BAREKENG: Journal of Mathematics and Its Applications
December 2024 Volume 18 Issue 4 Page 2747–2756
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol18iss4pp2747-2756

A COMPLETION THEOREM FOR COMPLEX VALUED *S*-METRIC SPACE

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ABSTRACT

Article History:

Received: 13th, June 2024
Revised: 8th, August 2024
Accepted: 6th, September 2024
Published: 14th, October 2024

Keywords:

Completion;
Complex;
S-metric Space.

Any complex valued S-metric space where each Cauchy sequence converges to a point in this space is said to be complete. However, there are complex valued S-metric spaces that are incomplete but can be completed. A completion of a complex valued S-metric space (X, S_c) is defined as a complete complex valued S-metric space $(\hat{X}, \hat{S_c})$ with an isometry $i: X \to \hat{X}$ such that $i(X)$ is dense in $\hat{X}$. In this paper, we prove the existence of a completion for a complex valued S-metric space. The completion is constructed using the quotient space of Cauchy sequence equivalence classes within a complex valued S-metric space. This construction ensures that the new space preserves the essential properties of the original S-metric space while being completeness. Furthermore, isometry and denseness are redefined regarding a complex valued S-metric space, generalizing those established in a complex valued metric space. In addition, an example is also presented to illustrate the concept, demonstrating how to find a unique completion of a complex valued S-metric space.

How to cite this article:

M. Kiftiah, Yundari, Suryani and N. Lauren., "A COMPLETION THEOREM FOR COMPLEX VALUED S-METRIC SPACE," *BAREKENG: J. Math. & App.*, vol. 18, iss. 4, pp. 2747-2756, December, 2024.

1. INTRODUCTION

One knows that the set of all rational numbers $\mathbb{Q}$ is incomplete because it contains Cauchy sequences that do not converge. Indeed, the set of all real numbers $\mathbb{R}$ can be defined as the completion of $\mathbb{Q}$ which is constructed using equivalence classes of Cauchy sequences. Due to the specific convergence properties of the metric and the fact that convergent sequences are Cauchy sequences, the completion of an incomplete metric space by a Cauchy sequence can also be obtained.

Several authors have proposed different types of metric space generalizations, such as [1], [2], [3], [4], [5], [6], [7], [8]. Completing these spaces has also been studied, and many results have been proven. For instance, Dun and Hang [9] demonstrated that the metrization theorem on b-metric spaces of Paluszyn´ski and Stempak [10] can be used to complete any b-metric space. This approach overcomes the restriction of using the quotient space of Cauchy sequence equivalence classes for constructing the completion of a metric space. Ge and Lin [11] proved the existence and uniqueness completion theorem for partial metric spaces by introducing symmetrically dense subsets. In response to Ge and Lin's question about the denseness property, Dung [12] provided an example of a partial metric space. Furthermore, Hasanah and Supeno [13] investigated the completion characteristics of complex valued metric space. They constructed the completion of this space by redefining the concepts of isometry and denseness, as defined in metric spaces. In their recent paper, Beg et al. [14] explored the completion of a complex valued strong b-metric space.

The completion of a metric space using Cauchy sequences can be obtained due to convergence properties possessed by the metric and the fact that convergent sequences are Cauchy sequences. Inspired by Azam et al. [4], a new space has recently been proposed as a generalization of the S-metric space, known as complex valued S-metric space [15]. However, there is no work discussing the notion of completion for complex valued S-metric space. Motivated by prior research, this paper aims to show the existence and uniqueness of completion in terms of complex valued S-metric space.

2. RESEARCH METHODS

First, we recall some fundamental concepts needed to discuss completion in complex valued S-metric space, including the partial order on the set of all complex numbers $\mathbb{C}$ given in Azam et. al. [4] and the complex valued S-metric space described in Mlaiki [15] and Tas and Ozgur [16].

Definition 1. (see [4]) *Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. The partial order $\precsim$ on $\mathbb{C}$ is defined as follows:*

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

From **Definition 1**, it implies that $z_1 \precsim z_2$ if one of these conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

It is written as $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of the conditions (i), (ii), or (iii) holds. If only the condition (iii) is satisfied then we can write $z_1 \prec z_2$.

The properties of partial order on $\mathbb{C}$ are given by

(i) $0 \precsim z_1 \lnsim z_2$ implies $|z_1| < |z_2|$,
(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$,
(iii) $0 \precsim z_1 \precsim z_2$ implies $|z_1| \leq |z_2|$
(iv) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2$, for all $z_1, z_2 \in \mathbb{C}$.

Definition 2. (see [15], [16]) *Let X be a nonempty set. A complex valued S-metric on X is a map $S_c: X^3 \to \mathbb{C}$ such that the following conditions hold for all $z_1, z_2, z_3, a \in X$.*

(i) $0 \precsim S_c(z_1, z_2, z_3)$,

(ii) $S_c(z_1, z_2, z_3) = 0$ *if and only if* $z_1 = z_2 = z_3$
(iii) $S_c(z_1, z_2, z_3) \precsim S_c(z_1, z_1, a) + S_c(z_2, z_2, a) + S_c(z_3, z_3, a)$.
Henceforth, the pair (X, S_c) *is called a complex valued* S*-metric space.*

Definition 3. (see [15], [16]) *Let* (X, S_c) *be a complex valued* S*-metric space.*

1) *A sequence* $\{z_n\} \subset X$ *converges to* $z \in X$ *if* $S_c(z_n, z_n, z) \to 0$ *as* $n \to \infty$. *That is for all c such that* $0 \prec c \in \mathbb{C}$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S_c(z_n, z_n, z) \prec c$. *We denote this by* $\lim_{n\to\infty} z_n = z$.
2) *A sequence* $\{z_n\} \subset X$ *is called a Cauchy sequence if* $S_c(z_n, z_n, z_m) \to 0$. *That is for all c such that* $0 \prec c \in \mathbb{C}$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m \geq n_0$ *we have* $S_c(z_n, z_n, z_m) \prec c$.
3) *A complex valued* S*-metric space* (X, S_c) *is said to be complete if every Cauchy sequence is convergent.*

Lemma 1. (see [15], [16]) *Let* (X, S_c) *be a complex valued* S*-metric space and* $\{z_n\}$ *be a sequence in* X. *Then* $\{z_n\}$ *converges to* z *if and only if* $|S_c(z_n, z_n, z)| \to 0$ *as* $n \to \infty$.

Lemma 2. (see [15], [16]) *Let* (X, S_c) *be a complex valued* S*-metric space and* $\{z_n\}$ *be a sequence in* X. *Then* $\{z_n\}$ *is a cauchy sequence if and only if* $|S_c(z_n, z_n, z_m)| \to 0$ *as* $n, m \to \infty$.

Lemma 3. (see [15], [16]) *If* (X, S_c) *be a complex valued* S*-metric space, then* $S_c(z_1, z_1, z_2) = S_c(z_2, z_2, z_1)$ *for all* $z_1, z_2 \in X$.

Following the steps to show the existence of completion. Let a complex valued $\boldsymbol{S}$-metric space $(\boldsymbol{X}, \boldsymbol{S_c})$ which is not $\boldsymbol{S}$-complete. First, we identify any two Cauchy sequences in $(\boldsymbol{X}, \boldsymbol{S_c})$ and define $\mathrm{C}[\boldsymbol{x}]$ as the equivalence classes of these sequences. Next, we form a new space consisting of these classes, denoted as $\widehat{\boldsymbol{X}}$. Furthermore, we define a map $\widehat{\boldsymbol{S_c}}$ that maps from the product space $\widehat{\boldsymbol{X}}^{\boldsymbol{3}}$ to $\mathbb{C}$ and prove that it is a complex valued $\boldsymbol{S}$-metric. Thus, $(\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{S_c}})$ is a complex valued $\boldsymbol{S}$-metric space. To show that $(\boldsymbol{X}, \boldsymbol{S_c})$ and $(\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{S_c}})$ are isometric, we define a map $\boldsymbol{i}: \boldsymbol{X} \to \widehat{\boldsymbol{X}}$. Moreover, we prove that $\boldsymbol{i}(\boldsymbol{X})$ is a dense in $\widehat{\boldsymbol{X}}$. Using this denseness, we prove that $(\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{S_c}})$ is a complete space.

The next step is to show that the completion is unique. We assume $(\hat{X}, \widehat{S_c})$ and $(X^*, {S_c}^*)$ are two completions of (X, S_c), where $f: X \to \hat{X}$ and $g: X \to X^*$ are respective isometries and $f(X)$ is a dense in $\hat{X}$ and $g(X)$ is a dense in X^*. Then, we define a map $h: \hat{X} \to X^*$. We prove that h is an isometry and that $h(\hat{X}) = X^*$.

3. RESULTS AND DISCUSSION

Before proving the existence and uniqueness of completion in complex valued S-metric space, we first give some new concepts in this space. Adopting the definitions for complex metric case in Dahliatul and Imam [13], we also define two definitions of isometry and dense in terms of complex valued S-metric space.

Definition 4. Let (X, S_c) *and* $(\hat{X}, \widehat{S_c})$ *be complex valued* S*-metric spaces.*

(i) *A mapping* $f: X \to \hat{X}$ *is called to be an isometry if* $\widehat{S_c}\big(f(z_1), f(z_1), f(z_2)\big) = S_c(z_1, z_1, z_2)$, *for all* $z_1, z_2 \in X$.
(ii) *The spaces* (X, S_c) *and* $(\hat{X}, \widehat{S_c})$ *are said to be isometric if there exists a bijective isometry.*

Definition 5. *Let* (X, S_c) *be complex valued* S*-metric spaces and* Y *be a subset of* X. *A subset* Y *is a dense in* X *if every* $z \in X$ *is a limit point of* Y. *This is equivalent to its closure* $\bar{Y}$ *is equal to* X.

The following lemmas, which characterize the denseness and the convergence by means of complex valued S-metric space.

Lemma 4. *Let* (X, S_c) *be complex valued* S*-metric spaces and* Y *be a subset of* X. *A subset* Y *is dense in* X *if and only if for every* $x \in X$ *there is a sequence in* Y *converging to* x.

Proof. $\Longrightarrow$ Let x be any point in X. Since Y is dense in X, then there are 2 cases:

- If $x \in Y$ then there is constant sequence $\{x, x, \cdots\} \subset Y$ converging to x.
- If $x \notin Y$ then x be a limit point of Y. This means that for all $c_n = \frac{1}{n} + i\frac{1}{n}$ there exists $x_n \in Y$ such that

$x_n \in B_{c_n}(x) \cap Y - \{x\}$.
Consider that $\{x_n\} \subset Y$ such that $S_c(x_n, x_n, x) \prec c_n$, for all $n \in \mathbb{N}$. Now, we prove that $\{x_n\}$ converges to x. For all $c \in \mathbb{C}, c \succ 0$ with $c = c_1 + ic_2$, since $Re(c) = c_1 > 0$ and $Im(c) = c_2 > 0$, then by the Archimedian property, there exist $n_1, n_2 \in \mathbb{N}$ such that $\frac{1}{c_1} < n_1$ and $\frac{1}{c_2} < n_2$.
Choose $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ we have

$$S_c(x_n, x_n, x_{n_0}) \prec \frac{1}{n_1} + i\frac{1}{n_2} \prec c_1 + ic_2$$

Therefore, $\{x_n\}$ converges to x.
$\Leftarrow$ If $x \in Y$ then $x \in \bar{Y}$. Hence, $Y \subseteq \bar{Y}$. Next, if $x \in X - Y$, by hypothesis there exist $\{y_n\} \subset Y$ such that $\{y_n\}$ converges to x. Hence, for every $c \in \mathbb{C}, c \succ 0$ we have $y_n \in B_c(x) \cap Y - \{x\}$. Therefore, x is a limit point of Y, implies $x \in \bar{Y}$. So, $X \subseteq \bar{Y}$.
Conversely, if $y \in Y$ then $y \in X$. Hence, $Y \subseteq X$. Morover, if $y \in Y'$, where Y' be a set of all limit point of Y, then for every $c \in \mathbb{C}$, $c \succ 0$ we have there exist $z \in Y$ such that $z \in B_c(y) \cap Y - \{y\}$. Consequently, $y \in X$. Since $\bar{Y} = Y \cup Y'$ then $\bar{Y} \subseteq X$.
Hence, $\bar{Y} = X$. ■

Lemma 5. *Let (X, S_c) and $(\hat{X}, \hat{S_c})$ be complex valued S-metric spaces and a map $f: X \to \hat{X}$. If $f(X)$ is a dense in $\hat{X}$ then there exist a sequence $\{x_n\} \subset X$ such that $\{f(x_n)\}$ converges to $\hat{x} \in \hat{X}$.*

Proof. Given $f(X)$ is a dense in $\hat{X}$.
For all $c_n = \frac{1}{n} + i\frac{1}{n}$, $n \in \mathbb{N}$, choose $x_n \in X$ such that

$$\hat{S_c}(f(x_n), f(x_n), \hat{x}) \prec c_n$$

We proof that $\{f(x_n)\}$ is convergent in $\hat{X}$.
For any $\hat{x} \in \hat{X}$, by **Lemma 4**, then there exists $\{y_n\} \subset \hat{X}$ converging to $\hat{x}$. In particular, for all $c \in \mathbb{C}, c \succ 0$ with $c = c_1 + ic_2$ there exist $n_1, n_2 \in \mathbb{N}$ such that $\frac{1}{c_1} < n_1$ and $\frac{1}{c_2} < n_2$. Choose $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$ we have

$$\hat{S_c}(y_n, y_n, \hat{x}) \prec c$$

Since $f(X)$ is a dense in $\hat{X}$, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $f(x_n) = y_n$. Moreover, for all $n \geq n_0$,

$$\hat{S_c}(f(x_n), f(x_n), \hat{x}) \prec \frac{1}{n_1} + i\frac{1}{n_2} \prec c_1 + ic_2 = c$$

Hence, $\{f(x_n)\}$ converges to $\hat{x} \in \hat{X}$. ■

Lemma 6. *Let (X, S_c) and $(\hat{X}, \hat{S_c})$ be complex valued S-metric spaces and an isometry $f: X \to \hat{X}$ with $f(X)$ being dense in $\hat{X}$. If $\{x_n\}$ is sequence in X then it is a Cauchy sequence.*

Proof. Let $\hat{x} \in \hat{X}$. Since $f(X)$ is a dense in $\hat{X}$, by **Lemma 5** there exist a sequence $\{x_n\} \subset X$ such that $\{f(x_n)\} \subset f(X)$ converges to $\hat{x} \in \hat{X}$. It implies $\{f(x_n)\}$ is Cauchy sequences in $f(X)$. In particular, for all $0 \prec c \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $S_c(f(x_n), f(x_n), f(x_m)) \prec c$.
Since f is an isometry, for all $m, n \geq n_0$ we have

$$S_c(x_n, x_n, x_m) = \hat{S_c}(f(x_n), f(x_n), f(x_m)) \prec c.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. ■

Furthermore, we introduce the concept of completion in complex valued S-metric space.

Definition 6. *Let (X, S_c) be a complex valued S-metric space. A complete complex valued S-metric space $(\hat{X}, \hat{S_c})$ is called a completion of (X, S_c) if there exists an isometry $i: X \to \hat{X}$ such that $i(X)$ is dense in $\hat{X}$.*

Lemma 7. *Let $\mathsf{C}[x]$ be the collection of all Cauchy sequences on a complex valued S-metric space, i.e. $\mathsf{C}[x] = \{\{x_n\} \subset X \mid \{x_n\}$ is Cauchy sequence in $X\}$. For any two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathsf{C}[x]$, define the relation "$\sim$" as*

$$\{x_n\} \sim \{y_n\} \Leftrightarrow \lim_{n\to\infty} |S_c(x_n, x_n, y_n)| = 0$$

then "$\sim$" is an equivalence relation.

Proof. Given (X, S_c) a complex valued S-metric space and $\mathsf{C}[x] = \{\{x_n\} | \{x_n\}$ is an arbitrary sequence in X$\}$. We proof that "$\sim$" is reflexive, symmetric and transitive.

1. Let $\{x_n\} \in C[x]$. By the property of S_c, we obtain $S_c(x_n, x_n, x_n) = 0$, which implies $\lim_{n\to\infty} |S_c(x_n, x_n, x_n)| = 0$. Hence, "$\sim$" is reflexive.
2. Let $\{x_n\}, \{y_n\} \in C[x]$. By Lemma 3, we have $S_c(x_n, x_n, y_n) = S_c(y_n, y_n, x_n)$, for all $n \in \mathbb{N}$. Taking modulus and limit as $n \to \infty$ on both sides, we have
$$0 = \lim_{n\to\infty} |S_c(x_n, x_n, y_n)| = \lim_{n\to\infty} |S_c(y_n, y_n, x_n)|$$
Hence, "$\sim$" is symmetric.
3. Let $\{x_n\}, \{y_n\}, \{z_n\} \in C[x]$ such that $\{x_n\}\sim\{y_n\}$ and $\{y_n\}\sim\{z_n\}$. Then
$$\lim_{n\to\infty} |S_c(x_n, x_n, y_n)| = 0 \text{ and } \lim_{n\to\infty} |S_c(y_n, y_n, z_n)| = 0.$$
By the property of S_c, we get
$$S_c(x_n, x_n, z_n) \precsim 2S_c(x_n, x_n, y_n) + S_c(z_n, z_n, y_n)$$
Taking modulus and limit as $n \to \infty$ on both sides, we obtain
$$\lim_{n\to\infty} |S_c(x_n, x_n, z_n)| \le 2 \lim_{n\to\infty} |S_c(x_n, x_n, y_n)| + \lim_{n\to\infty} |S_c(z_n, z_n, y_n)| = 0$$
Hence, "$\sim$" is transitive.

This proves that "$\sim$" is an equivalence relation on $C[x]$. ∎

Next, we define a set $\hat{X}$ be the set of all equivalence classes in $C[x]$ over the relation "$\sim$", that is
$$\hat{X} = \{[\{x_n\}] | \{x_n\} \in C[x]\}$$
In particular, $\{x_n\} \in [\{x_n\}]$ means that $\{x_n\}$ is a member of $[\{x_n\}]$, i.e. a representative of the class $[\{x_n\}]$.

Lemma 8. *Let $\hat{X}$ be the set of all equivalence classes in $C[x]$ over the relation "$\sim$". If for every $[\{x_n\}], [\{y_n\}] \in \hat{X}$ holds*
$$\widehat{S_c}([\{x_n\}], [\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} S_c(x_n, x_n, y_n)$$
then $\widehat{S_c}$ is a complex-valued S_c-metric and $(\hat{X}, \widehat{S_c})$ is a complex valued S-metric space.

Proof. We prove that $\widehat{S_c}$ is well-defined by proving that the limit exists and is independent of the choice of Cauchy sequences from equivalence classes.

(i) Let $\{x_n\},\{y_n\} \in X$.
Consider
$$S_c(x_n, x_n, y_n) \precsim 2S_c(x_n, x_n, x_m) + 2S_c(y_n, y_n, y_m) + S_c(x_m, x_m, y_m)$$
which implies
$$S_c(x_n, x_n, y_n) - S_c(x_m, x_m, y_m) \precsim 2[S_c(x_n, x_n, x_m) + S_c(y_n, y_n, y_m)]$$
Also
$$S_c(x_m, x_m, y_m) - S_c(x_n, x_n, y_n) \precsim (-2)[S_c(x_n, x_n, x_m) + S_c(y_n, y_n, y_m)]$$
Thus,
$$|S_c(x_n, x_n, y_n) - S_c(x_m, x_m, y_m)| \le 2|S_c(x_n, x_n, x_m) + S_c(y_n, y_n, y_m)|$$
Assume $m > n$. Taking limit as $n \to \infty$, we get
$$\lim_{n\to\infty} |S_c(x_n, x_n, y_n) - S_c(x_m, x_m, y_m)| \le 2\left[\lim_{n\to\infty} |S_c(x_n, x_n, x_m)| + \lim_{n\to\infty} |S_c(y_n, y_n, y_m)|\right]$$
Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X then
$$\lim_{n\to\infty} |S_c(x_n, x_n, x_m)| = \lim_{n\to\infty} |S_c(y_n, y_n, y_m)| = 0.$$
Hence,
$$\lim_{n\to\infty} |S_c(x_n, x_n, y_n) - S_c(x_m, x_m, y_m)| = 0.$$
This means that $\{S_c(x_n, x_n, y_n)\}$ is a Cauchy sequences in $\mathbb{C}$. By the completeness of $\mathbb{C}$, this sequence converges. Indeed, $\lim_{n\to\infty} S_c(x_n, x_n, y_n)$ exists.

(ii) Let $\{x_n\}, \{x'_n\}, \{y_n\}, \{y'_n\} \in X$ such that $\{x_n\}\sim\{x'_n\}$ and $\{y_n\}\sim\{y'_n\}$. This means that
$$\lim_{n\to\infty} |S_c(x_n, x_n, x'_n)| = 0 \text{ and } \lim_{n\to\infty} |S_c(y_n, y_n, y'_n)| = 0.$$
With the same argument on (i), we can easily verify that
$$\lim_{n\to\infty} |S_c(x_n, x_n, y_n) - S_c(x'_n, x'_n, y'_n)| = 0$$
This means that $\lim_{n\to\infty} S_c(x_n, x_n, y_n) = \lim_{n\to\infty} S_c(x'_n, x'_n, y'_n)$.
So, $\lim_{n\to\infty} S_c(x_n, x_n, y_n)$ is independent of the particular choice of the representatives.

Then, we prove that $\widehat{S_c}$ is a complex-valued S_c-metric.
If $[\{x_n\}], [\{y_n\}], [\{z_n\}] \in \hat{X}$ then

1) Since $S_c(x_n, x_n, y_n) \succsim 0$,

$$\widehat{S_c}([\{x_n\}],[\{x_n\}],[\{y_n\}]) = \lim_{n\to\infty} S_c(x_n,x_n,y_n) \succsim 0.$$

2) $\widehat{S_c}([\{x_n\}],[\{x_n\}],[\{y_n\}]) = 0$ implies $\lim_{n\to\infty} S_c(x_n,x_n,y_n) = 0$.
Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X,

$$\{x_n\}\sim\{y_n\} \Leftrightarrow [\{x_n\}] = [\{y_n\}]$$

3) Recall that $S_c(x_n,x_n,y_n) \precsim 2S_c(x_n,x_n,z_n) + S_c(y_n,y_n,z_n)$ which implies

$$\lim_{n\to\infty} S_c(x_n,x_n,y_n) \precsim 2\lim_{n\to\infty} S_c(x_n,x_n,z_n) + \lim_{n\to\infty} S_c(y_n,y_n,z_n)$$

This means

$$\widehat{S_c}([\{x_n\}],[\{x_n\}],[\{y_n\}]) \precsim 2\widehat{S_c}([\{x_n\}],[\{x_n\}],[\{z_n\}]) + \widehat{S_c}([\{y_n\}],[\{y_n\}],[\{z_n\}])$$

Hence, $\widehat{S_c}$ is a complex valued S-metric. Moreover, $(\hat{X},\widehat{S_c})$ is a complex valued S-metric space. ■

We can now state and prove the main theorem that every complex valued S-metric space has a unique completion.

Theorem 1. *Every complex valued S-metric space has a completion.*

Proof. Given (X,S_c) and $(\hat{X},\widehat{S_c})$ be any two complex valued S-metric spaces. For each $x\in X$, associate the class $\hat{x} = [\{x,x,x,\dots\}] \in \hat{X}$ which contains the constant Cauchy sequence $\{x,x,x,\dots\}$. Then, define $f: X \to \hat{X}$ with $f(x) = \hat{x}$.
It is clear that f is an isometry. Since for each $x,y\in X$,

$$\widehat{S_c}(f(x),f(x),f(y)) = \widehat{S_c}(\hat{x},\hat{x},\hat{y}) = \lim_{n\to\infty} S_c(x,x,y) = S_c(x,x,y)$$

Next, we show that $f(X)$ is a dense in $\hat{X}$.
Consider $\hat{x} = [\{x_n\}]$. Let $\{x_n\}\in\hat{x}$. Since $\{x_n\}$ is Cauchy, then for every $c \succ 0$ there is a $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ we have $S(x_n,x_n,x_{n_0}) \prec c$.
Let $\{x_{n_0},x_{n_0},x_{n_0},\dots\} \in \widehat{x_{n_0}} \in f(X)$. We obtain

$$\widehat{S_c}(\hat{x},\hat{x},\widehat{x_{n_0}}) = \lim_{n\to\infty} S_c(x_n,x_n,x_{n_0}) \prec c.$$

Thus, every neighbourhood of $\hat{x}\in\hat{X}$ contains an element of $f(X)$. This means that every $\hat{x}\in\hat{X}$ is a limit point of $\hat{X}$. Hence, by **Definition 5**, $f(X)$ is a dense subset of $\hat{X}$.
Furthermore, we show that $(\hat{X},\widehat{S_c})$ is complete.
Let $\{\widehat{x_n}\}$ be any Cauchy sequence in $\hat{X}$. Choose $c_n\in\mathbb{C}$ with $c_n = \frac{1}{2n} + i\frac{1}{2n}$. Since $f(X)$ is a dense in $\hat{X}$, then for all $\widehat{x_n}\in\hat{X}$ there is $\widehat{y_n}\in f(X)$ such that

$$0 \prec \widehat{S_c}(\widehat{x_n},\widehat{x_n},\widehat{y_n}) \prec \frac{1}{2n} + i\frac{1}{2n}$$

Taking modulus and limit as $n\to\infty$ on both sides,

$$\lim_{n\to\infty} |\hat{S}(\widehat{x_n},\widehat{x_n},\widehat{y_n})| = 0.$$

To show that $\{\widehat{y_n}\}$ is a Cauchy sequence, we use **Condition (iv)** in the **Definition of S_c**. Then for all $m,n\in\mathbb{N}$ (assuming $m>n$), we have

$$\begin{aligned}\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m}) &\preccurlyeq \widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{x_n}) + \widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{x_n}) + \widehat{S_c}(\widehat{y_m},\widehat{y_m},\widehat{x_n})\\ &\preccurlyeq 2\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{x_n}) + 2\widehat{S_c}(\widehat{y_m},\widehat{y_m},\widehat{x_m}) + \widehat{S_c}(\widehat{x_m},\widehat{x_m},\widehat{x_n})\\ &\prec 2\left[\frac{1}{2n} + i\frac{1}{2n}\right] + 2\left[\frac{1}{2m} + i\frac{1}{2m}\right] + \widehat{S_c}(\widehat{x_m},\widehat{x_m},\widehat{x_n})\end{aligned}$$

Since $\{\widehat{x_n}\}$ is a Cauchy sequence, we obtain

$$\lim_{m,n\to\infty} \widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m}) = 0$$

Consequently,

$$\lim_{m,n\to\infty} |\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m})| = 0$$

So, $\{\widehat{y_n}\}$ is a Cauchy sequence in $f(X)$.
Next, we will show that $\{y_n\}$ is a Cauchy sequence in X.
Suppose $y_n\in X$, such that $\widehat{y_n} = f(y_n)$. We have

$$\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m}) = \widehat{S_c}(f(y_n),f(y_n),f(y_m))$$

Using the fact that f is an isometry, we find

$$\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m}) = S_c(y_n,y_n,y_m), \text{for each } m,n\in\mathbb{N}$$

Therefore,

$$\lim_{m,n\to\infty} |\widehat{S_c}(\widehat{y_n},\widehat{y_n},\widehat{y_m})| = \lim_{m,n\to\infty} |S_c(y_n,y_n,y_m)| = 0$$

This means that $\{y_n\}$ is a Cauchy sequence in X.
Let $\{\widehat{y_n}\} \in \hat{x} \in \hat{X}$. We will show that $\{\widehat{x_n}\}$ converges to $\hat{x}$.
Recall that

$$\widehat{S_c}(\widehat{x_n}, \widehat{x_n}, \hat{x}) \preccurlyeq 2\widehat{S_c}(\widehat{x_n}, \widehat{x_n}, f(y_n)) + \widehat{S_c}(\hat{x}, \hat{x}, f(y_n))$$

For fixed n, constant sequence $\{y_n, y_n, y_n, ...\} \in \widehat{y_n} = f(y_n)$, and $\{y_m\} \in \hat{x}$, then

$$\widehat{S_c}(\widehat{x_n}, \widehat{x_n}, \hat{x}) \prec 2\left[\frac{1}{2n} + i\frac{1}{2n}\right] + \lim_{m,n\to\infty} S_c(y_m, y_m, y_n)$$

Since $\{y_n\}$ is a Cauchy sequence in X, then

$$\widehat{S_c}(\widehat{x_n}, \widehat{x_n}, \hat{x}) \prec \frac{1}{n} + i\frac{1}{n},$$

which implies

$$\lim_{n\to\infty} |S_c(\widehat{x_n}, \widehat{x_n}, \hat{x})| = 0$$

Hence, $\{\widehat{x_n}\}$ converges to $\hat{x}$. It means that $(\hat{X}, \widehat{S_c})$ is a complete space.
We proof that the completion is unique by showing $(\hat{X}, \widehat{S_c})$ is isometrically equivalent to $(X^*, {S_c}^*)$.
Let $(\hat{X}, \widehat{S_c})$ and $(X^*, {S_c}^*)$ be any two completions of complex valued metric space (X, S_c). It follows that there exist isometries $f: X \to \hat{X}$ and $g: X \to X^*$ such that $f(X)$ is a dense in $\hat{X}$ and $g(X)$ is a dense in X^*.
Next, consider a map $h: f(X) \to g(X)$ with $h(f(x)) = g(x)$, for all $x \in X$. Obviously, h is an isometry, since for all $x, y \in X$ then $\widehat{S_c}\big(f(x), f(x), f(y)\big) = S_c(x, x, y) = {S_c}^*\big(g(x), g(x), g(y)\big)$.

Let $\hat{x} \in \hat{X}$. Since $f(X)$ is a dense in $\hat{X}$, by **Lemma 4**, there exist a sequence $\{x_n\} \subset X$ such that it converges in X. Since f is an isometry, $\{f(x_n)\}$ is a Cauchy sequence in $\hat{X}$. Also, by the denseness of f, $\{f(x_n)\}$ is a Cauchy sequence in $f(X)$, which implies $\{x_n\}$ is a Cauchy sequence in X.
Moreover, since h is an isometry, then $\{g(x_n)\}$ is a Cauchy sequence in X^*. Using the fact that X^* is a complete space, we find that there exist $x^* \in X^*$ such that $\lim_{n\to\infty} g(x_n) = x^*$.
From these, we define a map i to extend h, i.e. $i: \hat{X} \to X^*$ as follow

$$i(\hat{x}) = \begin{cases} h(\hat{x}), & \hat{x} \in f(X) \\ \lim_{n\to\infty} h\big(f(x_n)\big) = \lim_{n\to\infty} g(x_n), & \hat{x} \in \hat{X} - f(X), f(x_n) \in f(X), \lim_{n\to\infty} f(x_n) = \hat{x}. \end{cases}$$

We show that i is an isometry and $i\big(\hat{X}\big) = X^*$.
For all $\hat{x}, \hat{y} \in \hat{X}$, there exist $\{x_n\}, \{y_n\} \subset X$ such that $\lim_{n\to\infty} f(x_n) = \hat{x}$ and $\lim_{n\to\infty} f(y_n) = \hat{y}$.
By the definition of i, we obtain

$${S_c}^*\big(i(\hat{x}), i(\hat{x}), i(\hat{y})\big) = \lim_{n\to\infty} {S_c}^*\Big(h\big(f(x_n)\big), h\big(f(x_n)\big), h\big(f(y_n)\big)\Big) = \lim_{n\to\infty} {S_c}^*\big(g(x_n), g(x_n), g(y_n)\big)$$

From the isometry of h, we find

$${S_c}^*\big(i(\hat{x}), i(\hat{x}), i(\hat{y})\big) = \lim_{n\to\infty} \widehat{S_c}\big(f(x_n), f(x_n), f(y_n)\big)$$

Since $\{f(x_n)\}$ and $\{f(y_n)\}$ are convergent sequences, we have

$${S_c}^*(i(\hat{x}), i(\hat{x}), i(\hat{y})) = \widehat{S_c}(\hat{x}, \hat{x}, \hat{y})$$

So, i is an isometry.
Finally, we show that $i\big(\hat{X}\big) = X^*$.
Let $x^* \in X^*$, based on **Lemma 5**, there exist $\{x_n\} \subset X$ such that $\{g(x_n)\}$ converges to x^*. Clearly, $\{g(x_n)\}$ is Cauchy sequence in $g(X)$. By **Lemma 6**, we have $\{x_n\}$ is a Cauchy sequence in X. Also, $\{g(x_n)\}$ is a Cauchy sequence in $g(X)$.
Since $h: f(X) \to g(X)$ is an isometry, $\{f(x_n)\}$ is also a Cauchy sequence in $\hat{X}$. This implies, $\{f(x_n)\}$ converges in $\hat{X}$, namely $\hat{x}$. Since i is an isometry, $\{i(f(x_n))\} = \{g(x_n)\}$ converges to $i(\hat{x})$.
Therefore,

$${S_c}^*(i(\hat{x}), i(\hat{x}), x^*) \preccurlyeq 2{S_c}^*(i(\hat{x}), i(\hat{x}), g(x_n)) + {S_c}^*(x^*, x^*, g(x_n)$$

Taking limit as $n \to \infty$ on both sides, we have

$$0 \preccurlyeq \lim_{n\to\infty} {S_c}^*(i(\hat{x}), i(\hat{x}), x^*) \preccurlyeq 2\lim_{n\to\infty} {S_c}^*(i(\hat{x}), i(\hat{x}), g(x_n)) + \lim_{n\to\infty} {S_c}^*(x^*, x^*, g(x_n)$$

which implies

$${S_c}^*(i(\hat{x}), i(\hat{x}), x^*) = 0$$

So, for all $x^* \in X^*$, there is $\hat{x} \in \hat{X}$ such that $i(\hat{x}) = x^*$. In particular, $i\big(\hat{X}\big) = X^*$. ∎

We provide an example to verify the existence of completion in a complex valued S-metric space.

Example 1. Consider $A \subset \mathbb{C}$ with $A = \{z \in \mathbb{C} | 0 < Re(z) \leq 1\} \cup \{z \in \mathbb{C} | 0 < Im(z) \leq 1\}$. If $z_k \in \mathbb{C}$ with $z_k = x_k + iy_k$ for all $k = 1,2,3$, then define a map $S_c: A^3 \to \mathbb{C}$ by

$$S_c(z_1,z_2,z_3) = (|x_1 - x_3| + |x_2 - x_3|) + i(|y_1 - y_3| + |y_2 - y_3|)$$

It is easy to verify that S_c is a complex valued S-metric. Moreover, (A, S_c) is a complex valued S-metric space.

Let $\{z_n\} \in A$ with $z_n = \left\{\frac{1}{n} + i\frac{1}{n}\right\}$ for all $n \in \mathbb{N}$. We show that $\{z_n\}$ is a Cauchy sequence in A.

Observe that

$$S_c(z_n,z_n,z_m) = 2\left|\frac{1}{n} - \frac{1}{m}\right| + 2i\left|\frac{1}{n} - \frac{1}{m}\right| = (2+2i)\left|\frac{m-n}{mn}\right| \prec (2+2i)\left|\frac{1}{n}\right|$$

Since $|2+2i| = 2\sqrt{2}$,

$$|S_c(z_1,z_2,z_3)| < \frac{2\sqrt{2}}{n}, \text{ for all } n \in \mathbb{N}.$$

Given a real number $\varepsilon > 0$. Let

$$c = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}$$

Obviously, $c \succ 0$. Choose $n_0 > \frac{2\sqrt{2}}{|c|}$. Then, for all $m, n \geq n_0$ we obtain

$$|S_c(z_n,z_n,z_m)| < \frac{2\sqrt{2}}{n} \leq \frac{2\sqrt{2}}{n} < \frac{2\sqrt{2}}{n_0} < |c| = \varepsilon$$

In particularly, $|S_c(z_n,z_n,z_m)| \to 0$ as $n \to \infty$. Using **Lemma 2**, $\{z_n\}$ is then a Cauchy sequence in A. However, $\{z_n\}$ does not convergent in A, because there is no $x \in A$ such that $\{z_n\}$ converges to x. Hence, (A, S_c) is incomplete complex valued S-metric space.

We will find a complete complex valued S-metric space (B, S_c) such that $(A, S_c) \subset (B, S_c)$.

Let $\bar{A} = \{z \in \mathbb{C} | 0 \leq Re(z) \leq 1\} \cup \{z \in \mathbb{C} | 0 \leq Im(z) \leq 1\}$. Claim that $\bar{A}$ is closure of A.

Let $z \in \mathbb{C}$ be a limit point of $\bar{A}$. Then, there is $\{z_n\} \subset \bar{A}$ such that $\lim_{n\to\infty} z_n = z$. Since $z_n \in \bar{A}$ for each $n \in \mathbb{N}$, it follows that $0 \leq Re(z_n) \leq 1$ and $0 \leq Im(z_n) \leq 1$. This implies $0 \leq Re(z) \leq 1$ and $0 \leq Im(z) \leq 1$. So, $z \in \bar{A}$. Therefore, any limit point of $\bar{A}$ must be in $\bar{A}$ then $\bar{A}$ is closed. Since $\bar{A}$ is closed, we have $\bar{A}$ is closure of A.

Next, consider $(B, S_c) = (\bar{A}, S_c)$. Then, every Cauchy $\{z_n\} \subset B$ is convergent on B. Hence, (B, S_c) contains (A, S_c) and is a complete complex valued S-metric space, as desired.

Moreover, since $A \subset B$ and B is closed then $B = \bar{A}$. Therefore, A is dense in B.

Next, we show that (A, S_c) and (B, S_c) are isometrics. Equivalent to showing that there exists a bijective isometry $f: A \to B$.

Define a map $f: A \to B$ by $f(z) = z$, for all $z \in A$.

Then, f is an isometry. Since for any $z_1, z_2 \in A$,

$$S_c(f(z_1), f(z_1), f(z_2)) = (|x_1 - x_3| + |x_2 - x_3|) + i(|y_1 - y_3| + |y_2 - y_3|) = S_c(z_1,z_2,z_3)$$

It is clearly that f is a bijective. So, (A, S_c) and (B, S_c) are isometrics.

Now, we can conclude that (B, S_c) is completion of (A, S_c).

4. CONCLUSIONS

Derived from the result and discussion, a proof establishes the existence and uniqueness of completion for complex valued S-metric space. It begins by introducing the fundamental concepts such as isometry and denseness through this space, followed by the proof of some lemmas characterizing denseness and convergence. The main theorem states that every complex valued S-metric space has a completion. Next, this completion is proven unique by establishing an isometric equivalence between two completions.

ACKNOWLEDGMENT

The authors would like to thank the Faculty of Mathematics and Natural Sciences, Universitas Tanjungpura, for the research grant provided through DIPA UNTAN funds with contract number 023.17.2.677517/2024.

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