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THE SUFFICIENT AND NECESSARY CONDITIONS FOR A MODULE TO BE A WEAKLY UNIQUE FACTORIZATION MODULE

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ABSTRACT

T A torsion-free module M over an integral domain R is called Unique Factorization Module (UFM) if satisfied some conditions: (1) Every non-zero element $x \in M$ has an irreducible factorization, that is $x = a_1 a_2 \dots a_n m$, with $a_1, a_2, \dots, a_n$ are irreducible in R and m is irreducible in M, and (2) if $x = a_1 a_2 \dots a_n m = b_1 b_2 \dots b_k m'$ are two irreducible factorizations of x, then $n = k$, $m \sim m'$ in M, and we can rearrange the order of the b_i's so that $a_i \sim b_i$ in R for every $i \in \{1, 2, \dots, n\}$. The definition of UFM is a generalization of the concept of factorization on the ring which is applied to the module. In this study, we will discuss another definition that is a generalization of UFM, namely by the Weakly Unique Factorization Module (w-UFM). First, some concepts that play an important role in defining w-UFM are given. After that, the definition and characterization of w-UFM is also given. The results of this study will provide the sufficient and necessary conditions of the w-UFM.

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1. INTRODUCTION

Let D be an integral domain. An integral domain D is called Unique Factorization Domain (UFD) if it satisfies: (1) every non-zero element in the set D has factorization, and (2) if $a = p_1 p_2 \dots p_n = q_1 q_2 \dots q_m$ are two factorizations of a on the set D, so $n = m$ and there is a permutation σ of $\{1,2,\dots,n\}$ such that p_i and $q_{\sigma(i)}$ are associated with each other for every $i = 1,2,\dots,n$ [1]. It is well-known that rings ang modules have many related properties. This gives motivation in applying the concept of factorization ring into the concept of factorization module. In [2] introduced the definition of Unique Factorization Modules (UFM) known as the factorial modules. In [3], it has been the concept of factorization in the module that underlies the definition of UFM along with their properties and characterization. Given M be a non-zero torsion-free module over an integral domain R. A module M is called UFM if: (1) every non-zero element $x \in M$ has a factorization, i.e. $x = a_1 a_2 \dots a_n m$ where $a_1, a_2, \dots, a_n$ is an irreducible element in R and m is an irreducible element in M, and if $x = a_1 a_2 \dots a_n m = b_1 b_2 \dots b_k m'$ are two factorizations of x, then $n = k$, $m \sim m'$ in M, and $a_i \sim b_i$ for each $i \in \{1,2,\dots,k\}$.

Studies regarding factorization in modules have been carried out several times. In [4] and [5], they generalized the theory of factorization in integral domains to commutative rings with zero divisors into the modules. In [6] they studied about unique factorization in modules and symmetric algebras. Further [7] define and study the properties of integral modules, they got a theorem on UFD analogue for UFM. Then, [8] shows that if R be a commutative ring with identity and M (not necessarily torsion-free) be an R-module, then M as a cyclic module is necessary but not suffcient condition for M to be an UFM. Most recently, [9] defines about a submodule approach for UFM.

Using the concept of factorization in rings, [10] defined the concept of weakly factorization in the module. The research also provides definitions of weakly prime elements and weakly prime submodule of a module. Based on that, it was defined more general concept of unique factorization module, they called that the Weakly Unique Factorization Module (w-UFM) which is a torsion-free module on a commutative ring with a unit element where every non-zero elements in the module have a unique weakly factorization. In this study, we will discuss the definition and properties of w-UFM, as well as its relationship with UFM. Then we will show the sufficient and necessary conditions for a module to be w-UFM.

2. RESEARCH METHODS

The method used in this research is literature study. The steps in this research are as follows:

1. Study the basic concepts of factorization and unique factorization domain in rings ang their properties.
2. Study the concept of factorization in modules ang unique factorization modules along with their properties, characterization, and examples.
3. Study the concept of prime elements in prime modules and submodules and their properties.
4. Study the definitions of weakly prime element and weakly prime submodules along with their properties and examples.
5. Study the definition of a weakly unique factorization modules and its properties.

3. RESULTS AND DISCUSSION

This chapter will present the research results that include the definition and properties of w-UFM, as well as its relationship with UFM.

3.1 Weakly Unique Factorization Modules

Before discussing the definition of w-UFM, several related concepts are explained first along with their properties.

Definition 1. [3] Let M be a module over integral domain R, and $U(R)$ is a set of all unit of R. For any $m, m' \in M$, then:

(1) An element m is said to divide m' in M (write $m|m'$) if there exists $r \in R$ such that rm. If $m|m'$, then m is called a factor of m' in M.
(2) An element $d \in R$ is said to divide m in M (write $d|m$) if there exists $m_0 \in M$ such that $m = dm_0$. If $d|m$, then d is called a factor of m in R.
(3) If $m|m'$ and $m'|m$, then those two elements are called **associates** in M (write $m \sim m'$). If $m|m'$ but m' is not associates, then m is called a proper factor of m' in M.

Based on [11], a non-zero element $m \in M$ is called **irreducible** in M if m does not have a proper factor in M. Clearly, a non-zero element $m \in M$ is irreducible if $m = rm'$ implies that $m \sim m'$ for every $m' \in M$ and a non-zero $r \in R$. The following theorem gives a necessary and sufficient condition of an element in a module to be irreducible.

Theorem 1. [8] *Let M be a module over R. A non-zero element $m \in M$ is irreducible if and only if $m = rn$ implies $r \in U(R)$ for some $r \in R$ and $n \in M$.*

Proof. ($\Longrightarrow$) Suppose that $m = rn$. Since $m \neq 0$ then $r \neq 0$, so $m \sim n$. Then we have $n = sm$ for $s \in R$. Futhermore, $(1 - rs)m = 0$ then $1 - rs = 0$ implies that $r \in U(R)$.

($\Longleftarrow$) It is easy to understand that $m \sim n$ because $r \in U(R)$. So, m is an irreducible element. ■

Definition 2. [3] Let M be a module over R and m is a non-zero element in M.

(1) An element m is called **primitive** if $m|am'$ implies $m|m'$ for every $m' \in M$ and non-zero $a \in R$.
(2) An irreducible element $p \in R$ is called prime to M if $p|am$ implies $p|a$ in R or $p|m$ in M for every $a \in R$ and $m \in M$.

The following theorem in [10] gives us a connection between primitive element and irreducible element in modules.

Theorem 2. [10] *Let M be a module over R. Every primitive element of M is irreducible.*

Proof. Let $m \in M$ primitive element with $m = rm'$ for some $r \in R$ and $m' \in M$. Then $m|rm'$ and since m is primitive, we have $m|m'$. Since $m = rm'$ and $m|m'$ then $m \sim m'$. So m is irreducible. ■

When talking about factorization in ring, the definition of greatest common divisor and least common multiple will certainly appear as on the following definition.

Definition 3. [10] Let M be a module over R, $r \in R$ and $m \in M$.

(1) An element $d \in R$ is called greatest common divisor of r and m (write $d = [r, m]$) if:
 (i) element $d|r$ in R and $d|m$ in M, and
 (ii) for each $c \in R$, if $c|r$ in R and $c|m$ in M then $c|d$ in R.
(2) An element $m' \in M$ is called the least common multiple of r and m (write $m' = \{r, m\}$) if:
 (i) element $r|m'$ and $m|m'$ in M, and
 (ii) for each $w \in M$ if $r|w$ and $m|w$ in M then $m'|w$.

Theorem 3. *Let M be a module over GCD-Domain R where $[a, m]$ exists for every $a \in R$ and $m \in M$. For every $b \in R$, the following statement satisfies:*

(1) $\big[[a, b], m\big] \sim [a, [b, m]]$,
(2) $[ba, bm] \sim b[a, m]$ *and*
(3) *If* $a|bm$ *and* $[a, m] = 1$, *then* $a|b$.

Proof.

(1) We will prove $\big[[a, b], m\big]|[a, [b, m]]$ and $\big[a, [b, m]\big]|[[a, b], m]$. It is clear that $\big[[a, b], m\big]|[a, b]$ and $\big[[a, b], m\big]|m$, which means $\big[[a, b], m\big]|a$ and $\big[[a, b], m\big]|b$. Based on that, we get $\big[[a, b], m\big]|[b, m]$. Then, we get $\big[[a, b], m\big]|[a, [b, m]]$. It is clear that $\big[a, [b, m]\big]|a$ and $\big[a, [b, m]\big]|[b, m]$. Based on that, we get $\big[a, [b, m]\big]|b$ and $\big[a, [b, m]\big]|m$. Then, we get $\big[a, [b, m]\big]|[a, b]$. Futhermore, we get $\big[a, [b, m]\big]|[[a, b], m]$.

(2) We will prove $[ba, bm] \sim b[a, m]$ by showing $[ba, bm] = b[a, m]$. Suppose $d = [a, m]$, we will show $bd|ba$, $bd|bm$ and if $a|ba$ then $e|bm$. It is clear that $d = ax + my$ for some x, y in R so $bd = bax + bmy$. It is clear that $e|bax$ and $e|bmy$, which means $e|bd$. Evidently, $[ba, bm] = b[a, m]$.
(3) We will prove $a|b$. Because $a|bm$, it means $bm = ak$ for some k in R. It is known that $[a, m] = 1$, means $1 = ax + my$ for some $x \in R$ and $y \in R$. We have $b = b.1 = b(ax + my) = bax + bmy = bax + aky = a(bx + ky)$. So, we get $a|b$. ■

Example 1. Given $\mathbb{Z}^3$ be a module over $\mathbb{Z}$. Pick $a = 8 \in \mathbb{Z}$ and $m = (4\ \ 12\ \ 8)^t \in \mathbb{Z}^3$. It is clear that $4 \in \mathbb{Z}$ is $[a, m]$ because $4|a$ and $4|m$, also because for every $c \in \mathbb{Z}$ if $c|a$ in $\mathbb{Z}$ and $c|m$ in $\mathbb{Z}^3$ then $c|4$ in $\mathbb{Z}$.

Definition 4. [10] Let M be a module over R. A non-zero element $m \in M$ is called **weakly prime** or **w-prime** if $m|abm'$ implies $m|am'$ or $m|bm'$ for some $a, b \in R$ and $m' \in M$.

Note that if m is w-prime element in M then rm is also w-prime element for every $r \in U(R)$ and every primitive element in M is a w-prime element [10]. Now we give an example of a w-prime element in modules.

Example 2. Given $M = \mathbb{Z}[x]$ be a module over $R = \mathbb{Z}$ and $2x \in M$. Let $a, b \in R$ and $f(x) \in M$ with $2x|abf(x)$, then there exists $c \in R$ so $abf(x) = c2x$. If $2x \nmid af(x)$, then $[2x, af(x)] = 1$ so there exists $g(x), h(x) \in M$ and implies

$$\begin{aligned} 1 &= 2xg(x) + af(x)h(x) \\ bf(x) &= 2xg(x)bf(x) + af(x)h(x)bf(x) \\ bf(x) &= b2xg(x)f(x) + abf(x)^2h(x) \end{aligned}$$

It is easy to understand that $2x|b2xg(x)f(x)$ and $2x|abf(x)^2h(x)$, so we have $2x|bf(x)$. So, it is proved that $2x$ is a w-prime element in M.

Theorem 4. *Every primitive element of R-module M is w-prime.*

Proof. Suppose that $m|abm'$ for $a, b \in R$ and $m' \in M$. Since m is primitive, then $m|m'$. Hence $m|am'$ and $m|bm'$, so m is w-prime element. ■

Let M be a module over R and N is a proper submodule of M. A submodule N is called prime submodule if $aRk \subseteq N$ implies $k \in N$ or $a \in Ann_R(M/N)$ [12]. If $a \in Ann_R(M/N)$, then for each $k \in M$ we have $a(k + N) = 0 + N$, which means $ak + N = 0 + N$, so we have $ak \in N$. This situation motivates the definition which is a weakness of the prime submodule, called the weakly prime submodule, which will be explained below.

Definition 5. [10] Let M be a module over R and N be a submodule of M. The submodule N is called **weakly prime submodule** if $abk \in N$ implies $ak \in N$ or $bk \in N$ for each $k \in M$ and $a, b \in R$.

According to [10], an element $m \in M$ is w-prime if and only if Rm is a weakly prime submodule in M as R-module. Weakly prime elements will play an important role in the concept of weakly factorization in the module, as described in the definition of w-UFM below. ■

Definition 6. [10] A module M over commutative ring with identity R is called **weakly unique factorization module** (w-UFM) if:

(1) Every non-zero $x \in M$ has a weakly factorization, so $x = a_1a_2 \ldots a_km$, with $a_1, a_2, \ldots a_k$ are irreducible elements in R and m is a w-prime element in M, and
(2) if $x = a_1a_2 \ldots a_km = b_1b_2 \ldots b_tm'$ are two weakly factorizations of x, then $k = t$, $m \sim m'$, and $a_i \sim b_i$ for each $i \in \{1,2, \ldots, k\}$.

It was previously known that every primitive element in M is a w-prime element. The next theorem will prove the opposite in the w-UFM case, as well as a necessary and sufficient condition for a module to be w-UFM.

Theorem 5. *Let M be a module over UFD R. Then M is a w-UFM if and only if every w-prime element in M is primitive.*

Proof. ($\Longrightarrow$) Let M be a w-UFM and $m \in M$ is a w-prime element. It will be shown that m is a primitive element. Suppose $m|abm'$ for a $m' \in M$ and $a, b \in R$, meaning that there is $r \in R$ such that $rm = abm'$. It

is known that M is a w-UFM and R is an UFD so that there are irreducible elements in R, namely $r_1, \dots, r_k, a_1, \dots, a_t, \dots, b_1, \dots, b_l, c_1, \dots, c_n$ and w-prime elements $m^* \in M$ such that $r = r_1 \dots r_k, a = a_1 \dots a_t$, $b = b_1 \dots b_l$ and $m' = c_1 \dots c_n m^*$. Furthermore, we get $r = r_1 \dots r_k m = a_1 \dots a_t b_1 \dots b_l c_1 \dots c_n m^*$. The set M is w-UFM, therefore we get $k = t + l + n$ and $r_i \sim a_i$, $r_j \sim b_j, r_s \sim c_s$ and $m \sim m^*$, therefore there is $r^* \in U(R)$ such that $m^* = r^* m$. We get $m' = c_1 \dots c_n m^* = c_1 \dots c_n r^* m$, so $m|m'$. Which is proved that m is primitive element.
($\Longleftarrow$) Note that every w-prime element of M is a primitive element. We will prove M is a w-UFM. Take any non-zero x in M where $x = a_1 \dots a_t m = a_1 \dots b_l m'$ is a factorization of x. We get $m|b_1 \dots b_l m'$, $m'|a_1 \dots a_t m$. It is clear that m and m' are w-prime elements so $m|m'$ and $m'|m$. Therefore, $m \sim m'$ so that there is $u \in U(R)$ such that $m = um'$. As a result, $a_1 \dots a_t um' = b_1 \dots b_l m'$ so $ua_1 \dots a_t = b_1 \dots b_l$. Based on what is known that R is UFD, it is proven that M is a w-UFM. ∎

3.2 Connections and Properties between w-UFM and UFM

It is known that UFM and w-UFM are modules with different definitions which are motivated by applying a factorization property in rings into modules. On this part we will show the relationship between UFM and w-UFM and their properties for module over UFD.

Theorem 6. *Let M be a module over UFD R. The module M is a UFM if and only M is a w-UFM.*

Proof.
($\Longrightarrow$) Note that M is a UFM, means that every irreducible element in M is a primitive element. It is also known that every w-prime element is an irreducible element from [3] and [10], so that every w-prime element in M is a primitive element. Evidently M is w-UFM (Based on **Theorem 5**.).

($\Longleftarrow$) Let M be a w-UFM. It is known that every w-prime element is irreducible, so by using a proof analogous to Theorem 2.1 in [3], it can be shown that every irreducible element in M is a primitive element. Therefore, it is proven that M is UFM. ∎

Based on **Theorem 6**., it is known that at w-UFM M over UFD R case, every irreducible element in M is also a w-prime element. From these properties it is also found that in a torsion-free module over an UFD, a w-UFM can also be viewed as an UFM. Using these properties, other properties of w-UFM can be determined based on [3].

Theorem 7. *Let M be a module over UFD R, then the following statements are equivalent:*

(1) M over R is a w-UFM (every w-prime element in M is a primitive element).
(2) For each $a \in R$ and $m \in M$, $[a, m]$ is in R.
(3) For each $a \in R$ and $m \in M$, $\{a, m\}$ is in M.
(4) Every irreducible element in R is prime to M.
(5) If a and b are elements of R with $aM \subseteq bM$, then $b|a$ and for each $a, b \in R$ there is $c \in R$ such that $aM \cap bM = cM$.

Proof.

(1) $\Longrightarrow$ (2) If $m = 0$, then $[a, m] \sim a$ for every $a \in R$. If $m = bn \neq 0$, with $b \in R$ and n are irreducible element in M, it can be claimed that $[a, m] \sim [a, b] = d \in R$. It is clear that d is the common divisor of a and m. Let d' be the other common divisor of a and m, and $m = bn = d'm'$ for some $m' \in M$. Note that n is primitive and $d = [a, b]$, so $d'|b$ results in $d'|d$. So, $d \sim [a, m]$ and point (2) are proved.

(2) $\Longrightarrow$ (3) If $a = 0$, then $\{a, m\} \sim 0 \in M$ for every $m \in M$. For any nonzero a in R and m in M, suppose $d \sim [a, m]$. We get $a = da'$ and $m = dm'$ for some $a' \in R$ and $m' \in M$ such that $[a', m'] \sim 1$. Then we get $\{a, m\} = m'' = a'm$.
(3) $\Longrightarrow$ (4) Take any irreducible element r in R with $r|am$ for some $a \in R$ and $m \in M$. If $r \nmid m$, then $am \in rM \cap Rm = Rrm$. Therefore, $r|a$ thus proves that r is prime to M.

(4) $\Longrightarrow$ (5) Take any $a, b \in R$ with $aM \subseteq bM$. First, we will show $b|a$. If $b = 0$, then $a = 0$ so $b|a$. If $b \neq 0$, then $b|an$ for an irreducible element n in M, with $p|an$ for each prime factor p of b. Based on known at point (4), $p|a$ for each prime factor p of b, so we have $b|a$. Second, we will show there is $c \in R$ such that $aM \cap bM = cM$. If $a = 0$ and $b = 0$, it is clear that $c = 0$. If a or b is nonzero elements, for example $c \sim \{a, b\}$ and $d \sim [a, b]$, it means $aM \cap bM \supseteq cM$ and $c = a'b = ab'$, where $a' = ad^{-1}$ and $b' = bd^{-1}$. If w is any nonzero element in M with $w = am = bm' \in aM \cap bM$, then $a'm = b'm'$. Based on $a'|b'm'$ and $[a, m] \sim 1$,

it means $a'|m'$ with the same arguments as in first proof. Therefore, $c = a'b$ divides by $w = bm'$. That means $aM \cap bM \subseteq cM$. Evidently $aM \cap bM = cM$.

(5) $\Longrightarrow$ (1) Take any irreducible element m in M with $am' = bm$ for some a, b in R and m' in M. Based on the known, we get $am' = bm \in aM \cap bM = cM$ for some $c \in R$. Then we get $b|c$ and $a|c$. Since m is irreducible, we get $b \sim c$ with $a|b$ and $m|m'$. Proven m primitive. Because any irreducible element in M is primitive, we get M is an UFM. It is known that the module M over UFD R is also an UFM. Then using the proof above, it is proven that the equivalence in points (1) to (5) holds. ∎

Example 3. Let $M = \mathbb{Z}[x]$ be a module over $R = \mathbb{Z}$. Take any irreducible element $p \in R$ with $p|af(x)$, for any $a \in R$ and $f(x) \in M$.
We will prove that p is prime to M. If $p \nmid a$ in R, it means that $[p, a] = 1$ which means there is $c, d \in R$ so

$$1 = cp + da$$
$$f(x) = cpf(x) + adf(x).$$

Since $p|cpf(x)$ and $p|adf(x)$, we get $p|f(x)$ so it proves that p is a prime to M. Since any irreducible element in R is prime to M, it is proved that $M = \mathbb{Z}[x]$ is a w-UFM based on Theorem 7.

We have known that every field is a UFD, every field is a vector space over itself, and every UFD is a module over the UFD, the following theorem show that the property can be generalize into vector space.

Theorem 8. *Every vector space is a w-UFM.*

Proof. Based on [3], we know that every vector space is a UFM. Since vector space is a module over field and field is an UFD, according to Theorem 6 it can be concluded that every vector space is also a w-UFM.

Now we will show how the w-UFMs relate to it submodule and its products as well as it direct sums. Firstly, it is introduced the notion of pure submodule. Research on pure submodule was first carried out by [13] and was continued by [14]. Then in [15] they make a generalization of pure submodules. However, this time we will only give a definition of pure submodule.

Definition 7. [13] Let M be a module over R and N be a submodule of M. A submodule N is called **pure submodule** if $aM \cap N = aN$ for each $a \in R$.

Example 4. Let M be a module over R. Any direct summand of M is a pure submodule.

Example 5. Let $M = \mathbb{Z}^2$ be a module over $\mathbb{Z}$ and $N = \{(x \ \ 0)^t \mid x \in \mathbb{Z}\} \subseteq \mathbb{Z}^2$. We will prove that $aM \cap N = aN$ for some $a \in \mathbb{Z}$.

It is clear that $aN \subseteq aM \cap N$. Take any $k \in aM \cap N$, so we have $k \in aM$ and $k \in N$. Furthermore, $k = a(p \ \ q)^t = (ap \ \ aq)^t$ and $k = (u \ \ 0)^t$ for some $p, q, u \in \mathbb{Z}$. Then $k = (ap \ \ 0)^t = a(p \ \ 0)^t$ and we have $k \in aN$. So, $aM \cap N = aN$ and N is a pure submodule.

Lu in [3] introduce the theorem showing the sufficient condition a submodule of UFM over UFD is also a UFM. Because in a module over UFD we already shown an UFM can be viewed as w-UFM, we have the following theorem.

Theorem 9. *If M is a module over UFD R which is a w-UFM and N is a pure submodule of M, then N is also a w-UFM.*

Proof. It is clear that N satisfies the first property of the UFM definition. For every $a \in R$ and $m \in M$, we have

$$aN \cap Rm = (aM \cap N) \cap Rm = (aM \cap Rm) \cap N = Rx \cap N = Rx$$

for an $x \in M$ based on the point (3) in Theorem 7 and the nature of N as a pure submodule. Therefore, it is proven that N is a UFM. Because M is a module over UFD R, by using Theorem 6 it is proven that N is a w-UFM. ∎

Theorem 10. *Let $\{ M_i \mid i \in I \}$ be the set of modules over UFD R. The following statements are equivalent:*

(1) $\prod_{i \in I} M_i$ is a w-UFM over R,
(2) $\bigoplus_{i \in I} M_i$ is a w-UFM over R,
(3) every M_i is a w-UFM over R.

Proof. Note that for every $i \in I$, if $M_i = \{(\dots, 0, 0, m_i, 0, 0, \dots) | m_i \in M_i\}$ then $M_i \cong M_i$. It was clear that $\bigoplus_{i\in I} M_i$ is a submodule of $\prod_{i\in I} M_i$. Therefore, it is easy to understand that $M_i \cong M_i \subseteq \bigoplus_{i\in I} M_i \subseteq \prod_{i\in I} M_i$.

It is also clear that M_i is a pure submodule of $\bigoplus_{i\in I} M_i$ and $\bigoplus_{i\in I} M_i$ is a pure submodule of $\prod_{i\in I} M_i$ because for any $a \in R$, we have a $a \bigoplus_{i\in I} M_i \cap M_i = aM_i$ and $a \prod_{i\in I} M_i \cap \bigoplus_{i\in I} M_i = a \bigoplus_{i\in I} M_i$. Based on **Theorem 9**, it is proved that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(3) $\Longrightarrow$ (1) Note that each M_i is a UFM over R. Suppose $\prod_{i\in I} M_i = M$. If $m = (m_i)_{i\in I} \in M$, with every $m_i = a_i m_i'$ for some $a_i \in R$ and the irreducible element $m_i' \in M_i$, then m is irreducible if and only if the set $\{\, a_i \mid i \in I \,\}$ doesn't have g.c.d. in R. Therefore, it is clear that M satisfies the first point in the UFM definition.

We will show that M is UFM by proving that every irreducible element in R is prime to M. Take any irreducible element $p \in R$ such that $p|am$ in M for some $a \in R$ and $m = (m_i)_{i\in I} \in M$, we get $p|am_i$ in M_i for every i. If $p \nmid a$ in R, then $p|m_i$ in M_i for every i, because every M_i is a UFM. Therefore, we get $p|m$ in M so that p is prime to M. Because of that, it is proven $M = \prod_{i\in I} M_i$ is a UFM. It is known that in module over UFD, an UFM is also w-UFM. Therefore, based on the proof in [3] Corollary 2.5, it is easy to see those equivalences (1) to (3) also can be applied. ■

4. CONCLUSIONS

The weakly unique factorization module (w-UFM) is a torsion-free module over commutative ring with identity in which each non-zero element has weakly factorization that is unique. It is known that UFM and w-UFM are modules with different definitions which are motivated by applying a rings factorization property into modules. In this study we find out that if R is an UFD and M be a module over R, then every w-prime element in M is a primitive element is a sufficient and necessary condition for M to be a w-UFM. However, in a torsion-free module over UFD, a module that is a w-UFM can also be viewed as an UFM. This causes every irreducible element in a w-UFM over an UFD to be a w-prime element. Because of these facts, it can be derived the properties of w-UFM based on the known properties of UFM.

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