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EDGE IRREGULAR REFLEXIVE LABELING OF DUMBBELL GRAPH, CORONA OF OPEN LADDER, AND NULL GRAPH

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ABSTRACT

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Graph G is a simple, connected, undirected graph with vertex set $V(G)$ and edge set $E(G)$. A graph G is called to have an edge irregular reflexive k-labeling if its vertices can be labeled with even numbers from 0 until 2_{kv} and its edges can be labeled with positive integers from 1 to k_e such that the weights for all the edges are different, where $k = max\{k_e, 2_{kv}\}$. The weight of edge uv in graph G with θ labeling, denoted by $wt_\theta(uv)$, is defined as sum of the edge label and all vertex labels incident to that edge. The reflexive edge strength of a graph G, denoted by $res(G)$, is the value of minimum k of the largest label. In this paper, edge irregular reflexive k-labeling for Dumbbell Graph $(D_{m,n,q})$ and corona of open ladder and null graph $(O(L_n) \odot N_m)$ will be determined. The reflexive edge strength of the Dumbbell Graph $D_{m,n,q}$ with $m, n \geq 3, m = n$ and $q = 3$ is $\left\lceil\frac{2m+2}{3}\right\rceil$ for $2m + 2 \equiv 0, 4 \ (mod\ 6)$ and $\left\lceil\frac{2m+2}{3}\right\rceil + 1$ for $2m + 2 \equiv 2 \ (mod\ 6)$. The reflexive edge strength of the corona of open ladder and null graph $O(L_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$ is $\left\lceil\frac{2nm+3n-4}{3}\right\rceil$ for $2nm + 3n - 4 \not\equiv 2,3 \ (mod\ 6)$ and $\left\lceil\frac{2nm+3n-4}{3}\right\rceil + 1$ for $2nm + 3n - 4 \equiv 2,3 \ (mod\ 6)$.

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1. INTRODUCTION

Graph labeling is often discussed in graph theory. Labeling was first introduced by Wallis [1] in 1963. According to Wallis, labeling on a graph is a function that pairs graph elements (vertices or edges) with numbers (generally positive or non-negative integers). If the domain of the mapping is the set of vertices, then it is called vertex labeling; if the domain of the mapping is the set of edges, then it is called edge labeling. If the domain of the mapping is the set of vertices and edges, then it is called vertex and edge (total) labeling. Over time there has been a development of various types of labeling in graphs, one of which is the irregular total k-labeling [2].

Vertex irregular total k-labeling and edge irregular total k-labeling were introduced in 2007 by Bača et al. [3]. Furthermore, in 2017, there is a new concept of irregular total k-labeling which is vertex irregular reflexive k-labeling and edge irregular reflexive k-labeling on a graph G introduced by Bača et al. [4]. A graph G is called to have an edge irregular reflexive k-labeling if its vertices can be labeled with even number from 0 to 2_{kv} and its edges can be labeled with positive integers from 1 to k_e such that the weights for all edges are different, with $k = \max\{k_e, 2_{kv}\}$.

Suppose graph G is a simple, connected, undirected graph with vertex set $V(G)$ and edge set $E(G)$. The weight of an edge uv in a graph G with θ labeling, denoted by $wt_\theta(uv)$, is the sum of vertex label u, edge label uv, and vertex label v, is defined as $wt_\theta(uv) = \theta(u) + \theta(uv) + \theta(v)$. The reflexive edge strength of a graph G, denoted by $res(G)$, is the value of minimum k of the largest vertex label or the largest edge label. In research conducted by Bača et al. [4] found a lemma that can be used as a guide to determine the reflexive edge strength of any graph G $(res(G))$.

Lemma 1. *For every graph G,*

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{for } |E(G)| \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{for } |E(G)| \equiv 2,3 \ (mod\ 6). \end{cases}$$

There are several research related to edge irregular reflexive k-labeling such as banana tree graphs $B_{2,n}$ and $B_{3,n}$ studied by Novelia and Indriati [5], alternate triangular snake graf $A(T_n)$ and double alternate quadrilateral snake graf $DA(Q_n)$ studied by Zalzabila et al. [6], wheel graph W_n and prism graph D_n studied by Tanna et al. [7], mongolian tent graph $M_{m,3}$ studied by Indriati and Azzahra [8], umbrella graphs $U_{3,n}$ and $U_{4,n}$ studied by Rahmawati and Indriati [9], corona of cycle and null graph (N_2) denoted by $(C_n \odot N_2)$ studied by Setiawan and Indriati [10], corona of path and other graphs studied by Indriati et al. [11], corona product of graphs with path studied by Yoong et al. [12]. Agustin et al. [13] have studied the edge irregular reflexive k-labeling on ladder graphs L_n and triangular ladder TL_n graphs. In this research, we determine edge irregular reflexive k-labeling for corona of open ladder and null graph $O(L_n) \odot N_m$ with n ≥ 3 and m ≥ 1. Dumbbell Graph $D_{m,n,q}$ have been studied by Saputri et al. [14] on odd harmonic labeling, but it has never been used for edge irregular reflexive k-labeling so in this research we determine edge irregular reflexive k-labeling for Dumbbell Graph $D_{m,n,q}$ with $m, n \geq 3, m = n$ and $q = 3$.

2. RESEARCH METHODS

The research method used in this research is a literature study that refers to various references, such as books, journals, and writings that discuss edge irregular reflexive k-labeling. From this method, we can determine $\boldsymbol{res(G)}$ of Dumbbell Graph and corona of open ladder and null graph. The steps of this research can be seen in **Figure 1**.

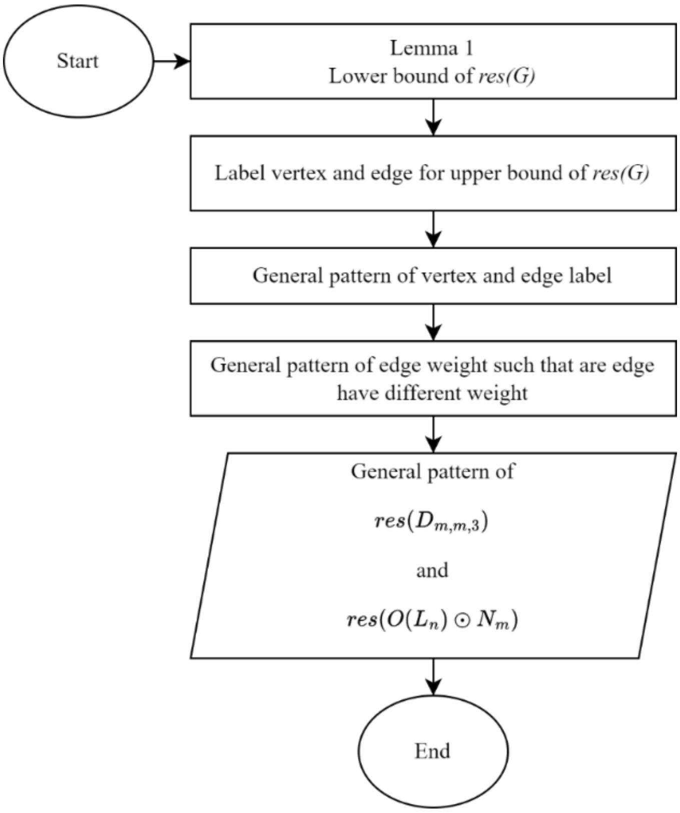


Figure 1. Flowchart of the research

3. RESULTS AND DISCUSSION

In this chapter, following the research methodology, the results and discussion of reflexive edge strength on the Dumbbell Graph $D_{m,m,3}$, and corona of open ladder and null graph $O(L_n) \odot N_m$ are given. In the first section we discuss for Dumbbell Graph, and in the second for corona of open ladder, and null graph.

3.1 Dumbbell Graph $D_{m,m,3}$

Dumbbell Graph is a graph formed from two circular graphs C_m and C_n connected by a path graph P_q, where the endpoints of the path graph P_q are one of the vertices of each circular graph [15]. So, the notation of Dumbbell Graph is $D_{m,n,q}$, where $m, n \geq 3$ denotes the number of vertices in the two circular graphs and $q \geq 2$ denotes the number of vertices in the path graph. The Dumbbell Graph $D_{m,n,q}$ has vertex set $V(D_{m,n,q}) = \{{v_i}^1 | 1 \leq i \leq m\} \cup \{u_j | 1 \leq j \leq q\} \cup \{{v_l}^2 | 1 \leq l \leq n\}$ and the set of edges $E(D_{m,n,q}) = \{{v_i}^1 {v_{i+1}}^1 | 1 \leq i \leq m-1\} \cup \{u_j u_{j+1} | 1 \leq j \leq q-1\} \cup \{{v_l}^2 {v_{l+1}}^2 | 1 \leq l \leq n-1\}$, ${v_i}^1$and ${v_l}^2$ are vertices of the circular graph, and u_j is a vertex of the path graph.

The reflexive edge strength of a Dumbbell Graph can be obtained through **Theorem 1**.

Theorem 1. *For any positive integer on the Dumbbell Graph $D_{m,n,q}$ with $m, n \geq 3, m = n$ and $q = 3$,*

$$res(D_{m,m,3}) = \begin{cases} \left\lceil \frac{2m+2}{3} \right\rceil, & \text{for } 2m+2 \equiv 0{,}4 \ (mod\ 6), \\ \left\lceil \frac{2m+2}{3} \right\rceil + 1, & \text{for } 2m+2 \equiv 2 \ (mod\ 6). \end{cases}$$

Proof. It is known that the number of edges of the Dumbbell Graph $D_{m,m,3}$ is $2m+2$, so based on **Lemma 1** we get

$$res(D_{m,m,3}) \geq \begin{cases} \left\lceil \frac{2m+2}{3} \right\rceil, & \text{for } 2m+2 \not\equiv 2{,}3 \ (mod\ 6), \\ \left\lceil \frac{2m+2}{3} \right\rceil + 1, & \text{for } 2m+2 \equiv 2{,}3 \ (mod\ 6). \end{cases}$$

Next, constructed θ on the edge irregular reflexive k-labeling of Dumbbell Graph $D_{m,m,3}$ with $k = \left\lceil\frac{2m+2}{3}\right\rceil$, for $2m+2 \equiv 0,4\ (mod\ 6)$ and $k = \left\lceil\frac{2m+2}{3}\right\rceil + 1$, for $2m+2 \equiv 2\ (mod\ 6)$ are as follows.

The vertex labels u_j for $1 \le j \le 3$ as follows,

$\theta(u_j) = 0.$

The vertex labels ${v_i}^1$ for $1 \le i \le m$ as follows,

$$\theta({v_i}^1) = \begin{cases} \dfrac{4i+2a}{3}, & \text{for } i \equiv a \text{ (mod 3)}, a = 1,2,3, 1 \le i \le \left\lfloor\dfrac{m}{2}\right\rfloor, \\ 2m - 2i + 2t - 2, & \text{for } i > \left\lfloor\dfrac{m}{2}\right\rfloor, 1 \le t \le \left\lceil\dfrac{m-2}{6}\right\rceil, \\ & m - 3 - 3t + 3 \le i \le m - 1 - 3t + 3, \\ 0, & \text{for } i = m. \end{cases}$$

The vertex labels ${v_l}^2$ for $1 \le l \le n$ as follows,

$$\theta({v_l}^2) = \begin{cases} 0, & \text{for } l = 1, \\ \dfrac{4l}{3}, & \text{for } l \equiv 0 \text{ (mod 3)}, 2 \le l \le \left\lceil\dfrac{n}{2}\right\rceil, \\ \dfrac{4l+2}{3}, & \text{for } l \equiv 1 \text{ (mod 3)}, 2 \le l \le \left\lceil\dfrac{n}{2}\right\rceil, \\ \dfrac{4l-2}{3}, & \text{for } l \equiv 2 \text{ (mod 3)}, 2 \le l \le \left\lceil\dfrac{n}{2}\right\rceil, \\ 2n - 2l - 2t + 4, & \text{for } l > \left\lceil\dfrac{n}{2}\right\rceil, 1 \le t \le \left\lceil\dfrac{n-1}{6}\right\rceil, n + 1 - 3t \le l \le n + 3 - 3t. \end{cases}$$

The edge labels $u_j u_{j+1}$ for $1 \le j \le 2$ as follows,

$\theta(u_j u_{j+1}) = j.$

The edge labels ${v_i}^1{v_{i+1}}^1$ for $1 \le i \le m-1$ as follows,

$$\theta({v_i}^1{v_{i+1}}^1) = \begin{cases} \dfrac{4i-3}{3}, & \text{for } i \equiv 0 \text{ (mod 3)}, 1 \le i < \left\lfloor\dfrac{m}{2}\right\rfloor \text{ and} \\ & \text{for } m \equiv 1 \text{ (mod 6) and } i \equiv 0 \text{ (mod 3)}, \\ \dfrac{4i-1}{3}, & \text{for } i \equiv 1 \text{ (mod 3)}, 1 \le i < \left\lfloor\dfrac{m}{2}\right\rfloor, \\ \dfrac{4i-5}{3}, & \text{for } i \equiv 2 \text{ (mod 3)}, 1 \le i < \left\lfloor\dfrac{m}{2}\right\rfloor, \\ 2m - 2i - 2t - 1, & \text{for } m \ge 5, i \ge \left\lfloor\dfrac{m}{2}\right\rfloor, 1 \le t \le \left\lfloor\dfrac{m+1}{6}\right\rfloor, \\ & m - 2 - 3t \le i \le m - 3t, \\ & m \not\equiv 1 \text{ (mod 6) and } i \not\equiv 0 \text{ (mod 3)}, \\ 3, & \text{for } i \ge m - 2. \end{cases}$$

The edge labels ${v_m}^1{v_1}^1$ as follows,

$\theta({v_m}^1{v_1}^1) = 1.$

The edge labels ${v_l}^2{v_{l+1}}^2$ for $1 \le l \le n-1$ as follows,

$$\theta(v_l^2 v_{l+1}^2) = \begin{cases} 4, & \text{for } l = 1, \\ \dfrac{4l+2a}{3}, & \text{for } l \equiv a \pmod 3 < \left\lceil \dfrac{n}{2} \right\rceil, a = 0,1,2, \\ \dfrac{4l+4}{3}, & \text{for } n \equiv 3,4 \pmod 6 \text{ and } l \equiv 2 \pmod 3), \\ \dfrac{4l}{3}, & \text{for } n \equiv 5 \pmod 6 \text{ and } l \equiv 0 \pmod 3), \\ 2n - 2l - 2t, & \text{for } n \geq 6, l \geq \left\lceil \dfrac{n}{2} \right\rceil, 1 \leq t \leq \left\lfloor \dfrac{n}{6} \right\rfloor, n-1-3t \leq l \leq n+1-3t, \\ & n \not\equiv 3,4 \pmod 6 \text{ and } l \not\equiv 2 \pmod 3), \\ & n \not\equiv 5 \pmod 6 \text{ and } l \not\equiv 0 \pmod 3), \\ 2, & \text{for } l = n-1. \end{cases}$$

The edge labels $v_n^2 v_1^2$ as follows,

$\theta(v_n^2 v_1^2) = 2.$

Based on the obtained labelling formula, the largest label which is the upper boundary of $res(D_{m,m,3})$ is located at vertex v_i^1 for $i = \frac{m}{2}$,

$$f\left(v_{\frac{m}{2}}^1\right) = \frac{4\frac{m}{2}+2a}{3}, \text{ for } \frac{m}{2} \equiv a \pmod 3), a = 1,2,3.$$

From the lower and upper bounds on the Dumbbell Graph $D_{m,m,3}$, the maximum value of the vertex label and edge labels are obtained, namely $\left\lceil\frac{2m+2}{3}\right\rceil$ for $2m+2 \equiv 0, 4 \ (mod\ 6)$ and $\left\lceil\frac{2m+2}{3}\right\rceil + 1$ for $2m + 2 \equiv 2\ (mod\ 6)$. Then, the following edge weights are obtained,

$wt_\theta(u_j u_{j+1}) = j, \quad$ for $1 \leq j \leq 2$.

$$wt_\theta(v_i^1 v_{i+1}^1) = \begin{cases} 4i+3, & \text{for } 1 \leq i < \left\lceil \dfrac{m}{2} \right\rceil, \\ 4m - 4i + 1, & \text{for } i \geq \left\lceil \dfrac{m}{2} \right\rceil. \end{cases}$$

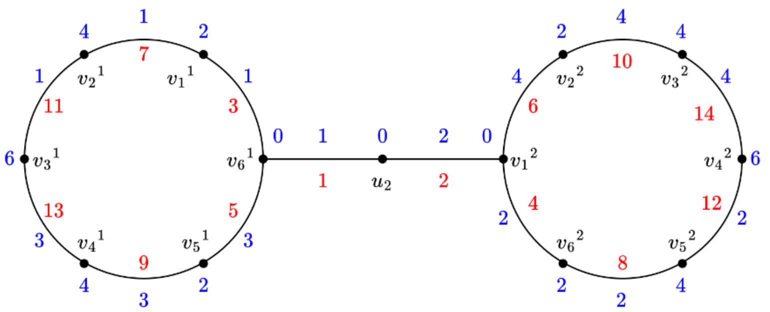
$wt_\theta(v_m^1 v_1^1) = 3.$

$$wt_\theta(v_l^2 v_{l+1}^2) = \begin{cases} 4l+2, & \text{for } 1 \leq l \leq \left\lfloor \dfrac{n}{2} \right\rfloor, \\ 4n - 4l + 4, & \text{for } l > \left\lfloor \dfrac{n}{2} \right\rfloor. \end{cases}$$

$wt_\theta(v_n^2 v_1^2) = 4.$

It can be seen that all the edge weights of the Dumbbell Graph $D_{m,m,3}$ are different, the lower bound and upper bound are equal to $res(D_{m,m,3})$. Therefore, θ satisfies the edge irregular reflexive k-labeling and has reflexive edge strength. Thus the theorem is proved.

Figure 2 shows the edge irregular reflexive 6-labeling on the graph Dumbbell Graph $D_{6,6,3}$. The label of each edge and vertex is shown with blue numbers. The weight of each edge is shown with a red number.



***Note**: (vertex v_6^1 can also be called vertex u_1, vertex v_1^2 can also be called vertex u_3)*

Figure 2. Edge Irregular reflexive 6-labeling on the graph dumbbell graph $D_{6,6,3}$

3.2 Corona of Open Ladder and Null Graph $O(L_n) \odot N_m$

The corona of open ladder and null graph denoted by $O(L_n) \odot N_m$. This graph is a connected graph with vertex set $V(O(L_n) \odot N_m) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u_{i,j}, v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set $E(O(L_n) \odot N_m) = \{u_i v_i : 2 \leq i \leq n-1\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_{i,j}, v_i v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. The corona of open ladder and null graph has $2nm + 3n - 4$ edges. The reflexive edge strength of the corona of open ladder and null graph can be obtained through **Theorem 2**.

Theorem 2. *For any positive integer $n \geq 3$ and $m \geq 1$,*

$$res(O(L_n) \odot N_m) = \begin{cases} \left\lceil \frac{2nm+3n-4}{3} \right\rceil, & \text{for } 2nm + 3n - 4 \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{2nm+3n-4}{3} \right\rceil + 1, & \text{for } 2nm + 3n - 4 \equiv 2,3 \ (mod\ 6). \end{cases}$$

Proof. It is known that the number of edges of the corona of open ladder and null graph $O(L_n) \odot N_m$ is $2nm + 3n - 4$, so based on **Lemma 1** we get

$$res(O(L_n) \odot N_m) \geq \begin{cases} \left\lceil \frac{2nm+3n-4}{3} \right\rceil, & \text{for } 2nm + 3n - 4 \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{2nm+3n-4}{3} \right\rceil + 1, & \text{for } 2nm + 3n - 4 \equiv 2,3 \ (mod\ 6). \end{cases}$$

Next, constructed θ on the edge irregular reflexive k-labeling of corona of open ladder and null graph $O(L_n) \odot N_m$ with $k = \left\lceil \frac{2nm+3n-4}{3} \right\rceil$, for $2nm + 3n - 4 \not\equiv 2,3 \ (mod\ 6)$ and $k = \left\lceil \frac{2nm+3n-4}{3} \right\rceil + 1$, for $2nm + 3n - 4 \equiv 2,3 \ (mod\ 6)$ are as follows.

The vertex labels $u_i, v_i, u_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$\theta(u_i) = \theta(v_i) = \theta(u_{i,j}) = \begin{cases} 0, & \text{for } i = 1, \\ m+1, & \text{for } i = 2, m \text{ odd}, \\ m, & \text{for } i = 2, m \text{ even}, \\ \frac{2im + 3i - 3}{3}, & \text{for } i \text{ odd}, i \neq 1, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \frac{2im + 3i}{3}, & \text{for } i \text{ even}, i \neq 2, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \frac{2im + 3i - 5}{3}, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \frac{2im + 3i - 4}{3}, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \frac{2im + 3i - 2}{3}, & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \frac{2im + 3i - 1}{3}, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3). \end{cases}$$

The vertex labels $v_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$\theta(v_{i,j}) = \begin{cases} 0, & \text{for } i = 1, \\ m+1, & \text{for } i = 2, m \text{ odd}, \\ m+2, & \text{for } i = 2, m \text{ even}, \\ \dfrac{2im+3i-3}{3}, & \text{for } i \text{ odd}, i \neq 1, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \dfrac{2im+3i}{3}, & \text{for } i \text{ even}, i \neq 2, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \dfrac{2im+3i-5}{3}, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{2im+3i-4}{3}, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{2im+3i-2}{3}, & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{2im+3i-1}{3}, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3). \end{cases}$$

The edge labels $u_i u_{i+1}$ for $1 \leq i \leq n-1$ as follows,

$$\theta(u_i u_{i+1}) = \begin{cases} m, & \text{for } i = 1, m \text{ odd}, \\ m+1, & \text{for } i = 1, m \text{ even}, \\ m+1, & \text{for } i = 2, m \text{ odd}, \\ m+2, & \text{for } i = 2, m \text{ even}, \\ \dfrac{(2i-2)m+3i-6}{3}, & \text{for } i \geq 3, m \equiv 0 \ (mod\ 3), \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i-4}{3}, & \text{for } i \equiv 0,3 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2,5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i}{3}, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i-2}{3}, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i-8}{3}, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 2 \ (mod\ 3). \end{cases}$$

The edge labels $v_i v_{i+1}$ for $1 \leq i \leq n-1$ as follows,

$$\theta(v_iv_{i+1}) = \begin{cases} m+1, & \text{for } i=1, m \text{ odd,} \\ m+2, & \text{for } i=1, m \text{ even,} \\ m+2, & \text{for } i=2, m \text{ odd,} \\ m+3, & \text{for } i=2, m \text{ even,} \\ \dfrac{(2i-2)m+3i-3}{3}, & \text{for } i \geq 3, m \equiv 0 \ (mod\ 3), \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i-1}{3}, & \text{for } i \equiv 0,3 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2,5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i+3}{3}, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i+1}{3}, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-2)m+3i-5}{3}, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 2 \ (mod\ 3). \end{cases}$$

The edge labels u_iv_i for $2 \leq i \leq n-1$ as follows,

$$\theta(u_iv_i) = \begin{cases} m+1, & \text{for } i=2, m \text{ odd,} \\ m+3, & \text{for } i=2, m \text{ even,} \\ \dfrac{(2i-3)m+3i-3}{3}, & \text{for } i \text{ odd}, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \dfrac{(2i-3)m+3i-9}{3}, & \text{for } i \text{ even}, i \neq 2, m \equiv 0 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\ \dfrac{(2i-3)m+3i+1}{3}, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-3)m+3i-1}{3}, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-3)m+3i-5}{3}, & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\ \dfrac{(2i-3)m+3i-7}{3}, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3). \end{cases}$$

The edge labels $u_iu_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$\theta(u_iu_{i,j}) = \begin{cases} j, & \text{for } i = 1, \\ j, & \text{for } i = 2, m \text{ odd}, \\ j+2, & \text{for } i = 2, m \text{ even}, \\ j+1, & \text{for } i = 3, \\ \dfrac{(2i-6)m+3i-6}{3} + j\,, & \text{for } i \text{ odd}, i \neq 1,3, m \equiv 0\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3\ (mod\ 6), m \equiv 1,2\ (mod\ 3), \\ \dfrac{(2i-6)m+3i-12}{3} + j\,, & \text{for } i \text{ even}, i \neq 2, m \equiv 0\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0\ (mod\ 6), m \equiv 1,2\ (mod\ 3), \\ \dfrac{(2i-6)m+3i-2}{3} + j\,, & \text{for } i \equiv 1\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 5\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-6)m+3i-4}{3} + j\,, & \text{for } i \equiv 2\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-6)m+3i-8}{3} + j\,, & \text{for } i \equiv 4\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-6)m+3i-10}{3} + j\,, & \text{for } i \equiv 5\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1\ (mod\ 6), m \equiv 2\ (mod\ 3). \end{cases}$$

The edge labels $v_iv_{i,j}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq m$ as follows,

$$\theta(v_iv_{i,j}) = \begin{cases} j+m, & \text{for } i = 1, \\ j+m+1, & \text{for } i = 2, \\ \dfrac{(2i-3)m+3i-3}{3} + j\,, & \text{for } i \text{ odd}, i \neq 1, m \equiv 0\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 3\ (mod\ 6), m \equiv 1,2\ (mod\ 3), \\ \dfrac{(2i-3)m+3i-9}{3} + j\,, & \text{for } i \text{ even}, i \neq 2, m \equiv 0\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 0\ (mod\ 6), m \equiv 1,2\ (mod\ 3), \\ \dfrac{(2i-3)m+3i+1}{3} + j\,, & \text{for } i \equiv 1\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 5\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-3)m+3i-1}{3} + j\,, & \text{for } i \equiv 2\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 4\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-3)m+3i-5}{3} + j\,, & \text{for } i \equiv 4\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 2\ (mod\ 6), m \equiv 2\ (mod\ 3), \\ \dfrac{(2i-3)m+3i-7}{3} + j\,, & \text{for } i \equiv 5\ (mod\ 6), m \equiv 1\ (mod\ 3) \text{ and} \\ & \text{for } i \equiv 1\ (mod\ 6), m \equiv 2\ (mod\ 3). \end{cases}$$

The edge labels $v_iv_{i,j}$ for $i = n, 3 \leq i \leq n$ and $1 \leq j \leq m$ as follows,

$$
\theta(v_i v_{i,j}) = \begin{cases}
\dfrac{(2i-3)m+3i-6}{3} + j, & \text{for } i \text{ odd}, i \neq 1, m \equiv 0 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 3 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\
\dfrac{(2i-3)m+3i-12}{3} + j, & \text{for } i \text{ even}, i \neq 2, m \equiv 0 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 0 \ (mod\ 6), m \equiv 1,2 \ (mod\ 3), \\
\dfrac{(2i-3)m+3i-2}{3} + j, & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\
\dfrac{(2i-3)m+3i-4}{3} + j, & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\
\dfrac{(2i-3)m+3i-8}{3} + j, & \text{for } i \equiv 4 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 2 \ (mod\ 6), m \equiv 2 \ (mod\ 3), \\
\dfrac{(2i-3)m+3i-10}{3} + j, & \text{for } i \equiv 5 \ (mod\ 6), m \equiv 1 \ (mod\ 3) \text{ and} \\
 & \text{for } i \equiv 1 \ (mod\ 6), m \equiv 2 \ (mod\ 3).
\end{cases}
$$

From the lower and upper bounds on the corona of open ladder and null graph $O(L_n) \odot N_m$, the maximum value of the vertex label and edge labels are obtained, namely $\left\lceil \frac{2nm+3n-4}{3} \right\rceil$ for $2nm + 3n - 4 \not\equiv 2,3 \ (mod\ 6)$ and $\left\lceil \frac{2nm+3n-4}{3} \right\rceil + 1$ for $2nm + 3n - 4 \equiv 2,3 \ (mod\ 6)$. The following edge weights are obtained,

$wt_\theta(u_i u_{i+1}) = 2im + 3i - 2,$ for $1 \le i \le n-1$.
$wt_\theta(v_i v_{i+1}) = 2im + 3i - 1,$ for $1 \le i \le n-1$.
$wt_\theta(u_i v_i) = (2i-1)m + 3i - 3,$ for $2 \le i \le n-1$.

$$
wt_\theta(u_i u_{i,j}) = \begin{cases} j, & \text{for } i = 1, 1 \le j \le m, \\ (2i-2)m + 3i - 4 + j, & \text{for } 2 \le i \le n, 1 \le j \le m. \end{cases}
$$

$$
wt_\theta(v_i v_{i,j}) = \begin{cases} (2i-1)m + 3i - 3 + j, & \text{for } 1 \le i \le n-1, 1 \le j \le m, \\ (2i-1)m + 3i - 4 + j, & \text{for } i = n, 3 \le i \le n, 1 \le j \le m. \end{cases}
$$

It can be seen that all the edge weights of the corona of open ladder and null graph $O(L_n) \odot N_m$ are different, the lower bound and upper bound are equal to $res(O(L_n) \odot N_m)$. Therefore, θ satisfies the edge irregular reflexive k-labeling and has reflexive edge strength. Thus the theorem is proved.

Figure 3 shows the edge irregular reflexive 4-labeling on the corona of open ladder and null graph $O(L_3) \odot N_1$. The label of each edge and vertex is shown with blue numbers. The weight of each edge is shown with a red number.

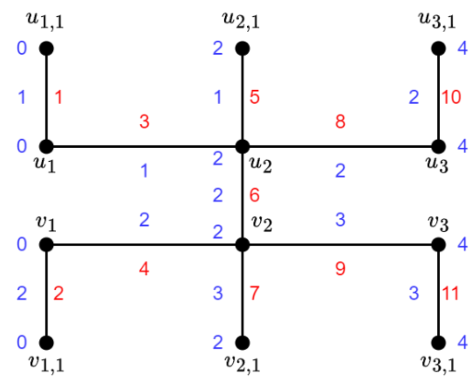


Figure 3. Edge irregular reflexive 4-labeling on the corona of open ladder and null graph $O(L_3) \odot N_1$

4. CONCLUSIONS

In this research, it has been discussed about edge irregular reflexive k-labeling of Dumbbell Graph $D_{m,n,q}$ with $m,n \geq 3, m=n$ and $q=3$ and edge irregular reflexive k-labeling of corona of open ladder and null graph $O(L_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$. Based on the results and discussion, the following conclusions are obtained

1. The reflexive edge strength of Dumbbell Graph $D_{m,n,q}$ with $m,n \geq 3, m=n$ and $q=3$ is

$$res(D_{m,m,3}) = \begin{cases} \left\lceil \frac{2m+2}{3} \right\rceil, & \text{for } 2m+2 \equiv 0,4 \ (mod\ 6), \\ \left\lceil \frac{2m+2}{3} \right\rceil + 1, & \text{for } 2m+2 \equiv 2 \ (mod\ 6). \end{cases}$$

2. The reflexive edge strength of corona of open ladder and null graph $O(L_n) \odot N_m$ with $n \geq 3$ and $m \geq 1$ is

$$res(O(L_n) \odot N_m) = \begin{cases} \left\lceil \frac{2nm+3n-4}{3} \right\rceil, & \text{for } 2nm+3n-4 \not\equiv 2,3 \ (mod\ 6), \\ \left\lceil \frac{2nm+3n-4}{3} \right\rceil + 1, & \text{for } 2nm+3n-4 \equiv 2,3 \ (mod\ 6). \end{cases}$$

For readers who feel interested in this research topic, they can conduct research to determine the edge irregular reflexive k-labeling of Dumbbell Graph $D_{m,n,q}$ with $m,n \geq 3$ and $q=2$, Dumbbell Graph $D_{m,n,q}$ with $m,n \geq 3$ and $q>3$, and other graph classes that have not been studied, such as the corona of open ladder and cycle graph $O(L_n) \odot C_m$ with $n,m \geq 3$.

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