldsymbol{e}^T_{1,2} \end{pmatrix} \text{ and } {B_n}^{*(p)} = \begin{pmatrix} E^*_n \\ F^*_n \end{pmatrix} \text{ for } n \geq 3 \tag{2}$$

where

$$E_n^* = \begin{pmatrix} \boldsymbol{e}_{1,n}^T & O_{1\times 1} & O_{1\times 1} & \cdots & \cdots & O_{1\times n-1} \\ O_{1\times 2} & \boldsymbol{e}_{1,n-1}^T & O_{1\times 1} & \cdots & \cdots & O_{1\times n-1} \\ O_{1\times 3} & O_{1\times 1} & \boldsymbol{e}_{1,n-2}^T & \vdots & \vdots & O_{1\times n-1} \\ O_{1\times 4} & O_{1\times 1} & O_{1\times 1} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{1\times n} & O_{1\times n-1} & O_{1\times n-2} & \cdots & O_{1\times 2} & \boldsymbol{e}_{1,n}^T \end{pmatrix} \text{ and}$$

$$F_n^* = \begin{pmatrix} F_2 & O_{1\times 1} & O_{1\times n} & O_{1\times n-1} & \cdots & \cdots & O_{1\times 3} \\ O_{2\times 2} & F_3 & O_{2\times 1} & O_{2\times n} & O_{2\times n-1} & \cdots & \ddots \\ O_{3\times 3} & O_{3\times 2} & F_4 & O_{3\times 1} & O_{3\times n} & \ddots & \vdots \\ O_{4\times 4} & O_{4\times 3} & O_{4\times 2} & F_5 & O_{4\times 1} & \ddots & O_{n-3\times n-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & O_{n-2\times n} \\ O_{n-1\times n-1} & O_{n-1\times n-2} & O_{n-1\times n-3} & \cdots & O_{n-1\times 2} & F_n & O_{n-1\times 1} \end{pmatrix}$$

where size of matrix E_n^* is $n \times \frac{n(n+1)}{2}$ and matrix F_n^* is $\frac{n(n-1)}{2} \times \frac{n(n+1)}{2}$.

For example, for $n = 4$,

$$B_4{}^{*(p)} = \begin{pmatrix} \boldsymbol{e}_{1,4}^T & O_{1\times 1} & O_{1\times 2} & O_{1\times 3} \\ O_{1\times 2} & \boldsymbol{e}_{1,3}^T & O_{1\times 2} & O_{1\times 3} \\ O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,2}^T & O_{1\times 3} \\ O_{1\times 4} & O_{1\times 3} & O_{1\times 2} & \boldsymbol{e}_{1,1}^T \\ O_{1\times 1} & \boldsymbol{e}_{1,4}^T & O_{1\times 2} & O_{1\times 3} \\ O_{2\times 2} & F_3 & O_{2\times 1} & O_{2\times 4} \\ O_{3\times 3} & O_{3\times 2} & F_4 & O_{3\times 1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The properties associate with $B_n{}^{*(p)}$ are as follows:

Theorem 6. *The $B_n{}^{*(p)}$ is a permutation matrix.*

Proof. A matrix is said to be a permutation matrix if each row and each column of the matrix contains one element 1 and the other elements are 0. In **Equation (2)**, matrix $B_n{}^{*(p)}$ is formed from matrices E_n^* and F_n^*. Matrix E_n^* contains one unit vector in each row and each column, while the other elements contain a zero matrix. The F_n^* matrix contains the F_n matrix, where $F_n = \left(O_{(n-1)\times 1}, I_{n-1}\right)$, meaning that F_n^* contains the identity matrix, which has one 1 element in each row and each column, while the other elements contain a zero matrix. So, the matrix $B_n{}^{*(p)}$ is a permutation matrix. ■

Corollary 1. *The $B_n{}^{*(p)}$ is an orthogonal matrix.*

Proof. The proof is analogous to the proof of Theorem 3.1 (see [24]). ■

Theorem 7. *The $B_n{}^{*(p)}$ is a unique matrix.*

Proof. Let A be an $n \times n$ matrix. Suppose $X = (x_{ij})$ and $Y = (y_{ij})$ are $B_n{}^{*(p)}$ of size $\frac{n(n+1)}{2}$ that transforms $vech^*(A)$ to $vecp^*(A)$ such that $Xvech^*(A) = vecp^*(A)$ and $Yvech^*(A) = vecp^*(A)$. From **Theorem 6**, it is obtained that $B_n{}^{*(p)}$ is a permutation matrix. That is, each element of $vech^*(A)$ is individually mapped to an element of $vecp^*(A)$. As a result, X and Y are permutation matrices with entries $x_{ij} = y_{ij}$ for each $i = 1,2,\cdots,\frac{n(n+1)}{2}$ and $j = 1,2,\cdots,\frac{n(n+1)}{2}$ or $X = Y$ so that it is proven that the $B_n{}^{*(p)}$ is a unique matrix. ■

Theorem 8. *Let A be an $n \times n$ matrix, then*

a. $tr\left({B_n}^{*(p)}\right) = 1,$

b. $det\left({B_n}^{*(p)}\right) = \begin{cases} -1, & if\ {B_n}^{*(p)}\ is\ an\ odd\ permutation\ matrix, \\ 1, & if\ {B_n}^{*(p)}\ is\ an\ even\ permutation\ matrix. \end{cases}$

Proof. (a) Based on **Equation (2)**, the ${B_n}^{*(p)}$ is formed from the partition matrix E_n^* and F_n^*, where the number 1 is always located on the main diagonal in the first row and first column while the main diagonal in the other rows and columns contain the number 0. Since a trace is the sum of the entries on the main diagonal, then $tr\left({B_n}^{*(p)}\right) = 1$. (b) Based on the article **[24]** it is found that if the permutation of the ${B_n}^{*(p)}$ matrix is odd then $det\left({B_n}^{*(p)}\right) = -1$ and otherwise. Consequently, it can be concluded that if ${B_n}^{*(p)}$ is an odd permutation matrix then $det\left({B_n}^{*(p)}\right) = -1$, and if ${B_n}^{*(p)}$ is an odd permutation matrix then $det\left({B_n}^{*(p)}\right) = 1$.■

Let $S = \left(s_{ij}\right)$ be an $n \times n$ symmetric matrix, which means $s_{ij} = s_{ji}$. Based on **Definition 5** and **Definition 6**, every element of $vec(S)$ is a duplicate of the elements $vech^*(S)$ and $vecp^*(S)$, except for the element on the main diagonal. Therefore, every $vech^*(S)$ and $vecp^*(S)$ element can be associated with $vec(S)$ elements. Consequently, there exists a unique matrix denoted as ${D_n}^{*(h)} = \left(h_{ij}\right)$ and ${D_n}^{*(p)} = \left(p_{ij}\right)$, with the elements 1 and 0, where $h_{ij} = 1$ represents the change from the jth row of $vech^*(S)$ to the ith row of $vec(S)$, and $p_{ij} = 1$ represents the change from the jth row of $vecp^*(S)$ to the ith row of $vec(S)$ such that

$$ {D_n}^{*(h)} vech^*(S) = vec(S) \tag{3} $$

and

$$ {D_n}^{*(p)} vecp^*(S) = vec(S). \tag{4} $$

The ${D_n}^{*(h)}$ of size $n^2 \times \frac{n(n+1)}{2}$ can be constructed in general as follows:

$$ {D_n}^{*(h)} = \begin{cases} \begin{pmatrix} I_i \\ O_{(n-i)\times i} \end{pmatrix}, \text{for } i = j, \\ \begin{pmatrix} \boldsymbol{e}_{j,j} \otimes \boldsymbol{e}_{i,j}^T \\ O_{(n-i)\times i} \end{pmatrix}, \text{for } i < j, \\ \left(O_{n\times j}\right), \text{for } i > j \end{cases} $$

where $i, j = 1,2, \dots, n$.

For example, $n = 3$,

$$ {D_3}^{*(h)} = \begin{pmatrix} \begin{pmatrix} I_1 \\ O_{2\times 1} \end{pmatrix} & \begin{pmatrix} \boldsymbol{e}_{2,2} \otimes \boldsymbol{e}_{1,2}^T \\ O_{1\times 2} \end{pmatrix} & \left(\boldsymbol{e}_{3,3} \otimes \boldsymbol{e}_{1,3}^T\right) \\ \left(O_{3\times 1}\right) & \begin{pmatrix} I_2 \\ O_{1\times 2} \end{pmatrix} & \left(\boldsymbol{e}_{3,3} \otimes \boldsymbol{e}_{2,3}^T\right) \\ \left(O_{3\times 1}\right) & \left(O_{3\times 2}\right) & \left(I_3\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. $$

Next, the ${D_n}^{*(p)}$ of size $n^2 \times \frac{n(n+1)}{2}$ can be constructed in general as follows:

$$D_n{}^{*(p)} = \begin{cases} \left(\boldsymbol{e}_{1,n} \otimes \boldsymbol{e}_{i,n}^T\right), & \text{for } j = 1, \\ \begin{pmatrix} I_{i-1} \\ O_{(n-i+1)\times(i-1)} \end{pmatrix}, \text{for } i = j, \text{where } i,j = 2,3, \dots n \\ \left(O_{n\times(j-1)}\right), \text{for } i > j, \text{where } j = 2,3, \dots, n-1, \\ \begin{pmatrix} O_{1\times(n-1)} \\ \boldsymbol{e}_{(n-1),(n-1)} \otimes \boldsymbol{e}_{i,(n-1)}^T \end{pmatrix}, \text{for } i = 1,2, \dots, n-1 \text{ and } j = n, \\ \left(\boldsymbol{e}_{2,n}\right), \text{for } i < j \text{ and } j = 2, \\ \begin{pmatrix} O_{2\times(j-1)} \\ \boldsymbol{e}_{(j-2),(j-1)} \otimes \boldsymbol{e}_{i,(j-1)}^T \\ O_{(n-1-j)\times(j-1)} \end{pmatrix}, \text{for } i < j, \text{where } j = 3,4, \dots, n-1. \end{cases}$$

For example, $n = 3$,

$$D_3{}^{*(p)} = \begin{pmatrix} \left(\boldsymbol{e}_{1,3} \otimes \boldsymbol{e}_{1,3}^T\right) & \left(\boldsymbol{e}_{2,3}\right) & \begin{pmatrix} O_{1\times 2} \\ \boldsymbol{e}_{2,2} \otimes \boldsymbol{e}_{1,2}^T \end{pmatrix} \\ \left(\boldsymbol{e}_{2,3} \otimes \boldsymbol{e}_{2,3}^T\right) & \begin{pmatrix} I_1 \\ O_{2\times 1} \end{pmatrix} & \begin{pmatrix} O_{1\times 2} \\ \boldsymbol{e}_{2,2} \otimes \boldsymbol{e}_{2,2}^T \end{pmatrix} \\ \left(\boldsymbol{e}_{3,3} \otimes \boldsymbol{e}_{3,3}^T\right) & \left(O_{3\times 1}\right) & \begin{pmatrix} I_1 \\ O_{2\times 1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The following section provides a theorem that explains the relationship between $B_n{}^{*(p)}$, $D_n{}^{*(h)}$, and $D_n{}^{*(p)}$.

Theorem 9. *Let S be an $n \times n$ symmetric matrix. Then*

a. $D_n{}^{*(h)} = D_n{}^{*(p)} B_n{}^{*(p)}$,

b. $D_n{}^{*(p)} = D_n{}^{*(h)} B_n{}^{*(p)T}$.

Proof. (a) Based on **Equation (3)** obtained

$$vec(S) = D_n{}^{*(h)} vech^*(S) \tag{5}$$

Equation (4) obtained

$$vec(S) = D_n{}^{*(p)} vecp^*(S) \tag{6}$$

and **Equation (1)** obtained

$$vecp^*(S) = B_n{}^{*(p)} vech^*(S) \tag{7}$$

Substituting **Equation (7)** into **Equation (6)**, so that

$$vec(S) = D_n{}^{*(p)} B_n{}^{*(p)} vech^*(S) \tag{8}$$

Furthermore, substituting **Equation (5)** into **Equation (8)** is obtained

$$D_n{}^{*(h)} vech^*(S) = D_n{}^{*(p)} B_n{}^{*(p)} vech^*(S) \tag{9}$$

So, we have $D_n{}^{*(h)} = D_n{}^{*(p)} B_n{}^{*(p)}$.

(b) From (a) obtained $D_n{}^{*(h)} = D_n{}^{*(p)} B_n{}^{*(p)}$. Since $B_n{}^{*(p)}$ is nonsingular then $B_n{}^{*(p)}$ is invertible. Thus, premultiplying both sides by $B_n{}^{*(p)-1}$ and obtained

$$D_n{}^{*(h)} B_n{}^{*(p)-1} = D_n{}^{*(p)} B_n{}^{*(p)} B_n{}^{*(p)-1} = D_n{}^{*(p)} I_{\frac{n(n+1)}{2}} = D_n{}^{*(p)} \tag{10}$$

From **Corollary 1**, $B_n{}^{*(p)}$ is an orthogonal matrix then $B_n{}^{*(p)-1} = B_n{}^{*(p)T}$. So,

$$D_n{}^{*(p)} = D_n{}^{*(h)} B_n{}^{*(p)T} \blacksquare$$

The properties of the ${D_n}^{*(h)}$ and ${D_n}^{*(p)}$ are discussed in the following theorem.

Theorem 10. *Let ${D_n}^{*(h)}$ and ${D_n}^{*(p)}$ both be analogous to the duplication matrix of size $n^2 \times \frac{n(n+1)}{2}$ and have a Moore-Penrose inverse, then*

a. $rank({D_n}^{*(h)}) = rank({D_n}^{*(p)}) = \frac{n(n+1)}{2}$,

b. ${D_n}^{*(h)+} = ({D_n}^{*(h)T}{D_n}^{*(h)})^{-1}{D_n}^{*(h)T}$,

c. ${D_n}^{*(p)+} = ({D_n}^{*(p)T}{D_n}^{*(p)})^{-1}{D_n}^{*(p)T}$,

d. ${D_n}^{*(h)+}{D_n}^{*(h)} = I_{\frac{n(n+1)}{2}}$ *and* ${D_n}^{*(p)+}{D_n}^{*(p)} = I_{\frac{n(n+1)}{2}}$,

e. ${D_n}^{*(h)+}vec(A) = vech^*(A)$ *for every* $n \times n$ *symmetric matrix* A,

f. ${D_n}^{*(p)+}vec(A) = vecp^*(A)$ *for every* $n \times n$ *symmetric matrix* A,

Proof. Part (a), since A is a symmetric matrix, means that $vech^*(A)$, $vecp^*(A)$, and $vec(A)$ have exactly the same elements. Suppose ${D_n}^{*(h)}vech^*(A) = \mathbf{0}$ and ${D_n}^{*(p)}vecp^*(A) = \mathbf{0}$. So according to the Equation (3) and Equation (4), it means $vec(A) = \mathbf{0}$. This implies that $vech^*(A) = vecp^*(A) = \mathbf{0}$. Thus, ${D_n}^{*(h)}vech^*(A) = \mathbf{0}$ and ${D_n}^{*(h)}vech^*(A) = \mathbf{0}$ if $vech^*(A) = vecp^*(A) = \mathbf{0}$, so the matrix ${D_n}^{*(h)}$ and ${D_n}^{*(p)}$ has full column rank. Parts (b), (c), and (d) follow immediately from (a) and Theorem 5.3(h) [1], whereas (e) is obtained by premultiplying ${D_n}^{*(h)}vech^*(A) = vec(A)$ by ${D_n}^{*(h)+}$ and then applying (d) and (f) is obtained by premultiplying ${D_n}^{*(p)}vecp^*(A) = vec(A)$ by ${D_n}^{*(p)+}$ and then applying (d). ■

The following section provides a theorem that explains the relationship between ${B_n}^{*(p)}$, ${D_n}^{*(h)}$, and ${D_n}^{*(p)}$.

Theorem 11. *Let S be an $n \times n$ symmetric matrix. Then,*

a. ${D_n}^{*(h)+} = {B_n}^{*(p)T}{D_n}^{*(p)+}$,

b. ${D_n}^{*(p)+} = {B_n}^{*(p)}{D_n}^{*(h)+}$.

Proof. Based on Theorem 10, we have

$$ {D_n}^{*(h)+} = ({D_n}^{*(h)T}{D_n}^{*(h)})^{-1}{D_n}^{*(h)T} \tag{11} $$

and

$$ {D_n}^{*(p)+} = ({D_n}^{*(p)T}{D_n}^{*(p)})^{-1}{D_n}^{*(p)T} \tag{12} $$

From Theorem 9, substitute ${D_n}^{*(h)} = {D_n}^{*(p)}{B_n}^{*(p)}$ into Equation (11) and ${D_n}^{*(p)} = {D_n}^{*(h)}{B_n}^{*(p)T}$ into Equation (12) so obtained the desired result. ■

We will give some theorems related to the relationship between $vech^*$ and $vecp^*$ with ${B_n}^{*(p)}$, ${D_n}^{*(h)}$, ${D_n}^{*(p)}$, and the commutation matrix, K_{mn}.

Theorem 12. *Let A be an $n \times m$ matrix, B be an $m \times m$ matrix, and C be an $m \times n$ matrix. If ABC is a symmetric matrix then*

$$ vech^*(ABC) = {D_n}^{*(h)+}(C^T \otimes A){D_m}^{*(h)}vech^*(B) = {D_n}^{*(h)+}(C^T \otimes A){D_m}^{*(p)}vecp^*(B) $$

and

$$ vecp^*(ABC) = {D_n}^{*(p)+}(C^T \otimes A){D_m}^{*(p)}vecp^*(B) = {D_n}^{*(p)+}(C^T \otimes A){D_m}^{*(h)}vech^*(B). $$

Proof. Based on Theorem 2, we have

$$ vec(ABC) = (C^T \otimes A)vec(B) \tag{13} $$

By using Definition 5, Equation (13) becomes

$$ {D_n}^{*(h)}vech^*(ABC) = (C^T \otimes A){D_m}^{*h}vech^*(B) \tag{14} $$

Premultiplying both sides by ${D_n}^{*(h)+}$ so that **Equation (14)** becomes

$$vech^*(ABC) = {D_n}^{*(h)+}(C^T \otimes A){D_m}^{*(h)}vech^*(B) \tag{15}$$

By using **Theorem 9(a)** and **Equation (1)**, **Equation (15)** becomes

$$vech^*(ABC) = {D_n}^{*(h)+}(C^T \otimes A){D_m}^{*(p)}vecp^*(B) \tag{16}$$

Premultiplying both sides by ${B_n}^{*(p)}$ so that **Equation (16)** becomes

$$vecp^*(ABC) = {D_n}^{*(p)+}(C^T \otimes A){D_m}^{*(p)}vecp^*(B) \tag{17}$$

By using **Theorem 9(b)**, **Equation (17)** becomes

$$vecp^*(ABC) = {D_n}^{*(p)+}(C^T \otimes A){D_m}^{*(h)}{B_m}^{*(p)T}vecp^*(B). \tag{18}$$

Based on **Corollary 1** then ${B_m}^{*(p)T} = {B_m}^{*(p)-1}$ so that **Equation (18)** becomes

$$vecp^*(ABC) = {D_n}^{*(p)+}(C^T \otimes A){D_m}^{*(h)}vech^*(B) \tag{19}$$

The proof is complete. ∎

Theorem 13. *Let A be an $n \times n$ matrix, B be an $n \times m$ matrix, C be an $m \times m$ matrix, and D be an $m \times n$ matrix. If A and C are symmetric matrices, then*

$$tr(ABCD) = \left(vecp^*(A)\right)^T {D_n}^{*(p)T}(D^T \otimes B){D_m}^{*(p)}vecp^*(C) = \left(vech^*(A)\right)^T {D_n}^{*(h)T}(D^T \otimes B){D_m}^{*(h)}vech^*(C).$$

Proof. Based on **Theorem 3**, we have

$$tr(ABCD) = \left(vec(A^T)\right)^T (D^T \otimes B)vec(C) \tag{20}$$

Since A is a symmetric matrix then $A^T = A$ and by using **Equation (4)** so that **Equation (20)** becomes

$$tr(ABCD) = \left({D_n}^{*(p)}vecp^*(A)\right)^T (D^T B){D_m}^{*(p)}vecp^*(C) \tag{21}$$

By using **Theorem 9(a)**, **Equation (21)** becomes

$$tr(ABCD) = \left({B_n}^{*(p)T}vecp^*(A)\right)^T {D_n}^{*(h)T}(D^T \otimes B){D_m}^{*(h)}vech^*(C) \tag{22}$$

Based on **Corollary 1**, we have ${B_n}^{*(p)T} = {B_n}^{*(p)-1}$ and by using **Equation (1)**, **Equation (22)** becomes

$$tr(ABCD) = \left(vech^*(A)\right)^T {D_n}^{*(h)T}(D^T \otimes B){D_m}^{*(h)}vech^*(C) \tag{23}$$

From **Equation (21)** and **Equation (23)**, the proof is complete. ∎

Theorem 14. *Let A be an $n \times n$ matrix. Then ${D_n}^{*(h)}vech^*(A) = {D_n}^{*(p)}vecp^*(A) = vec(A_U + A_U^T - D_A$, where A_U is the upper triangular matrix obtained from A by replacing a_{ij} by 0 for $i > j$, and D_A is the diagonal matrix having the same diagonal entries as A.*

Proof. Suppose A given a matrix of size $n \times n$ as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Based on **Equation (3)** and **Equation (4)**, A is a symmetric matrix. Thus, by following the rules of **Definition 5** and **Definition 6**, the elements selected are the main diagonal and the supra-diagonal elements. Therefore, the matrix formed is the upper triangular matrix from A which is denoted by A_U as follows:

$$A_U = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

By adding the A_U with its transpose, we have

$$A_U + A_U^T = A = \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & 2a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 2a_{nn} \end{pmatrix} \tag{24}$$

From **Equation (24)** the element of the main diagonal is $2a_{ij}$ for $i = j$ where $i, j = 1, 2, \cdots, n$. By paying attention to **Definition 5** and **Equation (6)**, the main diagonal of A acts as a reflection axis, so that the diagonal matrix is formed which is denoted by D_A as follows:

$$D_A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Equation (24) reduced by D_A obtained

$$(A_U + A_U^T) - D_A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} = A'$$

Consequently, ${D_n}^{*(h)} vech^*(A) = {D_n}^{*(p)} vecp^*(A) = vec(A') = vec(A_U + A_U^T - D_A)$. ■

Theorem 15. *Let A be an $n \times n$ matrix and B be an $m \times m$ matrix. Then*

a. $vecp^*(A \otimes B) = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} \otimes {D_m}^{*(p)}\right)(vecp^*(A) \otimes vecp^*(B)) = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(h)} \otimes {D_m}^{*(h)}\right)(vech^*(A) \otimes vech^*(B))$,

b. $vech^*(A \otimes B) = {D_{nm}}^{*(h)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(h)} \otimes {D_m}^{*(h)}\right)(vech^*(A) \otimes vech^*(B)) = {D_{nm}}^{*(h)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} \otimes {D_m}^{*(p)}\right)(vecp^*(A) \otimes vecp^*(B))$.

Proof. (a) Based on **Theorem 5**, we have

$$vec(A \otimes B) = (I_n \otimes K_{mn} \otimes I_m)(vec(A) \otimes vec(B)) \tag{25}$$

Analogously, by using **Equation (4)**, we have

$$vecp^*(A \otimes B) = (I_n \otimes K_{mn} \otimes I_m)(vecp^*(A) \otimes vecp^*(B))$$

$${D_{nm}}^{*(p)} vecp^*(A \otimes B) = (I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} vecp^*(A) \otimes {D_m}^{*(p)} vecp^*(B)\right)$$

$${D_{nm}}^{*(p)+} {D_{nm}}^{*(p)} vecp^*(A \otimes B) = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} vecp^*(A) \otimes {D_m}^{*(p)} vecp^*(B)\right)$$

$$vecp^*(A \otimes B) = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} vecp^*(A) \otimes {D_m}^{*(p)} vecp^*(B)\right) \tag{26}$$

By using **Theorem 5 [1]**, consider that

$$vecp^*(A \otimes B) = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} vecp^*(A) \otimes {D_m}^{*(p)} vecp^*(B)\right)$$

$$= {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} \otimes {D_m}^{*(p)}\right)\left({B_n}^{*(p)} vech^*(A) \otimes {B_m}^{*(p)} vech^*(B)\right)$$

$$= {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)\left({D_n}^{*(p)} \otimes {D_m}^{*(p)}\right)\left({B_n}^{*(p)} \otimes {B_m}^{*(p)}\right)(vech^*(A) \otimes vech^*(B))$$

Thus, **Equation (26)** becomes

$$vecp^*(A \otimes B) = XY \tag{27}$$

where

$$X = {D_{nm}}^{*(p)+}(I_n \otimes K_{mn} \otimes I_m)$$

$$Y = \left({D_n}^{*(h)} \otimes {D_m}^{*(h)}\right)(vech^*(A) \otimes vech^*(B))$$

(b) By using **Equation (1)** and premultiplying both sides by ${B_{nm}}^{*(p)-1}$, **Equation (27)** becomes

$$vech^*(A \otimes B) = {B_{nm}}^{*(p)-1} XY \tag{28}$$

Based on **Corollary 1**, we have ${B_{nm}}^{*(p)-1} = {B_{nm}}^{*(p)T}$ and by using **Theorem 11(a)** and **9(a)**, **Equation (28)** becomes

$$vech^*(A \otimes B) = PQ \tag{29}$$

where

$$P = D_{nm}{}^{*(h)+}(I_n \otimes K_{mn} \otimes I_m)$$

$$Q = \left(D_n{}^{*(p)}B_n{}^{*(p)} \otimes D_m{}^{*(p)}B_m{}^{*(p)}\right)(vech^*(A) \otimes vech^*(B))$$

By using **Theorem 5 [1]** and **Equation (1)**, **Equation (29)** becomes

$$vech^*(A \otimes B) = D_{nm}{}^{*(h)+}(I_n \otimes K_{mn} \otimes I_m)\left(D_n{}^{*(p)} \otimes D_m{}^{*(p)}\right)(vecp^*(A) \otimes vecp^*(B))$$

The proof is complete. ■

4. CONCLUSIONS

This article provides the properties of matrices that transform $vech^*$ to $vecp^*$, $vech^*$ to vec, and $vecp^*$ to vec, as well as the relationships of each of these matrices. It is suggested for further research is to find the relationship between the $vech^*$ and $vecp^*$ operators in the field of statistics.

ACKNOWLEDGMENT

This research was funded by DRTPM Dikti (Grand No. 115/EG/PG.02.00.PL/2023).

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