BAREKENG: Journal of Mathematics and Its Applications
March 2025 Volume 19 Issue 1 Page 0427–0440
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol19iss1pp0427-0440

A MATHEMATICAL APPROACH TO INVESTMENT WITH CHARGE ON BALANCE AND VOLUNTARY CONTRIBUTIONS UNDER WEIBULL MORTALITY FORCE FUNCTION

Edikan Edem Akpanibah[1*], Peter Benneth[2], Ase Matthias Esabai[3]

[1,2,3]Department of Mathematics, Faculty of Sciences, Federal University Otuoke P.M.B. 126, Yenagoa, Bayelsa State,562103, Nigeria.

*Corresponding author's e-mail: * **edikanakpanibah@gmail.com***

Article History:

Received: 26th June 2024
Revised: 14th November 2024
Accepted: 29th November 2024
Published: 13th January 2025

Keywords:

Additional Voluntary Contributions;
Charge on Balance;
Heston Volatilty Model Equation;
Investment Strategy;
Weibull Mortality Function.

ABSTRACT

One of the many challenges encountered by most pension fund administrators (PFAs) in the Defined Contribution (DC) pension plan is the determination of a sustainable and suitable investment plan for their members under mortality risk. To achieve this, there is need to develop an optimal portfolio which considers the volatility of the stock market price consisting of one risk-free asset and a risky asset which follows the Heston volatility model (HVM). Also, the portfolio considers additional voluntary contributions (AVC) by members, tax on the stock market price, charge on balance (CB), and the mortality risk of the pension scheme members (PSM) modeled by the Weibull mortality force function. Furthermore, an optimization problem is established from the extended Hamilton Jacobi Bellman (EHJB) equation using variational method. By applying the variable separation technique and mean variance utility, the optimal control strategy (OCS) and the efficient frontier are obtained. Finally, some numerical simulations are presented to study the behavior of the OCS with respect to some sensitive parameters. It was discovered that the composition of the OCS depends on the instantaneous volatility, tax on investment, AVC, risk aversion coefficient (RAC), CB and the correlation coefficient. Hence, the understanding of the behaviour these parameters are very crucial in the determination of OCS.

How to cite this article:

E. E. Akpanibah, P. Benneth and A. M. Esabai., "A MATHEMATICAL APPROACH TO INVESTMENT WITH CHARGE ON BALANCE AND VOLUNTARY CONTRIBUTIONS UNDER WEIBULL MORTALITY FORCE FUNCTION," *BAREKENG: J. Math. & App.*, vol. 19, iss. 1, pp. 0427-0440, March, 2025.

Journal homepage: *https://ojs3.unpatti.ac.id/index.php/barekeng/*
Journal e-mail: *barekeng.math@yahoo.com; barekeng.journal@mail.unpatti.ac.id*

Research Article · Open Access

1. INTRODUCTION

The role of pension scheme as it concerns employee's welfare during working period (accumulation phase) and after working period (distribution phase) cannot be over emphasized as it plays key role with respect to social security system and has become so popular among working class members [1]-[2]. Very importantly, there two basic types of pension scheme in which employees can be enroll in; they include the defined benefit (DB) pension scheme and the DC pension scheme. In the former, the employer bears the burden of contribution and set aside certain percentage of its yearly budget toward taking care of his retired employees. The benefit of members of this scheme depends on years spent during active service, rank before retirement and salary scale before retirement. As good as this scheme is to PSM, there are challenges of nonpayment and late payment of benefit to members due to poor management of the funds and insufficient funds. However, these challenges necessitate the introduction of the later scheme. In the DC pension scheme, there is joint contribution by both employer and employee where employees contribute 8% and employers contribute 10% [3]. These funds are paid into individual member's unique account known as the retirement savings account (RSA) under the custody of pension fund custodians (PFC) and managed by the pension fund administrators (PFAs) under the supervision of Nigerian pension commission (PENCOM). Unlike the former scheme, the retirement benefits of each member depend solely on the accumulations of the joint contributions and ultimately of investment returns during the accumulation period. However, this investment returns depend heavily on the investment strategy and efficiency of the pension fund managers. Hence, this has led to the study of OCS which simply described the best distributions of members' wealth into different assets with the goal of obtaining optimal returns. Several authors such as [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], and [17] have studied the OCS under different assumptions.

It is worthy of note that it is impossible to determine the OCS of any investor without the understanding of the satisfaction an investor intends to achieve at the expiration of such investment and this has led to the study of utility maximization and these utility functions are often refers to as objective functions [1][15] There are two classes of utility goals; they include utility functions which maximize the accumulation at the expiration of an investment or retirement and the one that balance the return of investment and the risk involvement of such investment, i.e maximizing the accumulated fund and minimizing risk. The first class includes exponential utility function which exhibits constant absolute risk aversion (CARA) [6][18][19][20]. Next is logarithm or power utility functions which exhibit constant relative risk aversion (CRRA) [8][9][20] and the quadratic loss functions [21][22]. The second class includes the mean variance utility function and value at risk (VAR) utility. The first utility function, which originates from [23], studied a single-phase portfolio optimization problem under mean variance utility. [24], extended the work of [23] to a continuous time model and determined the efficient frontier under mean variance utility. Several authors such as [1], [25], [26], and [27] all used mean variance utility under different assumptions. The second utility equals the problem of maximizing the terminal accumulation with minimum guarantee [28].

In DC pension scheme, the accumulation period requires that members of the scheme contribute and wait till after retirement for distribution. However, this has not been the case in recent years due to mortality risk involved in the scheme. This implies that PFAs are mandated to return the accumulated funds of death members to their next of kin during the accumulation phase; this has somehow affected the dynamics of the investment strategies already in existence. Hence this has led to the study of OCS with return of premium clauses under different assumptions.

Several authors have since engaged in this study under different assumptions. The OCS with return of premium clause was first studied in [1] under mean variance utility function. In their paper, they considered investment in bond and equity where the price process of the equity followed the geometric Brownian motion (GBM). In [25], the OCS with return of premium clause was studied under the Heston volatility model; they considered investment in a risk-free asset and a risky asset where the price process of the risky asset followed the Heston volatility model and went on to determine the efficient frontier of the members. The OCS with return of premium clause was also studied in [26]. They considered investment in one risk free asset and two risky assets modeled by GBM and constant elasticity of variance (CEV) model and solved for the OCS under mean variance utility. [29] obtained the OCS in a DC scheme where the price process of the risky asset followed the jump diffusion process. In [30], the OCS for inflation index bond and the stock market price was studied for a portfolio with a risk free asset, stock and inflation index bond where the stock market price followed the Heston's volatility model. [31] further extended the work of [1] by studying the OCS with return of premium for a portfolio with assets such as fixed deposit, stock and loan. Furthermore, [32] studied the explicit solution of an OCS with voluntary contribution and return clause under logarithm utility. However, in the afore mentioned literature, the returned funds were without interest from the invested funds but in [33] they studied the OCS with return of premium including predetermined interest; they used the Abraham De Moivre model as their mortality force function. Very recently, some authors have used the Weibull force function to model their mortality rate and went on to study the OCS for a pension plan under different assumptions. They used Weibull force function because the mortality rate is proportional to the power of the PSM's age unlike in Abraham De Moivre's force function where the mortality rate is the ratio of difference between the terminal age of the PSM's and PSM's present age to the PSM's age. [34] and [35] studied OCS with return premium with proportional administrative charges; in their work, they used the Weibull force function to model the mortality rate of a DC pension scheme and obtained the OCS under exponential and mean variance utilities respectively.

All through the literature and to the best of our knowledge, the OCS for a DC scheme have not yet been studied with return of premium clause for portfolio with a combination of charge on balance and additional voluntary contribution under Heston volatility model when the mortality risk is modeled by Weibull force function. We use the mean variance utility as our objective force function and determine the OCS and the efficient frontier of members under this condition. Furthermore, we will study the impact of some sensitive parameters on the OCS. The main contribution in this paper is that we modified the work of [25] by modeling our mortality risk using Weibull force function instead of Abraham De Moivre model, introduce charge on balance similar to [34] and additional voluntary contributions similar to [36], to determine the OCS.

2. RESEARCH METHODS

2.1 The Weibull Mortality Force Function

In this section, we discuss a pension fund system with return clause of premium based on mortality risk where the mortality risk is modeled using Weibull mortality force function.

Suppose ℓ is the pension scheme member's (PSM's) monthly contributions, w_0 the initial age of accumulation phase, T period of investment such that $w_0 + T$ is the terminal age of the PSM.

Also, let $\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}$ be the mortality rate of PPMs from time t to $t+\frac{1}{\mathfrak{n}}$, $t\ell$ is the accumulated contributions at time t, $at\ell\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}$ is the returned contributions to the death members' families within the accumulation period. Let a represents the presence of return clause; if $a = 0$, the PFA returns nothing to the death member's family, if $a\epsilon(0,1)$, the PFA returns a fraction of PPM's contributions to the death member's family and if $a = 1$, the PFA returns all the contribution to the death member's family. The return clause could be without interest [1][20][25][26]

From the work of [1], [25], and [26], we have

$$\begin{cases} \dfrac{\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}}{1-\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}} = \mathfrak{U}(w_0+t)dt, \\ \mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t} = \mathfrak{U}(w_0+t)dt, \end{cases} \tag{1}$$

See [1], and [20] for more details.

Where $\mathfrak{U}(t)$ is the force function and w is the maximal age of the life table. From [34], the Weibull force function formula is given as

$$\mathfrak{U}(t) = \mathfrak{u}t^{z}, \quad 0 \le t < T, \mathfrak{u} > 0, z > 0 \tag{2}$$

This implies that

$$\mathfrak{U}(w_0+t) = \mathfrak{u}(w_0+t)^{z} \tag{3}$$

2.2 PSM's Portfolio with CB and AVC

In this section, we formulate a PSM's wealth by considering a financial market comprising of an investment in a fixed deposit and a stock market. Also, we consider a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$ over a real space Ω, probability measure $\mathcal{P}$ and filtration $\mathcal{F}_t$which represents the available information from the market generated by the Brownian motions $\mathcal{W}_s$ and $\mathcal{W}_k$. These two Brownian motions correlate thus; $E[\mathcal{W}_s \mathcal{W}_k] = \rho$.

Let $\mathcal{S}_t^0(t)$ represent the value of the fixed deposit at time t and its model is given as

$$\begin{cases} d\mathcal{S}_t^0(t) = \mathcal{S}_t^0(t) r dt \\ \mathcal{S}_t^0(0) = 1 \end{cases}, \tag{4}$$

Also, let $\mathcal{S}_t^1(t)$ represents the value of the stock at time t which follows the Heston volatility model [25], [36] whose dynamics is given by the system of stochastic differential equation below

$$\begin{cases} d\mathcal{S}_t^1(t) = \mathcal{S}_t^1(t)\left((r + \mathfrak{H}\mathcal{C}_t)dt + \sqrt{\mathcal{C}_t} d\mathcal{M}_s\right) \\ \mathcal{S}_t^1(t)(0) = s_0 \end{cases}, \tag{5}$$

$$\begin{cases} d\mathcal{C}_t(t) = \mathfrak{T}(\mathfrak{b}\varphi - \mathcal{C}_t)dt + \sigma\sqrt{\mathcal{C}_t} d\mathcal{M}_c \\ \mathcal{C}_t(0) = c_0 \end{cases}, \tag{6}$$

where $r > 0$ is the interest rate of the risk-free asset, $\sqrt{\mathcal{C}_t}$ is the volatility of the stock market price, $\mathfrak{b}$ is the long-term price variance, $\mathfrak{T}$ is the rate of reversion to the long-term price variance, σ is the volatility and $\mathfrak{H}$ is the expected appreciation rate of the stock market price.

Furthermore, we consider a case where the PSM is allowed to invest in one risk-free asset and a risky asset. Let $(\mathfrak{Y}(t), \mathfrak{Y}_0(t))$ be the proportion of the PPM wealth to be invested in stock and fixed deposit where $\mathfrak{Y}_0(t) = 1 - \mathfrak{Y}(t)$. Let $\mathcal{B}$ be the charge on balance which is determined based on the value of the stock by the pension fund administrators [34], β the tax rate on the stock [14] and θ is the additional voluntary contribution made by the PPM during the accumulation period [15][33].

From [1] and [25], the differential form associated with the PPM wealth $\mathcal{Q}(t)$ corresponding to investment strategies $(\mathfrak{Y}(t), \mathfrak{Y}_0(t))$ and the accumulation phase period $[t, t + \frac{1}{\mathfrak{n}}]$, is given as:

$$\mathcal{Q}\left(t + \frac{1}{\mathfrak{n}}\right) = \begin{bmatrix} \mathcal{Q}(t)\left(\mathfrak{Y}_0(t)\frac{\mathcal{S}^0_{t+\frac{1}{\mathfrak{n}}}}{\mathcal{S}^0_t} + \mathfrak{Y}(t), \frac{\mathcal{S}^1_{t+\frac{1}{\mathfrak{n}}}}{\mathcal{S}^1_t}\right) + \theta d\mathcal{M}_s + \ell\frac{1}{\mathfrak{n}} \\ -\mathcal{B}\mathcal{Q}(t)\frac{1}{\mathfrak{n}} - \beta\frac{1}{\mathfrak{n}}\mathcal{Q}(t) - at\ell\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t} \end{bmatrix}\left(\frac{1}{1 - \mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}}\right) \tag{7}$$

$$\mathcal{Q}\left(t + \frac{1}{\mathfrak{n}}\right) - \mathcal{Q}(t) = \begin{bmatrix} \mathcal{Q}(t)\begin{pmatrix} (1 - \mathfrak{Y}(t))\left(\frac{\mathcal{S}^0_{t+\frac{1}{\mathfrak{n}}} - \mathcal{S}^0_t}{\mathcal{S}^0_t}\right) \\ +\mathfrak{Y}(t)\left(\frac{\mathcal{S}^1_{t+\frac{1}{\mathfrak{n}}} - \mathcal{S}^1_t}{\mathcal{S}^1_t}\right) \end{pmatrix} \\ -(\mathcal{B} + \beta)\mathcal{Q}(t)\frac{1}{\mathfrak{n}} + \theta d\mathcal{M}_s + \ell\frac{1}{\mathfrak{n}} - at\ell\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t} \end{bmatrix}\left(1 + \frac{\mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}}{1 - \mathfrak{X}_{\frac{1}{\mathfrak{n}}, w_0+t}}\right) \tag{8}$$

From [1], we have

$$\left\{ (\mathcal{B} + \beta)\frac{1}{\mathfrak{n}} = (\mathcal{B} + \beta)dt, \quad \ell\frac{1}{\mathfrak{n}} \to \ell dt, \frac{\mathcal{S}^0_{t+\frac{1}{\mathfrak{n}}} - \mathcal{S}^0_t}{\mathcal{S}^0_t} \to \frac{d\mathcal{S}^0_t}{\mathcal{S}^0_t}, \quad \frac{\mathcal{S}^1_{t+\frac{1}{\mathfrak{n}}} - \mathcal{S}^1_t}{\mathcal{S}^1_t} \to \frac{d\mathcal{S}^1_t}{\mathcal{S}^1_t} \right. \tag{9}$$

Substituting Equation (1) and Equation (9) into Equation (8), we have

$$dQ(t)=\left[\begin{matrix}Q(t)\left(\mathfrak{Y}(t)\frac{dS_t^1}{S_t^1}+\left(1-\mathfrak{Y}(t)\right)\frac{dS_t^0}{S_t^0}-(\mathcal{B}+\beta)dt\right)\\ +\theta d\mathcal{M}_s+\ell dt-at\ell\mathfrak{U}(w_0+t)dt\end{matrix}\right](1+\mathfrak{U}(w_0+t)dt) \tag{10}$$

Substituting **Equation (3)**, **Equation (4)**, **Equation (5)** and **Equation (6)** into **Equation (10)**, we have

$$dQ(t)=\left[\begin{matrix}\left\{\begin{matrix}Q(t)\left(\begin{matrix}\mathfrak{Y}(t)(\mathfrak{H}-r)\mathcal{C}_t\\ +u(w_0+t)^z+r-(\mathcal{B}+\beta)\end{matrix}\right)\\ +\ell(1-atu(w_0+t)^z)\end{matrix}\right\}dt+\left(Q(t)\mathfrak{Y}(t)\sqrt{\mathcal{C}_t}+\theta\right)d\mathcal{M}_s\\ Q(0)=q_0\end{matrix}\right] \tag{11}$$

2.3 Mean Variance Utility and EHJB Equation

The mean variance utility is best suitable for time inconsistent problem which is similar to our type. In this section, we take into consideration a PFA who is interested to maximize the wealth of his members especially the surviving ones by protecting their wealth in the presence return clause and minimize the volatility of the accumulated wealth. Hence, there is need to develop an optimal control problem using the mean-variance utility as follows:

$$\mathcal{I}(t,q,c)=\sup_{\ell}\{E_{t,q,c}Q^{\mathfrak{Y}}(T)-Var_{t,q,c}Q^{\mathfrak{Y}}(T)\} \tag{12}$$

Next, we follow the approach in [1], [26] by using the variational inequality technique. The control problem in **Equation (12)** is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{I}(t,q,c)$

$$\begin{cases}\mathcal{J}(t,q,c,\mathfrak{Y})=E_{t,q,c}[Q^{\mathfrak{Y}}(T)]-\frac{\gamma}{2}Var_{t,q,c}[Q^{\mathfrak{Y}}(T)]\\ =(E_{t,q,c}[Q^{\mathfrak{Y}}(T)]-\frac{\gamma}{2}\left(E_{t,q,c}[Q^{\mathfrak{Y}}(T)^2]-(E_{t,q,c}[Q^{\mathfrak{Y}}(T)])^2\right))\\ \mathcal{I}(t,q,c)=\sup_{\mathfrak{Y}}\mathcal{J}(t,q,c,\mathfrak{Y})\end{cases} \tag{13}$$

Following [1], the OCS $\mathfrak{Y}^*$ satisfies:

$$\mathcal{I}(t,q,c)=\sup_{\mathfrak{Y}}\mathcal{J}(t,q,c,\mathfrak{Y}^*) \tag{14}$$

Where γ is the risk-averse coefficient of the PPM.

Let

$$i^{\mathfrak{Y}}(t,q,c)=E_{t,q,c}[Q^{\mathfrak{Y}}(T)]$$

$$j^{\mathfrak{Y}}(t,q,c)=E_{t,q,c}[Q^{\mathfrak{Y}}(T)^2]$$

Then

$$\mathcal{I}(t,q,c)=\sup_{\ell}x\left(t,q,c,\ i^{\mathfrak{Y}}(t,q,c),j^{\mathfrak{Y}}(t,q,c)\right)$$

where

$$x(t,q,c,i,j)=i-\frac{\gamma}{2}(j-i^2) \tag{15}$$

Our interest here is to maximize the PSM utility in **Equation (12)** subject to his wealth in **Equation (11)** and applying the Ito's lemma and maximum principle, we obtain the EHJB equation summarized by the verification theorem below.

Theorem 1. **(verification theorem)** *If there exist three real functions $\mathcal{X},\mathcal{Y},\mathcal{Z}:[0,T]\times\mathfrak{R}\to\mathfrak{R}$ satisfying the following EHJB equations:*

$$\begin{cases} \sup_{\mathfrak{Y}} \begin{Bmatrix} \mathcal{X}_t - x_t + (\mathcal{X}_q - x_q)\left[q\begin{pmatrix} \mathfrak{Y}c(\mathfrak{H}-r) \\ +\mathfrak{u}(w_0+t)^z + r - (\mathcal{B}+\beta) \end{pmatrix} \\ +\ell(1-at\mathfrak{u}(w_0+t)^z)\right] \\ +\mathfrak{T}(\mathfrak{b}-c)(\mathcal{X}_c - x_c) + \frac{1}{2}\left(q\mathfrak{Y}(t)\sqrt{c}+\theta\right)^2(\mathcal{X}_{qq} - \mathcal{L}_{qq}) \\ +\frac{1}{2}\sigma^2 c(\mathcal{X}_{cc} - \mathcal{L}_{cc}) + \left(\sigma q c\mathfrak{Y}\rho + \theta\sigma\sqrt{c}\right)(\mathcal{X}_{qc} - \mathcal{L}_{qc}) \end{Bmatrix} = 0 \\ \mathcal{X}(T,q,c) = x(T,q,c,q,q^2) \end{cases} \tag{16}$$

$$\begin{cases} \begin{Bmatrix} \mathcal{Y}_t + \mathcal{Y}_q\left[q\begin{pmatrix} \mathfrak{Y}c(\mathfrak{H}-r) \\ +\mathfrak{u}(w_0+t)^z + r - (\mathcal{B}+\beta) \end{pmatrix} \\ +\ell(1-at\mathfrak{u}(w_0+t)^z)\right] \\ +\mathfrak{T}(\mathfrak{b}-c)\mathcal{Y}_c + \frac{1}{2}\left(q\mathfrak{Y}(t)\sqrt{c}+\theta\right)^2\mathcal{Y}_{qq} \\ +\frac{1}{2}\sigma^2 c\mathcal{Y}_{cc} + \left(\sigma q c\mathfrak{Y}\rho + \theta\sigma\sqrt{c}\right)\mathcal{Y}_{qc} \end{Bmatrix} = 0 \\ \mathcal{Y}(T,q,c) = q \end{cases} \tag{17}$$

$$\begin{cases} \begin{Bmatrix} \mathcal{Z}_t + \mathcal{Z}_q\left[q\begin{pmatrix} \mathfrak{Y}c(\mathfrak{H}-r) \\ +\mathfrak{u}(w_0+t)^z + r - (\mathcal{B}+\beta) \end{pmatrix} \\ +\ell(1-at\mathfrak{u}(w_0+t)^z)\right] \\ +\mathfrak{T}(\mathfrak{b}-c)\mathcal{Z}_c + \frac{1}{2}\left(q\mathfrak{Y}(t)\sqrt{c}+\theta\right)^2\mathcal{Z}_{qq} \\ +\frac{1}{2}\sigma^2 c\mathcal{Z}_{cc} + \left(\sigma q c\mathfrak{Y}\rho + \theta\sigma\sqrt{c}\right)\mathcal{Z}_{qc} \end{Bmatrix} = 0 \\ \mathcal{Z}(T,q,c) = q^2 \end{cases} \tag{18}$$

where:

$$\mathcal{L}_{qq} = \gamma\mathcal{Y}_q^2,\ \mathcal{L}_{xc} = \gamma\mathcal{Y}_q\mathcal{Y}_c,\ \mathcal{L}_{cc} = \gamma\mathcal{Y}_c^2 \tag{19}$$

$$\begin{cases} x_t = x_q = x_c = x_{cc} = x_{qq} = x_{qi} = x_{qj} = x_{ij} = x_{jj} = 0 \\ x_i = 1+\gamma a, x_{ii} = \gamma, x_j = -\frac{\gamma}{2} \end{cases} \tag{20}$$

Then $\mathcal{J}(t,q,c) = \mathcal{X}(t,q,c), i^{\mathfrak{Y}^*} = \mathcal{Y}(t,q,c), j^{\mathfrak{Y}^*} = \mathcal{Z}(t,q,c)$ *for the OCS*$\mathfrak{Y}^*$.

Proof. The proof is similar to one in [37], [38], and [39].

3. RESULTS AND DISCUSSION

3.1 Main Results

Proposition 1. *The OCS for the stock market price is given by*

$$\mathfrak{Y}^* = \frac{1}{\gamma q}\left[\mathfrak{H} - r - \rho\sigma(\mathfrak{H}-r)^2\left(\frac{1-e^{[(t-T)(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))]}}{\mathfrak{T}+\rho\sigma}\right)\right]e^{\begin{bmatrix} (r-\mathcal{B}-\beta)(t-T) \\ +\frac{\mathfrak{u}}{z+1}((w_0+t)^{z+1} \\ -(w_0+T)^{z+1} \end{bmatrix}} - \frac{\theta}{x\sqrt{c}} \tag{21}$$

Proof. First, we simplify Equation (16) by substituting Equation (19) and Equation (20) into Equation (16) to have

$$\begin{cases} \sup\limits_{\mathfrak{Y}} \left\{ \begin{array}{c} \mathcal{X}_t + \mathcal{X}_q \left[\begin{array}{c} q \left(\begin{array}{c} \mathfrak{Y}c(\mathfrak{H} - r) \\ +u(w_0 + t)^z + r - (\mathcal{B} + \beta) \end{array} \right) \\ +\ell(1 - atu(w_0 + t)^z) \end{array} \right] \\ +\mathfrak{T}(b - c)\mathcal{X}_c + \frac{1}{2}\left(q\mathfrak{Y}(t)\sqrt{c} + \theta\right)^2 \left(\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2\right) \\ +\frac{1}{2}\sigma^2 c(\mathcal{X}_{cc} - \gamma \mathcal{Y}_c^2) + \left(\sigma q c \mathfrak{Y}\rho + \theta\sigma\sqrt{c}\right)\left(\mathcal{X}_{qc} - \gamma \mathcal{Y}_q \mathcal{Y}_c\right) \end{array} \right\} = 0 \\ \mathcal{X}(T, q, c) = x(T, q, c, q, q^2) \end{cases} \tag{22}$$

Differentiating **Equation (22)** with respect to $\mathfrak{Y}(t)$ and solve for $\mathfrak{Y}(t)$, we have

$$\mathfrak{Y}^* = -\left[\frac{(\mathfrak{H} - r)\mathcal{X}_q + \sigma\rho(\mathcal{X}_{qc} - \gamma \mathcal{Y}_q \mathcal{Y}_c) + \frac{\theta\sqrt{k}}{k}(\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2)}{q(\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2)} \right], \tag{23}$$

substituting **Equation (23)** into **Equation (17)** and **Equation (22)**, we have

$$\begin{cases} \left\{ \begin{array}{c} \mathcal{X}_t + \mathcal{X}_q \left[\begin{array}{c} q\left(u(w_0 + t)^z + r - (\mathcal{B} + \beta)\right) \\ -\theta\sqrt{c}(\mathfrak{H} - r) + \ell(1 - atu(w_0 + t)^z) \end{array} \right] + \mathfrak{T}(b - c)\mathcal{X}_c \\ +\frac{1}{2}\sigma^2 c(\mathcal{X}_{cc} - \gamma \mathcal{Y}_c^2) - \frac{c}{2} \frac{\left[\mathcal{X}_q(\mathfrak{H} - r) + (\mathcal{X}_{qc} - \gamma \mathcal{Y}_q \mathcal{Y}_c)\sigma\rho\right]^2}{(\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2)} \end{array} \right\} = 0 \\ \mathcal{X}(T, q, c) = x(T, q, c, q, q^2) \end{cases} \tag{24}$$

$$\begin{cases} \left\{ \begin{array}{c} \mathcal{Y}_t + \mathcal{Y}_q \left[\begin{array}{c} q\left(u(w_0 + t)^z + r - (\mathcal{B} + \beta)\right) \\ -\theta\sqrt{c}(\mathfrak{H} - r) + \ell(1 - atu(w_0 + t)^z) \end{array} \right] + \mathfrak{T}(b - c)\mathcal{Y}_c \\ -c(\mathcal{Y}_q(\mathfrak{H} - r) + \sigma\rho \mathcal{Y}_{qc}) \left[\frac{\mathcal{X}_q(\mathfrak{H} - r) + (\mathcal{X}_{qc} - \gamma \mathcal{Y}_q \mathcal{Y}_c)\sigma\rho}{\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2} \right] \\ +\frac{1}{2} c \mathcal{Y}_{qq} \left[\frac{\mathcal{X}_q(\mathfrak{H} - r) + (\mathcal{X}_{qc} - \gamma \mathcal{Y}_q \mathcal{Y}_c)\sigma\rho}{(\mathcal{X}_{qq} - \gamma \mathcal{Y}_q^2)} \right]^2 \end{array} \right\} = 0 \\ \mathcal{Y}(T, q, c) = q \end{cases} \tag{25}$$

Next, we conjecture a solution for $\mathcal{X}(t, q, c)$ and $\mathcal{Y}(t, q, c)$ as follows

$$\begin{cases} \mathcal{X}(t, q, c) = u_1(t)q + v_1(t)\frac{c}{\gamma} + \frac{w_1(t)}{\gamma}, \\ u_1(T) = 1, v_1(T) = 0, w_1(T) = 0 \end{cases} \tag{26}$$

$$\begin{cases} \mathcal{Y}(t, q, c) = u_2(t)q + v_2(t)\frac{c}{\gamma} + \frac{w_2(t)}{\gamma}, \\ u_2(T) = 1, v_2(T) = 0, w_2(T) = 0 \end{cases} \tag{27}$$

Differentiating **Equation (26)** and **Equation (27)**, we have

$$\left\{ \mathcal{X}_t = u_{1t}q + \frac{c v_{1t}}{\gamma} + \frac{w_{1t}(t)}{\gamma}, \mathcal{X}_q = u_1, \mathcal{X}_c = \frac{v_1(t)}{\gamma},\ \mathcal{X}_{qq} = \mathcal{X}_{qc} = \mathcal{X}_{cc} = 0 \right. \tag{28}$$

$$\left\{ \mathcal{Y}_t = u_{2t}q + \frac{c v_{2t}}{\gamma} + \frac{w_{2t}(t)}{\gamma}, \mathcal{Y}_q = u_2, \mathcal{Y}_c = \frac{v_2(t)}{\gamma},\ \mathcal{Y}_{qq} = \mathcal{Y}_{qc} = \mathcal{Y}_{cc} = 0 \right. \tag{29}$$

Substituting **Equation (28)** and **Equation (29)** into **Equation (26)** and **Equation (27)**, we have:

$$\begin{cases} \left\{ \begin{array}{c} q u_{1t}(t) + \frac{c v_{1t}(t)}{\gamma} + \frac{w_{1t}(t)}{\gamma} + \left[\begin{array}{c} q\left(u(w_0 + t)^z + r - (\mathcal{B} + \beta)\right) \\ -\theta\sqrt{c}(\mathfrak{H} - r) + \ell(1 - atu(w_0 + t)^z) \end{array} \right] u_1(t) \\ +\frac{\mathfrak{T}(b - c)v_1(t)}{\gamma} - \frac{\sigma^2 c v_2^2}{2\gamma} - \frac{c[u_1(\mathfrak{H} - r) - u_2 v_2 \sigma\rho]^2}{2\gamma u_2^2} \end{array} \right\} = 0 \\ u_1(T) = 1, v_1(T) = 0, w_1(T) = 0 \end{cases} \tag{30}$$

$$\begin{cases}\left\{\begin{array}{c} q u_{2t}(t)+\frac{c v_{2t}}{\gamma}+\frac{w_{2t}(t)}{\gamma}+u_2(t)\left[\begin{array}{c} q\left(\mathfrak{u}(w_0+t)^{z}+r-(\mathcal{B}+\beta)\right)\\ -\theta\sqrt{c}(\mathfrak{H}-r)+\ell(1-at\mathfrak{u}(w_0+t)^{z})\end{array}\right]\\ +\frac{\mathfrak{T}(\mathfrak{b}-c)v_2(t)}{\gamma}+c(\mathfrak{H}-r)\left[\frac{u_1(\mathfrak{H}-r)-u_2v_2\sigma\rho}{\gamma u_2}\right]\end{array}\right\}=0\\ u_2(T)=1, v_2(T)=0, w_2(T)=0\end{cases} \tag{31}$$

Simplifying **Equation (30)** and **Equation (31)**, we have

$$\begin{cases}\left\{\begin{array}{c}\left[u_{1t}(t)+\left(\begin{array}{c}\mathfrak{u}(w_0+t)^{z}\\ +r-(\mathcal{B}+\beta)\end{array}\right)u_1(t)\right]q+\left[\begin{array}{c}v_{1t}(t)-\mathfrak{T}v_1(t)-\frac{1}{2}\sigma^2v_2^2\\ -\frac{[u_1(\mathfrak{H}-r)-u_2v_2\sigma\rho]^2}{2u_2^2}\end{array}\right]\frac{c}{\gamma}\\ +\left[w_{1t}(t)+\gamma\left(\ell(1-at\mathfrak{u}(w_0+t)^{z})-\theta\sqrt{c}(\mathfrak{H}-r)\right)u_1(t)+\mathfrak{T}\mathfrak{b}v_1(t)\right]\frac{1}{\gamma}\end{array}\right\}=0\\ u_1(T)=1, v_1(T)=0, w_1(T)=0\end{cases} \tag{32}$$

$$\begin{cases}\left\{\begin{array}{c}\left[u_{2t}(t)+\left(\begin{array}{c}\mathfrak{u}(w_0+t)^{z}\\ +r-(\mathcal{B}+\beta)\end{array}\right)u_2(t)\right]q+\left[\begin{array}{c}v_{2t}(t)-\mathfrak{T}v_2(t)+\frac{u_1(\mathfrak{H}-r)^2}{u_2}\\ -v_2\sigma\rho(\mathfrak{H}-r)\end{array}\right]\frac{c}{\gamma}\\ +\left[w_{2t}(t)+\gamma\left(\ell(1-at\mathfrak{u}(w_0+t)^{z})-\theta\sqrt{c}(\mathfrak{H}-r)\right)u_2(t)+\mathfrak{T}\mathfrak{b}v_2(t)\right]\frac{1}{\gamma}\end{array}\right\}=0\\ u_2(T)=1, v_2(T)=0, w_2(T)=0\end{cases} \tag{33}$$

Since $q\neq 0,\ c\neq 0,\ \gamma\neq 0$, **Equation (32)** and **Equation (33)** reduces to

$$\begin{cases}u_{1t}(t)+\left(\mathfrak{u}(w_0+t)^{z}+r-(\mathcal{B}+\beta)\right)u_1(t)=0\\ u_1(T)=1\end{cases} \tag{34}$$

$$\begin{cases}v_{1t}(t)-\mathfrak{T}v_1(t)-\frac{1}{2}\sigma^2v_2^2-\frac{[u_1(\mathfrak{H}-r)-u_2v_2\sigma\rho]^2}{2u_2^2}=0,\\ v_1(T)=0\end{cases} \tag{35}$$

$$\begin{cases}\left[w_{1t}(t)+\gamma\left(\ell(1-at\mathfrak{u}(w_0+t)^{z})-\theta\sqrt{c}(\mathfrak{H}-r)\right)u_1(t)+\mathfrak{T}\mathfrak{b}v_1(t)\right]=0\\ w_1(T)=0\end{cases} \tag{36}$$

$$\begin{cases}u_{2t}(t)+\left(\mathfrak{u}(w_0+t)^{z}+r-(\mathcal{B}+\beta)\right)u_2(t)=0\\ u_2(T)=1\end{cases} \tag{37}$$

$$\begin{cases}v_{2t}(t)-\mathfrak{T}v_2(t)+\frac{u_1(\mathfrak{H}-r)^2}{u_2}-v_2\sigma\rho(\mathfrak{H}-r)=0,\\ v_2(T)=0\end{cases} \tag{38}$$

$$\begin{cases}w_{2t}(t)+\gamma\left(\ell(1-at\mathfrak{u}(w_0+t)^{z})-\theta\sqrt{c}(\mathfrak{H}-r)\right)u_2(t)+\mathfrak{T}\mathfrak{b}v_2(t)=0\\ w_2(T)=0\end{cases} \tag{39}$$

Solving **Equation (34)** – **Equation (39)**, we obtain:

$$u_1(t)=e^{\left[(r-\mathcal{B}-\beta)(T-t)+\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]} \tag{40}$$

$$u_2(t)=e^{\left[(r-\mathcal{B}-\beta)(T-t)+\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]} \tag{41}$$

$$v_1(t)=\left(\begin{array}{c}\left(\frac{\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}+\frac{\sigma^2(\rho^2-1)}{2\mathfrak{T}}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2+\frac{(\mathfrak{H}-r)^2}{2\mathfrak{T}}\right)e^{\mathfrak{T}(t-T)}\\ +\left(\begin{array}{c}\left(\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\left(1-e^{\rho\sigma(\mathfrak{H}-r)(t-T)}\right)\\ -\frac{1}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left[1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}\right]\end{array}\right)e^{\mathfrak{T}(t-T)}\end{array}\right) \tag{42}$$

$$v_2(t)=\frac{\left(1-e^{\left((\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)\right)}\right)(\vartheta-\mathcal{R})^2}{\mathfrak{T}+\rho\sigma} \tag{43}$$

$$
w_1(t) = \begin{pmatrix} \gamma\left(\ell - \theta\sqrt{c}(\mathfrak{H} - r)\right)\mathfrak{u}e^{\frac{\mathfrak{u}}{z+1}(w_0+T)^{z+1}}\left(\int_t^T (w_0+T)^z\, e^{\left[\begin{matrix}(r-\mathcal{B}-\beta)(T-\tau)\\ -\frac{\mathfrak{u}}{z+1}(w_0+\tau)^{z+1}\end{matrix}\right]}d\tau\right) \\ +\gamma\left(\ell - \theta\sqrt{c}(\mathfrak{H} - r)\right)\left[\frac{e^{\left[(r-\mathcal{B}-\beta)(T-t)-\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]}}{\mathfrak{u}(w_0+t)^z} - \frac{1}{\mathfrak{u}(w_0+T)^z}\right] \\ \frac{1}{\mathfrak{T}^2}\left(\frac{\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)} + \frac{\sigma^2(\rho^2-1)}{2}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2 + \frac{(\mathfrak{H}-r)^2}{2}\right)\left(e^{\mathfrak{T}(t-T)} - 1\right) \\ \frac{1}{\mathfrak{T}}\left(\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)} + \frac{\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\left(e^{\mathfrak{T}(t-T)} - 1\right) \\ -\frac{1}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\begin{pmatrix} e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)} + \frac{1}{\mathfrak{T}}\left(e^{\mathfrak{T}(t-T)} - 1\right) - 1 \\ -\frac{1}{2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left(e^{(2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)} - 1\right) \end{pmatrix} \end{pmatrix} \tag{44}
$$

$$
w_2(t) = \begin{pmatrix} \gamma\left(\ell - \theta\sqrt{c}(\mathfrak{H} - r)\right)\mathfrak{u}e^{\frac{\mathfrak{u}}{z+1}(w_0+T)^{z+1}}\left(\int_t^T (w_0+T)^z\, e^{\left[\begin{matrix}(r-\mathcal{B}-\beta)(T-\tau)\\ -\frac{\mathfrak{u}}{z+1}(w_0+\tau)^{z+1}\end{matrix}\right]}d\tau\right) \\ +\gamma\left(\ell - \theta\sqrt{c}(\mathfrak{H} - r)\right)\left[\frac{e^{\left[(r-\mathcal{B}-\beta)(T-t)-\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]}}{\mathfrak{u}(w_0+t)^z} - \frac{1}{\mathfrak{u}(w_0+T)^z}\right] \\ \frac{(\mathfrak{H}-r)^2(T-t)}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)} + \left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2\left(e^{\left((\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)\right)} - 1\right) \end{pmatrix} \tag{45}
$$

Substituting **Equation (40), Equation (42)** and **Equation (44)** into **Equation (26)** and **Equation (41), Equation (43)**, and **Equation (45)** into **Equation (27)**, $\mathcal{X}(t, q, c)$ and $\mathcal{Y}(t, q, c)$ are solved.

From **Equation (28)** and **Equation (29)**, we have

$$
\begin{cases} \mathcal{X}_q = u_1 = e^{\left[(r-\mathcal{B}-\beta)(T-t)+\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]} \\ \mathcal{Y}_q = u_2 = e^{\left[(r-\mathcal{B}-\beta)(T-t)+\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]} \\ \qquad \mathcal{Y}_c = \frac{v_2(t)}{\gamma} = \frac{1}{\gamma}\frac{\left(1-e^{\left((\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)\right)}\right)(\vartheta-\mathcal{R})^2}{\mathfrak{T}+\rho\sigma} \\ \qquad \mathcal{X}_{qq} = \mathcal{X}_{qc} = 0 \end{cases} \tag{46}
$$

Substituting **Equation (46)** into **Equation (23)**, we obtain **Equation (21)** which completes the proof.

Next, we proceed to solve the efficient frontier which shows the relationship between the expectation and variance.

Proposition 2. *The efficient frontier of the pension fund is given as follows*

$$
E_{t,q,c}[\mathcal{Q}^{\mathfrak{Y}^*}(T)]
$$

$$
\begin{aligned}
& q e^{\left[(r-\mathcal{B}-\beta)(T-t)+\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]} \\
& +\left(\ell-\theta\sqrt{c}(\mathfrak{H}-r)\right)\mathfrak{u}e^{\frac{\mathfrak{u}}{z+1}(w_0+T)^{z+1}}\left(\int_t^T (w_0+T)^z e^{\left[(r-\mathcal{B}-\beta)(T-\tau)-\frac{\mathfrak{u}}{z+1}(w_0+\tau)^{z+1}\right]}d\tau\right) \\
& +\left(\ell-\theta\sqrt{c}(\mathfrak{H}-r)\right)\left[\frac{e^{\left[(r-\mathcal{B}-\beta)(T-t)-\frac{\mathfrak{u}}{z+1}\left((w_0+T)^{z+1}-(w_0+t)^{z+1}\right)\right]}}{\mathfrak{u}(w_0+t)^z}-\frac{1}{\mathfrak{u}(w_0+T)^z}\right]+ \\
& \left(\begin{aligned}&\left[\begin{aligned}& c\frac{\left(1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}\right)(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{(\mathfrak{H}-r)^2(T-t)}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)} \\ & +\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2\left(e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}-1\right)\end{aligned}\right]\times \\
&\sqrt{\frac{Var_{t,q,c}[Q^{\mathfrak{Y}^*}(T)]}{\left(\begin{aligned}&2\left[\begin{aligned}&c\frac{\left(1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}\right)(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{(\mathfrak{H}-r)^2(T-t)}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\\&+\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2\left(e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}-1\right)\end{aligned}\right]-ce^{\mathfrak{T}(t-T)}\times\\
&\left(\left(\begin{aligned}&\left(\frac{2\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}+\frac{\sigma^2(\rho^2-1)}{\mathfrak{T}}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2+\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}}\right)\\&+\left(\frac{2(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{2\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\left(1-e^{\rho\sigma(\mathfrak{H}-r)(t-T)}\right)\\&-\frac{2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left[1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}\right]\end{aligned}\right)-\left(\begin{aligned}&\frac{2}{\mathfrak{T}^2}\left(\frac{\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{\sigma^2(\rho^2-1)}{2}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2+\frac{(\mathfrak{H}-r)^2}{2}\right)\left(e^{\mathfrak{T}(t-T)}-1\right)\\&-\frac{2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left(\begin{aligned}&e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}+\frac{1}{\mathfrak{T}}\left(e^{\mathfrak{T}(t-T)}-1\right)-1\\&-\frac{1}{2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left(e^{(2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}-1\right)\end{aligned}\right)\\&+\frac{2}{\mathfrak{T}}\left(\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\end{aligned}\right)\right)\end{aligned}\right)}}\end{aligned}\right)
\end{aligned} \tag{47}
$$

Proof. Recall that

$$E_{t,q,c}[Q^{\mathfrak{Y}^*}(T)] = i^{\mathfrak{Y}^*}(t,q,c) = \mathcal{Y}(t,q,c), \tag{48}$$

From **Equation (15)** and **Theorem 1**, we have

$$Var_{t,q,c}[Q^{\mathfrak{Y}^*}(T)] = \left(E_{t,q,c}[Q^{\mathfrak{Y}^*}(T)^2] - (E_{t,q,c}[Q^{\mathfrak{Y}^*}(T)])^2\right) = \frac{2}{\gamma}\left(\mathcal{Y}(t,q,c) - \mathcal{X}(t,q,c)\right) \tag{49}$$

Substituting **Equation (40)** - **Equation (45)** into **Equation (28)** and **Equation (29)** and simplify it, we have

$$
\frac{1}{\gamma} = \sqrt{\frac{Var_{t,q,c}[Q^{\mathfrak{Y}^*}(T)]}{\left(\begin{aligned}&2\left[\begin{aligned}&c\frac{\left(1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}\right)(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{(\mathfrak{H}-r)^2(T-t)}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\\&+\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2\left(e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}-1\right)\end{aligned}\right]-\\
&\left(\begin{aligned}&\left(\frac{2\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}+\frac{\sigma^2(\rho^2-1)}{\mathfrak{T}}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2+\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}}\right)\\&+\left(\frac{2(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{2\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\left(1-e^{\rho\sigma(\mathfrak{H}-r)(t-T)}\right)\\&-\frac{2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left[1-e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))}\right]\end{aligned}\right)ce^{\mathfrak{T}(t-T)}-\left(\begin{aligned}&\frac{2}{\mathfrak{T}^2}\left(\frac{\rho\sigma(\mathfrak{H}-r)^3}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{\sigma^2(\rho^2-1)}{2}\left(\frac{\mathfrak{H}-r}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\right)^2+\frac{(\mathfrak{H}-r)^2}{2}\right)\left(e^{\mathfrak{T}(t-T)}-1\right)\\&-\frac{2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left(\begin{aligned}&e^{(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}+\frac{1}{\mathfrak{T}}\left(e^{\mathfrak{T}(t-T)}-1\right)-1\\&-\frac{1}{2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}\left(e^{(2\mathfrak{T}+\rho\sigma(\mathfrak{H}-r))(t-T)}-1\right)\end{aligned}\right)\\&+\frac{2}{\mathfrak{T}}\left(\frac{(\mathfrak{H}-r)^2}{\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)}+\frac{\sigma(\rho^2-1)(\mathfrak{H}-r)}{\rho(\mathfrak{T}+\rho\sigma(\mathfrak{H}-r)^2)}\right)\end{aligned}\right)\end{aligned}\right)}} \tag{50}
$$

Substituting **Equation (41)**, **Equation (43)**, **Equation (45)** and **Equation (50)** into **Equation (48)**, we obtain **Equation (47)** which completes the proof.

3.2 Numerical Simulations

In this section, numerical simulations showing the relationship between the OCS and some sensitive parameters are presented. To achieve this, the following data are used similar to [25], [34] unless otherwise stated: $\vartheta = 0.5$, $\mathcal{B} = 0.01$, $r = 0.01$, $\gamma = 0.5$, $x = 1$, $\rho = 0.1$, $\sigma = 0.1$, $\aleph = 0.01$, $\beta = 0.01$, $a = 0.01$, $n = 0.001$, $w_0 = 20$, $T = 30$.

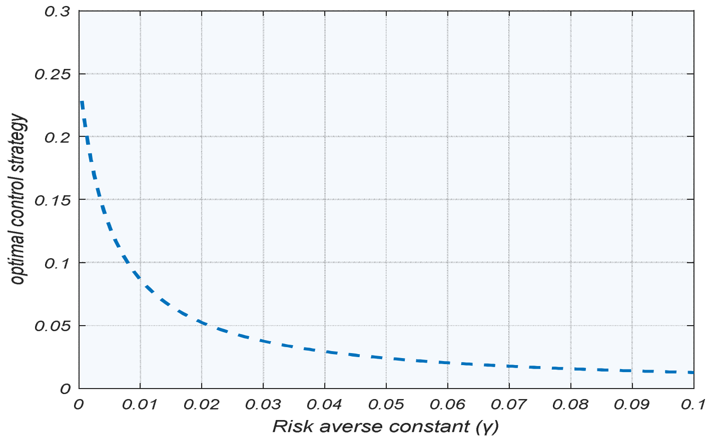

Figure 1. Relationship Between the OCS and RAC

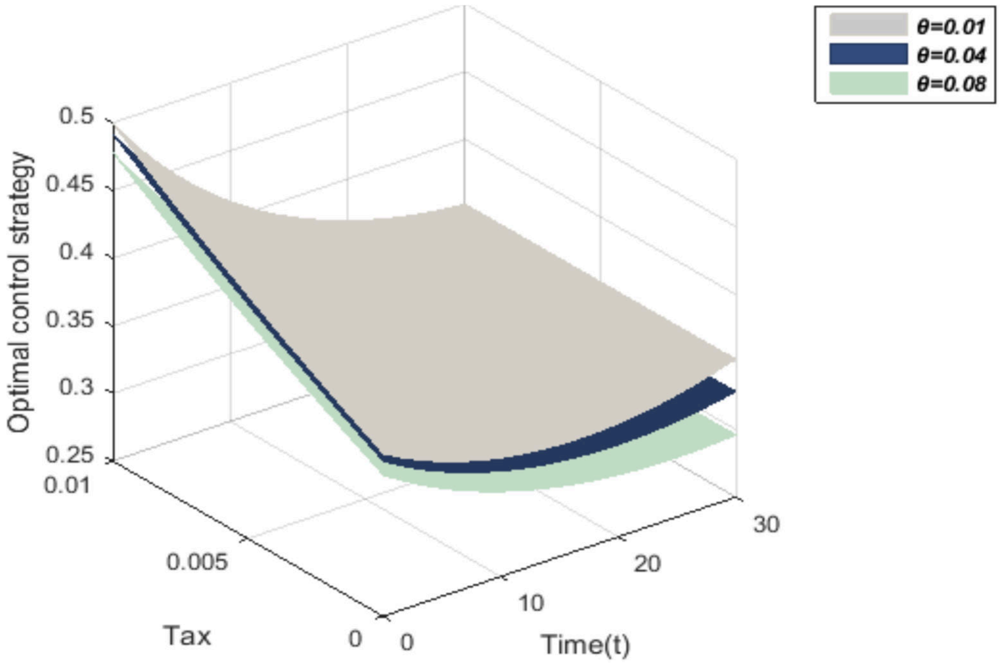

Figure 2. Relationship Between OCS and tax with Different AVC

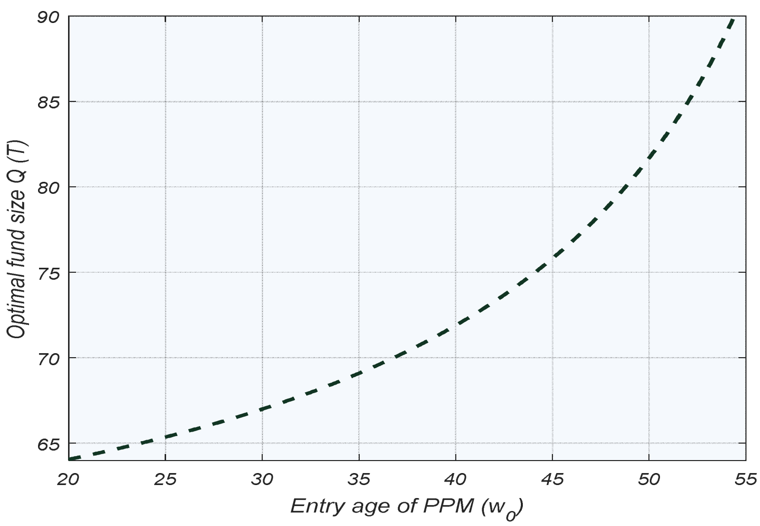

Figure 3. Relationship Between the Optimal Fund Size and Entry Age Members

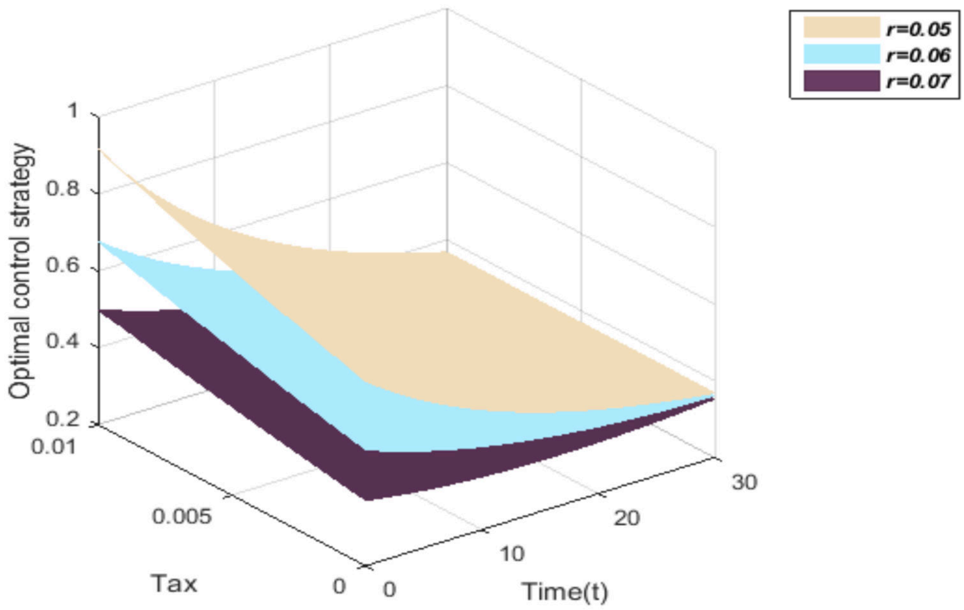

Figure 4. **Relationship Between OCS and Tax with Different Predetermined Interest**

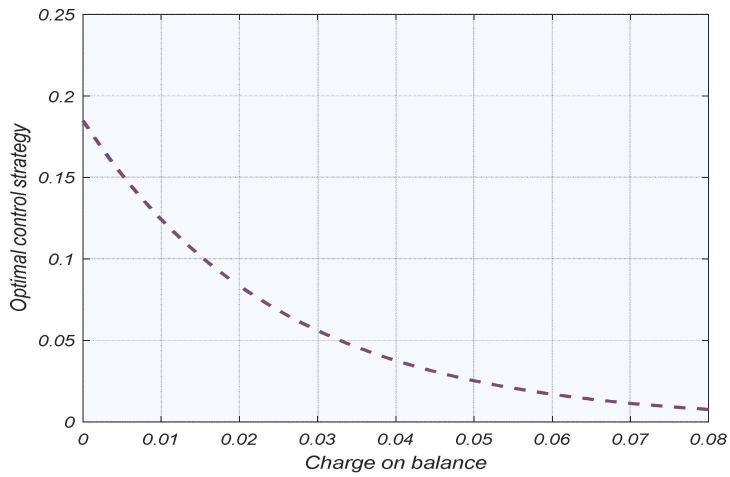

Figure 5. **Relationship Between OCS and CB**

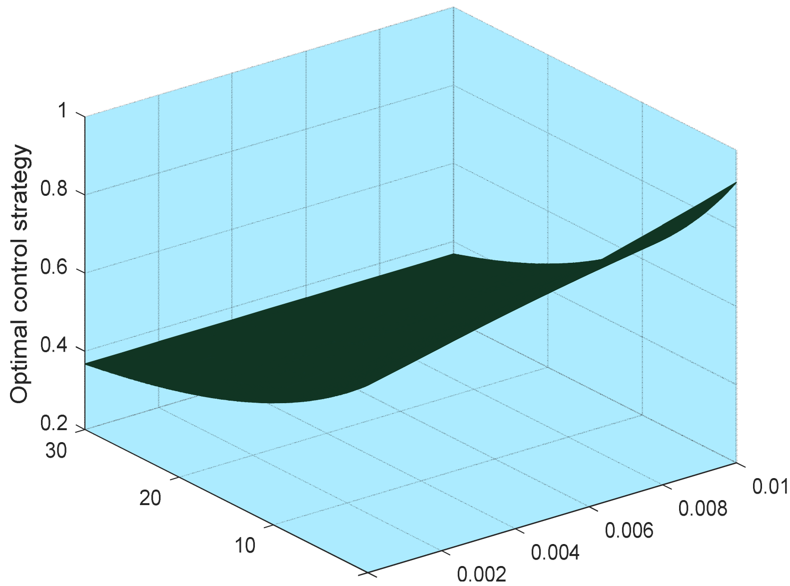

Figure 6. **Relationship between OCS and Appreciation Rate of the Stock Price**

3.3 Summary of Results and Discussion

In **Figure 1**, the relationship between OCS and the RAC was presented. It was observed that the OCS for the risky asset is a decreasing function of the RAC. Also, from the graph in figure 4, it was observed that the PSM with higher RAC will invest a smaller proportion of his wealth in the risky asset (stock) and increase investment in risk free asset while members with lower RAC will invest more in the risky assets while reducing investment in the risk-free asset. In **Figure 2**, the relationship between OCS and tax with different AVC was presented. It was observed that as the OCS increases, the tax on the risky asset increases likes wise. Also, it was observed that the OCS for the risky asset is a decreasing function of the AVC. It was also observed, that if PSM have more AVC in his RSA, such member may tend to invest less proportion of his wealth in the risky asset (stock) and increase investment in risk-free asset and vice versa. This is consistent with other literature [36].

In **Figure 3**, the relationship between the optimal fund size and entry age of the PSM. It was observed that the optimal fund size grows as the entry age of the PSM grows. This implies that with late entry into the scheme, PSM tens to increase the percentage of their investment in risky asset, thereby leading to an increase in the optimal fund size of the PSM. Also, in **Figure 4**, the relationship between the OCS and tax with different risk-free interest rate of the PSM was simulated. It was seen that the OCS falls as the risk-free interest rate grows and grows when the risk-free interest rate falls. This simply indicates that PSM will likely wish to invest less in risky asset if the interest rate from the risk-free asset is very high and attractive. However, if the risk-free interest rate is less attractive, PSM member may be advised by their fund administrators to move to fund I investment platform for more profitable investment in risky assets, thereby reducing their investment in risk free asset. Also, we observed similar to **Figure 2**, that OCS is an increasing function of tax and vice versa.

In **Figure 5**, the relationship between the OCS and CB was presented. It was seen that the OCS is a decreasing function of CB. This implies that if the CB imposed on investment in risky asset is high, this may discourage PSM from investing in risky asset, hence invest more in risk free assets. In **Figure 6**, the relationship between the OCS and appreciation rate of the stock market was presented. It was seen that the OCS is an increasing function of the appreciation rate. This implies that if the stock appreciates impressively over time, the PSM may be advised by the PFA to increase his investment in such asset for the purpose of increasing his returns on investment and may do otherwise if the value of the stock depreciates.

4. CONCLUSION

In conclusion, we developed an optimal portfolio considering the volatility of the stock market price consisting of a risk-free asset and a risky asset under the Heston volatility model (HVM). Also, the PFA takes into consideration the additional voluntary contributions (AVC) by PSM, tax on the stock market price, charge on balance (CB), and the mortality risk of the PSM using Weibull model. The investment model obtained under this assumption in (21) is strongly dependent on the instantaneous volatility, tax on investment, AVC, risk aversion coefficient (RAC), CB and the correlation coefficient. Also, the efficient frontier which shows the relationship between the expectation and the risk was also obtained in (47). Hence, the understanding of the behavior these parameters are very essential, efficient and necessary in the formation of OCS for any PSM by PFAs.

REFERENCES

[1] L. He and Z. Liang, "Optimal investment strategy for the DC plan with the return of premiums clauses in a mean--variance framework," *Insur. Math. Econ.*, vol. 53, no. 3, pp. 643–649, 2013.

[2] P. Antolin, S. Payet, and J. Yermo, "OECD Journal: Financial Market Trends: Assessing Default Investment Strategies in Defined Contribution Pension Plans," 2010.

[3] C. C. Ihejirka, "Critical Evaluation of Nigeria's Pension Reform Act of 2004: Focus on the Contributory Pension Scheme (CPS)," Northeastern University, 2020.

[4] E. E. Akpanibah and S. K. Samaila, "Stochastic strategies for optimal investment in a defined contribution (DC) pension fund," *Int. J. Appl. Sci. Math. Theory*, vol. 3, no. 3, pp. 48–55, 2017.

[5] P. Battocchio and F. Menoncin, "Optimal pension management in a stochastic framework," *Insur. Math. Econ.*, vol. 34, no. 1, pp. 79–95, 2004.

[6] D. Blake, D. Wright, and Y. Zhang, "Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion," *J. Econ. Dyn. Control*, vol. 37, no. 1, pp. 195–209, 2013.

[7] J.-F. Boulier, S. Huang, and G. Taillard, "Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund," *Insur. Math. Econ.*, vol. 28, no. 2, pp. 173–189, 2001.

[8] J. Gao, "Stochastic optimal control of DC pension funds," *Insur. Math. Econ.*, vol. 42, no. 3, pp. 1159–1164, 2008.

[9] A. J. G. Cairns, D. Blake, and K. Dowd, "Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans," *J. Econ. Dyn. Control*, vol. 30, no. 5, pp. 843–877, 2006.

[10] H. Xiao, T. Ren, Y. Bai, and Z. Zhou, "Time-consistent investment-reinsurance strategies for the insurer and the reinsurer under the generalized mean-variance criteria," *Mathematics*, vol. 7, no. 9, p. 857, 2019.

[11] J. Zhu and S. Li, "Time-consistent investment and reinsurance strategies for mean-variance insurers under stochastic interest rate and stochastic volatility," *Mathematics*, vol. 8, no. 12, p. 2183, 2020.

[12] G. A. Egbe, A. Awogbemi, and O. Osu, "Portfolio optimization of pension fund contribution in Nigeria," *Math. Theory Model.*, vol. 3, no. 8, pp. 42–52, 2013.

[13] B. O. Osu, E. E. Akpanibah, and K. N. C. Njoku, "On the effect of stochastic extra contribution on optimal investment strategies for stochastic salary under the affine interest rate model in a DC Pension Fund," *Gen. Lett. Math.*, vol. 2, no. 3, pp. 138–149, 2017.

[14] C. Zhang and X. Rong, "Optimal investment strategies for DC pension with stochastic salary under the affine interest rate model," *Discret. Dyn. Nat. Soc.*, vol. 2013, no. 1, p. 297875, 2013.

[15] O. Basimanebotlhe and X. Xue, "Stochastic optimal investment under inflationary market with minimum guarantee for DC pension plans," *J. Math. Res.*, vol. 7, no. 3, p. 1, 2015.

[16] K. N. C. Njoku and B. Osu, "O \& Philip, U. U,(2019). On the Investment Approach in a DC Pension Scheme for Default Fund Phase IV under the Constant Elasticity of Variance (CEV) model," *Int. J. Adv. Math.*, vol. 3, pp. 65–76, 2018.

[17] J. Gao, "Optimal portfolios for DC pension plans under a CEV model," *Insur. Math. Econ.*, vol. 44, no. 3, pp. 479–490, 2009.

[18] A. Prabowo, "Optimal Reinsurance and Investment Strategy Under CEV Model with Fractional Power Utility Function".

[19] P. Devolder, M. B. Princep, and I. D. Fabian, "Stochastic optimal control of annuity contracts," *Insur. Math. Econ.*, vol. 33, no. 2, pp. 227–238, 2003.

[20] E. E. Akpanibah and U. Ini, "Portfolio strategy for an investor with logarithm utility and stochastic interest rate under constant elasticity of variance model," *J. Niger. Soc. Phys. Sci.*, pp. 186–196, 2020.

[21] E. Vigna and S. Haberman, "Optimal investment strategy for defined contribution pension schemes," *Insur. Math. Econ.*, vol. 28, no. 2, pp. 233–262, 2001.

[22] D. Li, X. Rong, and H. Zhao, "Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model," *WSEAS Trans. Math.*, vol. 12, no. 3, pp. 243–245, 2013.

[23] H. M. Markowitz and G. P. Todd, *Mean-variance analysis in portfolio choice and capital markets*, vol. 66. John Wiley \& Sons, 2000.

[24] H. R. Richardson, "A minimum variance result in continuous trading portfolio optimization," *Manage. Sci.*, vol. 35, no. 9, pp. 1045–1055, 1989.

[25] D.-L. Sheng and X. Rong, "Optimal Time-Consistent Investment Strategy for a DC Pension Plan with the Return of Premiums Clauses and Annuity Contracts," *Discret. Dyn. Nat. Soc.*, vol. 2014, no. 1, p. 862694, 2014.

[26] D. Li, X. Rong, H. Zhao, and B. Yi, "Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model," *Insur. Math. Econ.*, vol. 72, pp. 6–20, 2017.

[27] E. E. Akpanibah, B. O. Osu, and S. A. Ihedioha, "On the optimal asset allocation strategy for a defined contribution pension system with refund clause of premium with predetermined interest under Heston's volatility model," *J. Nonlinear Sci. Appl*, vol. 13, no. 1, pp. 53–64, 2020.

[28] Z.-F. Li, K. W. Ng, and X.-T. Deng, "Continuous-time optimal portfolio selection using mean-CaR models," *Nonlinear Dyn. Syst. theory*, vol. 7, pp. 35–49, 2007.

[29] Z. Chai, X. Rong, and H. Zhao, "Optimal investment strategy for the DC plan with the return of premiums," *Syst. Eng.*, vol. 37, no. 7, pp. 1688–1696, 2017.

[30] W. Yuanye, F. A. N. Shunhou, and C. Hao, "DC pension plan with the return of premium clauses under inflation risk and volatility risk," *J. Syst. Sci. Math. Sci.*, vol. 38, no. 4, p. 423, 2018.

[31] E. E. Akpanibah and O. Oghen'Oro, "Optimal portfolio selection in a DC pension with multiple contributors and the impact of stochastic additional voluntary contribution on the optimal investment strategy," *Int. J. Math. Comput. Sci.*, vol. 12, no. 1, pp. 14–19, 2018.

[32] P. A. Azor, A. Igodo, and E. M. Ase, "Explicit Solution of an Investment Plan for a DC Pension Scheme with Voluntary Contributions and Return Clause under Logarithm Utility," *Int. J. Phys. Math. Sci.*, vol. 17, no. 8, pp. 115–121, 2023.

[33] E. Akpanibah and B. O Osu, "Optimal Portfolio Selection for a Defined Contribution Pension Fund with Return Clauses of Premium with Predetermined Interest Rate under Mean-variance Utility," *Asian J. Math. Sci.*, vol. 2, no. 2, 2018.

[34] C. Lai, S. Liu, and Y. Wu, "Optimal portfolio selection for a defined-contribution plan under two administrative fees and return of premium clauses," *J. Comput. Appl. Math.*, vol. 398, p. 113694, 2021.

[35] K. N. C. Njoku and E. E. Akpanibah, "Modeling and Optimization of Portfolio in a DC scheme with Return of Contributions and Tax using Weibull Force Function," *Asian J. Probab. Stat.*, vol. 16, no. 3, pp. 1–12, 2022.

[36] E. E. Akpanibah and B. O. Osu, "Assets allocation strategy in a DC pension scheme with refund clause of contributions with predetermined interest under Heston's Volatility model," *J. Niger. Assoc. Math. Phys.*, vol. 49, no. 1, pp. 31–40, 2019.

[37] L. He and Z. Liang, "Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs," *Insur. Math. Econ.*, vol. 44, no. 1, pp. 88–94, 2009.

[38] Z. Liang and J. Huang, "Optimal dividend and investing control of an insurance company with higher solvency constraints," *Insur. Math. Econ.*, vol. 49, no. 3, pp. 501–511, 2011.

[39] Y. Zeng and Z. Li, "Optimal time-consistent investment and reinsurance policies for mean-variance insurers," *Insur. Math. Econ.*, vol. 49, no. 1, pp. 145–154, 2011.