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THE REFLEXIVE EDGE STRENGTH OF THE PENTAGONAL SNAKE GRAPH AND CORONA OF THE OPEN TRIANGULAR LADDER AND NULL GRAPH

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ABSTRACT

Assume that G be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. The edge irregular reflexive k-labeling of graph G is a labeling selects positive integers from 1 to k_e as edge labels and non negative even numbers from 0 to $2k_v$ as vertex labels, and the weights assigned to each edge are distinct, where $k = max\ \{2k_v, k_e\}$. On graph G with φ labeling, the weight of edge uv is represented by $wt_\varphi(uv)$, which is defined as the sum of edge label and all vertex labels incident to that edge. Reflexive edge strength of graph G is the minimum k of the highest label, denoted by $res(G)$. In this research, reflexive edge strength for pentagonal snake graph (PS_n) and corona of open triangular ladder and null graph ($O(TL_n) \odot N_m$) will be determined. The method of this research is literature study, the lower bound of $res(G)$ determined by Ryan's lemma and the upper bound by labeling. The reflexive edge strength of pentagonal snake graph PS_n with $n \geq 2$ is $\left\lceil \frac{5n-5}{3} \right\rceil$ for $5n - 5 \not\equiv 2,3 \ (mod\ 6)$ and $\left\lceil \frac{5n-5}{3} \right\rceil + 1$ for $5n - 5 \equiv 2,3\ (mod\ 6)$. The reflexive edge strength of corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$ is $\left\lceil \frac{2nm+4n-5}{3} \right\rceil$ $for\ 2nm + 4n - 5 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1\ for\ 2nm + 4n - 5 \equiv 2,3\ (mod\ 6)$.

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1. INTRODUCTION

Let's G be an undirected simple graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$. The quantity of vertex is known as the *order*, denoted by $|V(G)|$, while the quantity of edge is known as the *size*, denoted by $|E(G)|$.

The labeling of a graph can be defined as a function φ that pairs graph elements (vertices and edges) into positive or non-negative integers [1]. Labeling can be divided into vertex labeling, edge labeling, and total labeling (vertex and edge labeling). Various kinds of graph labeling have been developed, one of which is irregular total k-labeling [2].

In 2007, Baĉa *et al.* [3] introduced some new concepts in graph theory, called total irregular edge k-labeling and total irregular vertex k-labeling. Then, edge irregular reflexive k-labeling and vertex irregular reflexive k-labeling in a graph were introduced in 2017 by Ryan in Baĉa *et al.* [4].

The edge irregular reflexive k-labeling of graph G is a labeling selects positive integers from 1 to k_e as edge labels and non negative even numbers from 0 to $2k_v$ as vertex labels, and the weights assigned to each edge are distinct, where $k = \max\{2k_v, k_e\}$. On graph G with φ labeling, the weight of edge uv is represented by $wt_\varphi(uv)$, which is defined as the sum of edge label and all vertex labels incident to that edge, as $wt_\varphi(uv) = \varphi(u) + \varphi(uv) + \varphi(v)$.

The reflexive edge strength of graph G is the minimum k of the highest label, denoted by $res(G)$. Ryan in Baĉa *et al.* [4] have proven the lemma to determine a lower bound of $res(G)$. The following lemma proven by Ryan in Baĉa *et al.* [4].

Lemma 1. *For any graph G,*

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{for } |E(G)| \not\equiv 2,3 \pmod 6 \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{for } |E(G)| \equiv 2,3 \pmod 6 \end{cases}$$

The result $res(G)$ of several graphs has been obtained such as wheel graph W_n and prims graph D_n [5], palm tree graphs $C_3 - B_2, r$ and $C_3 - B_3, r$ [6], double quadrilateral snake graph and mongolian tent graph $(M_{m,3})$ [7], convex polytope graphs and corona product of cycle with path [8]. Indriati *et al.*[9] determined the $res(G)$ of corona of path and complete graph K_1 and corona of path and path graph P_2. In 2021, Agustin *et al.* [10] determined of the ladder graph L_n and triangle ladder graph TL_n, Setiawan and Indriati [11] determined of sun graph and corona of cycle and null graph with two vertices N_2, $C_n \odot N_2$. In 2022, Saputri and Diari [12] determined the $res(G)$ of triangular chain graph mC_3. Then in 2023, Zalzabila *et al.* [13] determined the $res(G)$ of double alternate quadrilateral snake graph $DA(Q_n)$ and alternate triangular snake graph $A(T_n)$, in the same year Nurhayati and Susanti [14] determined of comb graphs with additional 2 pendants, Wijaya and Ningrum [15] determined of lobster graph. Because pentagonal snake PS_n with $n \geq 2$ has been determined in mean square cordial labeling [16] and corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$ [17] has been determined in quotient labeling, so in this article we did these graphs by another labeling, that is edge irregular reflexive labeling.

2. RESEARCH METHODS

The references used in this article based on books, papers, articles, and journals related to reflexive edge strength. From thus method, it can be determined reflexive edge strength of pentagonal snake, denoted by $\boldsymbol{res(PS_n)}$ with $\boldsymbol{n \geq 2}$ and reflexive edge strength of corona of open triangular ladder and null graph, denoted by $\boldsymbol{res(O(TL_n) \odot N_m)}$ with $\boldsymbol{n \geq 3}$ and $\boldsymbol{m \geq 1}$. This research used the following method.

1. Determine the lower bound of $res(PS_n)$ and $res(O(TL_n) \odot N_m)$ based on **Lemma 1.**
2. Label the vertices and edges of the pentagonal snake graph PS_n and the corona of open triangular ladder and null graph $O(TL_n) \odot N_m$.
3. Calculate the weight of each edge of pentagonal snake PS_n and corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ so that the weight for all edges are different.

4. Determine the general pattern of vertex labels, edge labels, and edge weights of edge irregular reflexive k-labeling of pentagonal snake graph PS_n and corona of open triangular ladder and null graph $O(TL_n) \odot N_m$.
5. Formulate the general pattern $res(PS_n)$ and $res(O(TL_n) \odot N_m)$.

3. RESULTS AND DISCUSSION

The discussion and results of reflexive edge strength of pentagonal snake graph PS_n and corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ are presented.

3.1 Pentagonal Snake Graph PS_n

The pentagonal snake graph denoted by PS_n is a graph obtained from a path $u_1, u_2, \dots, u_n$ by connecting vertex u_i to vertex v_i and vertex u_{i+1} to w_i, for $1 \le i \le n-1$. Then, connect vertex v_i and w_i to vertex x_i, for $1 \le i \le n-1$. The pentagonal snake graph PS_n can also be constructed from the path P_n, where each edge is converted into a cycle C_5 [16].

The pentagonal snake graph PS_n consists of vertex set $V(PS_n) = \{x_i, v_i, w_i : 1 \le i \le n-1\} \cup \{u_i : 1 \le i \le n\}$ and edge set $E(PS_n) = \{x_i v_i, x_i w_i, v_i u_i, w_i u_{i+1}, w_i u_{i+1} : 1 \le i \le n\}$. The pentagonal snake graph has $|V(PS_n)| = 4n-3$ and $|E(PS_n)| = 5n-5$.

Theorem 1. *For any positive integer $n \ge 2$,*

$$res(PS_n) = \begin{cases} \left\lceil \frac{5n-5}{3} \right\rceil, & \text{for } 5n-5 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{5n-5}{3} \right\rceil + 1, & \text{for } 5n-5 \equiv 2,3 \pmod 6. \end{cases} \quad (1)$$

Proof. It is known that the number of edges of the pentagonal snake graph PS_n is $5n-5$, so based on **Lemma 1** we get

$$res(PS_n) \ge \begin{cases} \left\lceil \frac{5n-5}{3} \right\rceil, & \text{for } 5n-5 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{5n-5}{3} \right\rceil + 1, & \text{for } 5n-5 \equiv 2,3 \pmod 6. \end{cases} \quad (2)$$

Next, we formulate the labeling function φ on the edge irregular reflexive k-labeling of pentagonal snake graph PS_n with $k \left\lceil \frac{5n-5}{3} \right\rceil$ for $5n-5 \not\equiv 2,3 \pmod 6$ and $k = \left\lceil \frac{5n-5}{3} \right\rceil + 1$ for $5n-5 \equiv 2,3 \pmod 6$ are as follows.

The vertex labels $\varphi(x_i)$ for $1 \le i \le n-1$ as follows,

$$\varphi(x_i) = \begin{cases} \left\lceil \frac{5i-5}{3} \right\rceil, & \text{for } i = 1,2,3, \\ \left\lceil \frac{5i}{3} \right\rceil, & \text{for } i \equiv 0 \pmod 6, \\ \left\lceil \frac{5i+1}{3} \right\rceil, & \text{for } i \equiv 1,2,3 \pmod 6 \text{ and } i \ne 1,2,3, \\ \left\lceil \frac{5i+4}{3} \right\rceil, & \text{for } i \equiv 4 \pmod 6 \text{ and } i \ne 4, \\ \left\lceil \frac{5i-2}{3} \right\rceil, & \text{for } i \equiv 5 \pmod 6 \text{ and } i = 4. \end{cases}$$

The vertex labels $\varphi(v_i)$ for $1 \le i \le n-1$ as follows,

$$\varphi(v_i)=\begin{cases}\left\lceil\frac{5i-6}{3}\right\rceil, & \text{for } i\equiv 0 \pmod 6,\\ \left\lceil\frac{5i-5}{3}\right\rceil, & \text{for } i\equiv 1,2,3 \pmod 6,\\ \left\lceil\frac{5i-2}{3}\right\rceil, & \text{for } i\equiv 4 \pmod 6,\\ \left\lceil\frac{5i-7}{3}\right\rceil, & \text{for } i\equiv 5 \pmod 6.\end{cases}$$

The vertex labels $\varphi(w_i)$ for $1\le i\le n-1$ as follows,

$$\varphi(w_i)=\begin{cases}\left\lceil\frac{5i}{3}\right\rceil, & \text{for } i\equiv 0 \pmod 6,\\ \left\lceil\frac{5i+1}{3}\right\rceil, & \text{for } i\equiv 1,2,3 \pmod 6,\\ \left\lceil\frac{5i+3}{3}\right\rceil, & \text{for } i\equiv 4 \pmod 6,\\ \left\lceil\frac{5i-1}{3}\right\rceil, & \text{for } i\equiv 5 \pmod 6.\end{cases}$$

The vertex labels $\varphi(u_i)$ for $1\le i\le n$ as follows,

$$\varphi(u_i)=\begin{cases}\left\lceil\frac{5i-6}{3}\right\rceil, & \text{for } i\equiv 0 \pmod 6,\\ \left\lceil\frac{5i-5}{3}\right\rceil, & \text{for } i\equiv 1,2,3 \pmod 6,\\ \left\lceil\frac{5i-4}{3}\right\rceil, & \text{for } i\equiv 4 \pmod 6,\\ \left\lceil\frac{5i-1}{3}\right\rceil, & \text{for } i\equiv 5 \pmod 6.\end{cases}$$

The edge labels $\varphi(x_iv_i)$ and $\varphi(x_iw_i)$ for $1\le i\le n-1$ as follows,

$$\varphi(x_iv_i)=\varphi(x_iw_i)=\begin{cases}i+1, & \text{for } i=1,2,3,4,\\ \left\lfloor\frac{5i}{3}\right\rfloor-1, & \text{for } i\equiv 0,1 \pmod 6 \text{ and } i\neq 1,\\ \left\lfloor\frac{5i}{3}\right\rfloor-2, & \text{for } i\equiv 2 \pmod 6 \text{ and } i\neq 2,\\ \left\lfloor\frac{5i}{3}\right\rfloor-3, & \text{for } i\equiv 3,4 \pmod 6 \text{ and } i\neq 3,4,\\ \left\lfloor\frac{5i}{3}\right\rfloor, & \text{for } i\equiv 5 \pmod 6.\end{cases}$$

The edge labels $\varphi(v_iu_i)$ for $1\le i\le n-1$ as follows,

$$\varphi(v_iu_i)=\begin{cases}\left\lfloor\frac{5i}{3}\right\rfloor-2, & \text{for } i\equiv 3,4 \pmod 6,\\ \left\lfloor\frac{5i}{3}\right\rfloor-1, & \text{for } i\equiv 2,5 \pmod 6,\\ \left\lfloor\frac{5i}{3}\right\rfloor, & \text{for } i\equiv 0,1 \pmod 6.\end{cases}$$

The edge labels $\varphi(w_iu_{i+1})$ for $1\le l\le n-1$ as follows,

$$\varphi(w_i u_{i+1}) = \begin{cases} \left\lceil \frac{5i-2}{3} \right\rceil, & \text{for } i \equiv 0,1 \pmod 6, \\ \left\lceil \frac{5i-5}{3} \right\rceil, & \text{for } i \equiv 2 \pmod 6, \\ \left\lceil \frac{5i-7}{3} \right\rceil, & \text{for } i \equiv 3 \pmod 6, \\ \left\lceil \frac{5i-9}{3} \right\rceil, & \text{for } i \equiv 4 \pmod 6, \\ \left\lceil \frac{5i+2}{3} \right\rceil, & \text{for } i \equiv 5 \pmod 6. \end{cases}$$

The edge labels $\varphi(v_i u_{i+1})$ for $1 \le l \le n-1$ as follows,

$$\varphi(v_i u_{i+1}) = \begin{cases} \left\lfloor \frac{5i}{3} \right\rfloor - 2, & \text{for } i \equiv 3,4 \ (mod\ 6), \\ \left\lfloor \frac{5i}{3} \right\rfloor - 1, & \text{for } i \equiv 2,5 \ (mod\ 6), \\ \left\lfloor \frac{5i}{3} \right\rfloor, & \text{for } i \equiv 0,1 \ (mod\ 6). \end{cases}$$

From the upper bound of the labels on the pentagonal snake graph PS_n, the maximum value obtained from the vertex labels and edge labels is $\left\lceil \frac{5n-5}{3} \right\rceil$ for $5n-5 \not\equiv 2,3\ (mod\ 6)$ and $\left\lceil \frac{5n-5}{3} \right\rceil + 1$ for $5n-5 \equiv 2,3\ (mod\ 6)$. Because the upper bound of $res(PS_n)$ is equal with the lower bound, then this value is the $res(PS_n)$. Furthermore, the following edge weights are obtained.

$$\begin{aligned} wt_\varphi(v_i u_i) &= 5i-4, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(x_i v_i) &= 5i-3, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(u_i u_{i+1}) &= 5i-2, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(x_i w_i) &= 5i-1, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(w_i u_{i+1}) &= 5i, && \text{for } 1 \le i \le n-1. \end{aligned}$$

The pentagonal snake graph PS_n exhibits distinct edge weights, with both the upper and lower bounds matching $res(PS_n)$. Thus, φ satisfies the edge irregular reflexive k-labeling and has a reflexive edge strength. The theorem is thus proven. ■

Example 1. Ilustrates of an edge irregular reflexive 6-labeling on the pentagonal snake graph PS_4 is shown in **Figure 1**. Every vertex has a black number shown on its label, while every edge has a blue number. Each edge's weight is indicated by a red number.

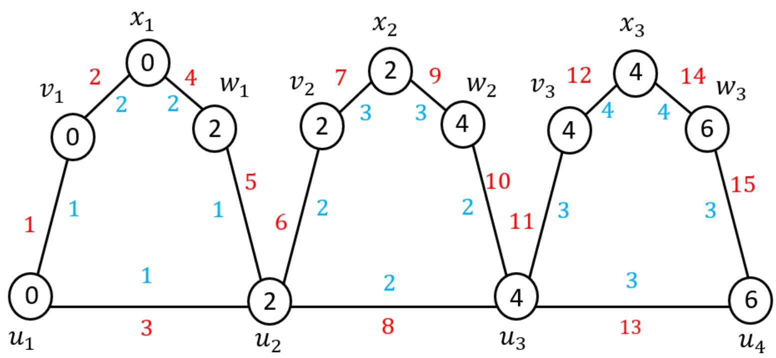

Figure 1. Edge irregular reflexive 6-labeling of graph pentagonal snake graph PS_4

3.2 Corona of Open Triangular Ladder and Null Graph $O(TL_n) \odot N_m$

The Corona of the open triangular ladder and null graph denoted by $O(TL_n) \odot N_m$ is the graph resulting from the corona operation between open triangular ladder $O(TL_n)$ graph and the null graph N_m. This graph consists of vertex set $V(O(TL_n) \odot N_m) = \{v_i : 1 \le i \le n-1\} \cup \{u_i, : 1 \le i \le n\} \cup \{v_{ij}, u_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and edge set $E(O(TL_n) \odot N_m) = \{v_i v_{i+1}, u_i u_{i+1}, u_i v_{i+1}, : 1 \le i \le n-1\} \cup \{u_i v_i : 2 \le i \le n-1\} \cup \{v_i v_{ij}, u_i u_{ij} : 1 \le i \le n, 1 \le j \le m\}$. The corona of open triangular ladder and null graph has $|V(O(TL_n) \odot N_m)| = 2nm + 2n$ and $|E(O(TL_n) \odot N_m)| = 2nm + 4n - 5$.

Theorem 2. *For any positive integer $n \ge 3$ and $m \ge 1$,*

$$res(O(TL_n) \odot N_m) = \begin{cases} \left\lceil \frac{2nm+4n-5}{3} \right\rceil, & \text{for } 2nm+4n-5 \not\equiv 2,3 \pmod 6 \\ \left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1, & \text{for } 2nm+4n-5 \equiv 2,3 \pmod 6 \end{cases} \qquad (3)$$

Proof. It is known that the number of edges of the corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ is $2nm + 4n - 5$, so based on **Lemma 1** we get

$$res(O(TL_n) \odot N_m) \ge \begin{cases} \left\lceil \frac{2nm+4n-5}{3} \right\rceil, & \text{for } 2nm+4n-5 \not\equiv 2,3 \pmod 6 \\ \left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1, & \text{for } 2nm+4n-5 \equiv 2,3 \pmod 6 \end{cases} \qquad (4)$$

Next, we formulate the labeling function φ on the edge irregular reflexive k-labeling of the corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ with $k = \left\lceil \frac{2nm+4n-5}{3} \right\rceil$ for $2nm + 3n - 4 \not\equiv 2,3 \pmod 6$ and $k = \left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1$ for $2nm + 3n - 4 \equiv 2,3 \pmod 6$ are as follows.

The vertex labels $\varphi(v_i)$ for $1 \le i \le n$ as follows,

$$\varphi(v_i) = \begin{cases} 0, & \text{for } i = 1, \\ 2, & \text{for } i = 2, m = 1, \\ 4, & \text{for } i = 2, m \ne 1, \\ \frac{2im+4i-6}{3}, & \text{for } i \equiv 0 \pmod 3;\ i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \ne 1; \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \ne 2, \\ \frac{2im+4i-4}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 0 \pmod 3, i \ne 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \ne 2, \\ \frac{2im+4i-2}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 2 \pmod 3, i \ne 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \ne 2. \end{cases}$$

The vertex labels $\varphi(u_i)$ for $1 \le i \le n$ and $\varphi(v_{ij})$ for $1 \le i \le n$, $1 \le j \le m$ as follows,

$$\varphi(u_i) = \varphi(v_{ij}) = \begin{cases} 0, & \text{for } i = 1, \\ 2m, & \text{for } i = 2, \\ \frac{2im+4i-6}{3}, & \text{for } i \equiv 0 \pmod 3;\ i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \ne 1; \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \ne 2, \\ \frac{2im+4i-4}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 0 \pmod 3, i \ne 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \ne 2, \\ \frac{2im+4i-2}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 2 \pmod 3, i \ne 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \ne 2. \end{cases}$$

The vertex labels $\varphi(u_{ij})$ for $1 \le i \le n$ and $1 \le j \le m$ as follows,

$$\varphi(u_{ij}) = \begin{cases} 0, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \\ \dfrac{2im + 4i - 6}{3}, & \text{for } i \equiv 0 \pmod 3;\ i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \neq 1; \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \neq 2, \\ \dfrac{2im + 4i - 4}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 0 \pmod 3, i \neq 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \neq 2, \\ \dfrac{2im + 4i - 2}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 2 \pmod 3, i \neq 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \neq 2. \end{cases}$$

The edge labels $\varphi(v_i v_{i+1})$for $1 \leq i \leq n-1$ as follows,

$$\varphi(v_i v_{i+1}) = \begin{cases} 3, & \text{for } i = 1, m = 1, \\ 2m - 1, & \text{for } i = 1, m \neq 1, \\ 2, & \text{for } i = 2, m = 1, \\ 2m + 1, & \text{for } i = 2, m \neq 1, \\ \dfrac{2im - 2m + 4i + 3}{3}, & \text{for } i \equiv 0 \pmod 3, m \equiv 0 \pmod 3 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \neq 2, \\ \dfrac{2im - 2m + 4i + 5}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \neq 1; \\ & \text{for } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \neq 2; \\ & \text{and } i \equiv 0 \pmod 3, m \equiv 1 \pmod 3, \\ \dfrac{2im - 2m + 4i + 1}{3}, & \text{for } i \equiv 0 \pmod 3, m \equiv 2 \pmod 3 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \neq 2, \\ \dfrac{2im - 2m + 4i - 1}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 0,2 \pmod 3, i \neq 1. \end{cases}$$

The edge labels $\varphi(u_i u_{i+1})$ for $2 \leq i \leq n-1$ as follows,

$$\varphi(u_i u_{i+1}) = \begin{cases} 1, & \text{for } i = 1, \\ 3, & \text{for } i = 2, \\ \dfrac{2im - 2m + 4i - 3}{3}, & \text{for } i \equiv 0 \pmod 3, m \equiv 0 \pmod 3 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \neq 2, \\ \dfrac{2im - 2m + 4i - 1}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \neq 1; \\ & \text{for } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \neq 2; \\ & \text{and } i \equiv 0 \pmod 3, m \equiv 1 \pmod 3, \\ \dfrac{2im - 2m + 4i - 5}{3}, & \text{for } i \equiv 0 \pmod 3, m \equiv 2 \pmod 3 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \neq 2, \\ \dfrac{2im - 2m + 4i - 1}{3}, & \text{for } i \equiv 1 \pmod 3, m \equiv 0,2 \pmod 3, i \neq 1. \end{cases}$$

The edge labels $\varphi(u_i v_{i+1})$ for $1 \leq i \leq n$ as follows,

$$\varphi(u_iv_{i+1}) = \begin{cases} 2, & \text{for } i = 1, m = 1, \\ 2m-2, & \text{for } i = 2, m \neq 1, \\ 4, & \text{for } i = 2, \\ \dfrac{2im-2m+4i}{3}, & \text{for } i \equiv 0 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3) \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3), i \neq 2, \\ \dfrac{2im-2m+4i+2}{3}, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), i \neq 1; \\ & \text{for } i \equiv 2 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), i \neq 2; \\ & \text{and } i \equiv 0 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), \\ \dfrac{2im-2m+4i-2}{3}, & \text{for } i \equiv 0 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3) \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3), i \neq 2, \\ \dfrac{2im-2m+4i-4}{3}, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 0,2 \ (\text{mod } 3), i \neq 1. \end{cases}$$

The edge labels $\varphi(u_iv_i)$ for $2 \leq i \leq n-1$ as follows,

$$\varphi(u_iv_i) = \begin{cases} 4, & \text{for } i = 2, m = 1, \\ 2m, & \text{for } i = 2, m \neq 1, \\ \dfrac{2im+4i}{3}, & \text{for } i \equiv 0 \ (\text{mod } 3); i \equiv 1 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3); \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), i \neq 2, \\ \dfrac{2im+4i-4}{3}, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3) \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3), i \neq 2, \\ \dfrac{2im+4i-8}{3}, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3) \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3), i \neq 2. \end{cases}$$

The edge labels $\varphi(v_iv_{ij})$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$\varphi(v_iv_{ij}) = \begin{cases} m+j, & \text{for } i = 1, \\ 3, & \text{for } i = 2, m = 1, \\ m+j-1, & \text{for } i = 2, m \neq 1, \\ \left(\dfrac{2im-3m+4i}{3}\right)+j-1, & \text{for } i \equiv 0 \ (\text{mod } 3); i \equiv 1 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), i \neq 1; \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 1 \ (\text{mod } 3), i \neq 2, \\ \left(\dfrac{2im-3m+4i-4}{3}\right)+j-1, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3), i \neq 1 \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3), i \neq 2, \\ \left(\dfrac{2im-3m+4i-8}{3}\right)+j-1, & \text{for } i \equiv 1 \ (\text{mod } 3), m \equiv 2 \ (\text{mod } 3), i \neq 1 \\ & \text{and } i \equiv 2 \ (\text{mod } 3), m \equiv 0 \ (\text{mod } 3), i \neq 2. \end{cases}$$

The edge labels $\varphi(u_iu_{ij})$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$\varphi(u_iu_{ij}) = \begin{cases} j, & \text{for } i = 1, \\ j+1, & \text{for } i = 2, \\ \left(\dfrac{2im-3m+4i}{3}\right)+j-1, & \text{for } i \equiv 0 \pmod 3; i \equiv 1 \pmod 3, m \equiv 1 \pmod 3, i \neq 1; \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 1 \pmod 3, i \neq 2, \\ \left(\dfrac{2im-3m+4i-4}{3}\right)+j-1, & \text{for } i \equiv 1 \pmod 3, m \equiv 0 \pmod 3, i \neq 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 2 \pmod 3, i \neq 2, \\ \left(\dfrac{2im-3m+4i-8}{3}\right)+j-1, & \text{for } i \equiv 1 \pmod 3, m \equiv 2 \pmod 3, i \neq 1 \\ & \text{and } i \equiv 2 \pmod 3, m \equiv 0 \pmod 3, i \neq 2. \end{cases}$$

From the upper bound of the labels on the corona of open triangular ladder and null graph $O(TL_n) \odot N_m$, the maximum value obtained from the vertex labels and edge labels is $\left\lceil \frac{2nm+4n-5}{3} \right\rceil$ for $2nm + 4n - 5 \not\equiv 2, 3 \pmod 6$ and $\left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1$ for $2nm + 4n - 5 \equiv 2, 3 \pmod 6$. Because the upper bound of $res(O(TL_n) \odot N_m)$ is equal with the lower bound, then this value is the $res(O(TL_n) \odot N_m)$. Furthermore, the following edge weights are obtained.

$$\begin{aligned} wt_\varphi(v_iv_{i+1}) &= 2im + 4i - 1, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(u_iu_{i+1}) &= 2im + 4i - 3, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(u_iv_{i+1}) &= 2im + 4i - 2, && \text{for } 1 \le i \le n-1. \\ wt_\varphi(u_iv_i) &= 2im + 4i - 4, && \text{for } 2 \le i \le n-1. \end{aligned}$$

$$wt_\varphi(v_iv_{i,j}) = \begin{cases} m+j, & \text{for } i = 1, 1 \le j \le m, \\ 2im - m + 4i + j - 5, & \text{for } 2 \le i \le n, 1 \le j \le m. \end{cases}$$

$$wt_\varphi(u_iu_{i,j}) = \begin{cases} j, & \text{for } i = 1, 1 \le j \le m, \\ 2im - 2m + 4i + j - 5, & \text{for } 2 \le i \le n, 1 \le j \le m. \end{cases}$$

The corona of open triangular ladder and null graph $O(TL_n) \odot N_m$ exhibits distinct edge weights, with both the upper and lower bounds matching $res(O(TL_n) \odot N_m)$. Thus, φ satisfies the edge irregular reflexive k-labeling and has a reflexive edge strength. The theorem is thus proven. ■

Example 2. Ilustrates an edge irregular reflexive 7-labeling on the corona of open triangular ladder and null graph $O(TL_3) \odot N_2$ is shown in Figure 2. Every vertex has a black number shown on its label, while every edge has a blue number. Each edge's weight is indicated by a red number.

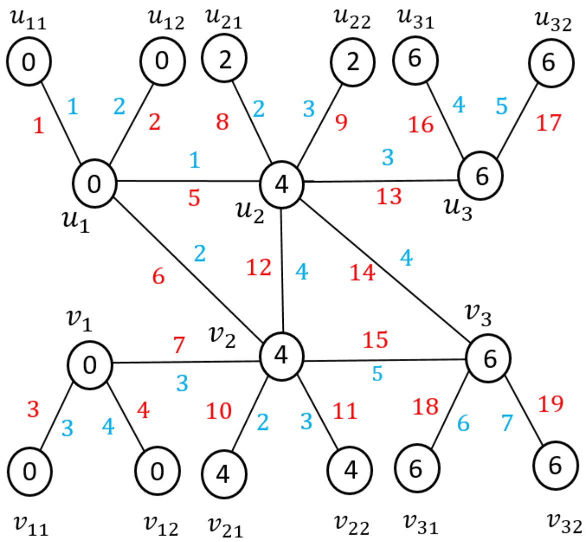

Figure 2. Edge irregular reflexive 7-labeling on the corona of open triangular ladder and null graph $O(TL_3) \odot N_2$.

4. CONCLUSIONS

The results and discussion lead to the following conclusions.

1. The reflexive edge strength of pentagonal snake graph PS_n with $n \geq 2$ is
$$res(PS_n) = \begin{cases} \left\lceil \frac{5n-5}{3} \right\rceil, & \text{for } 5n-5 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{5n-5}{3} \right\rceil + 1, & \text{for } 5n-5 \equiv 2,3 \pmod 6. \end{cases}$$
2. The reflexive edge strength of the corona of the open triangular ladder and null graph $O(TL_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$ is
$$res(O(TL_n) \odot N_m) = \begin{cases} \left\lceil \frac{2nm+4n-5}{3} \right\rceil, & \text{for } 2nm+4n-5 \not\equiv 2,3 \pmod 6, \\ \left\lceil \frac{2nm+4n-5}{3} \right\rceil + 1, & \text{for } 2nm+4n-5 \equiv 2,3 \pmod 6. \end{cases}$$

Open problem: How is the reflexive edge strength of the corona of the open triangular ladder and another graph, for example, the corona of the open triangular ladder and path graph?

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