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HOLT-WINTER METHOD FOR FORECASTING LIQUID ALUMINIUM SULFATE USAGE FOR PROBABILISTIC INVENTORY MODELING Q WITH ERLANG DISTRIBUTION

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ABSTRACT

Water is a natural resource important for life and daily activities. Water distributed by the Regional Drinking Water Company (PDAM) should include a coagulation process using liquid aluminium sulfate as a coagulant before it can be consumed. Therefore, this research aims to predict the need for liquid aluminium sulfate in PDAM from 2023 to 2024 using Holt-Winter's method. It also aims to evaluate the optimum liquid aluminium sulfate chemical inventory policy using Q probabilistic inventory model with Normal and erlang probabilistic distributions in PDAM. The data was obtained from Tirta Musi PDAM in Palembang City, Indonesia. The results of forecasting liquid aluminium sulfate demand level data with the Holt-Winter multiplicative method provide the smallest MAPE value. The erlang probability distribution assumption has been met through the Kolmogorov Smirnov test method. The erlang probabilistic inventory model provides a more optimal policy solution than the normal probabilistic inventory model, with minimum total cost and higher service level.

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1. INTRODUCTION

Regional Drinking Water Company (PDAM) engaged in the distribution of clean water is available in every province, regency, and municipality in Indonesia. For example in Palembang City, PDAM Tirta Musi distributes clean water to the people for daily needs. [1] stated that water distributed by PDAM Tirta Musi is sourced from Musi River with two intakes, namely Karang Anyar and Ilir 1. However, not all existing water are suitable for direct usage, some needs to undergo a coagulation process. This process entails the collection and precipitation of particles that cause turbidity by adding coagulant [2]. The commonly used coagulant in water purification is liquid aluminium sulfate $(Al_2(SO_4)_3)$ [3].

The availability of liquid aluminium sulfate ensures the smooth operation of the production processes, including maintaining the quality of the water produced. Therefore, to ensure the availability of this coagulant, appropriate inventory management is needed, enabling water treatment companies to avoid supply shortages that can disrupt the production process. It also allows companies to manage costs effectively by avoiding last-minute purchases that are often more expensive [4].

The most suitable method for inventory management is a probabilistic model, because the demand for liquid aluminium sulfate by the production section at PDAM tends to be unstable but has a certain probabilistic distribution. However, the Fuzzy inventory model can be used, supposing the reverse was the case [5][6]. Several previous research on probabilistic inventory models reported that the demand for this coagulant had a normal probabilistic distribution, as in the research at PDAM Tirta Mayang in Jambi city and PDAM Nganjuk regency [7][8]. Inventory models with normal probabilistic levels assume that demand is symmetrically distributed, and comprised variations around the mean value. When requesting a normal distribution pattern, several methods are applied in various research fields, such as food, industry, engineering, etc, including simple probabilistic, *P*-Backorder, *P*-Lost Sales, and *Q*-backorder models, as well as *Q*-Lost Sales [9]–[11]. According to [12], the Probabilistic *Q* Model was used to solve inventory policy problems related to determining the amount of operational (operating stock) and safety stock. Back-Order *Q* Probabilistic Model is a development of a simpler model with a constant order size. However, [13] stated that orders are placed when the reorder point is reached.

The demand for liquid aluminium sulfate by the production department at PDAM passes through the warehouse and procurement sections. A model with a certain demand level represents a case occurring in several stages or phases, resulting in a skewed distribution towards the minimum or maximum value. A typical example of the model is erlang probability distribution; therefore, this research aims to compare the *Q* probabilistic inventory model with normal and erlang probability distribution.

The preferable model was expected to provide optimum policies in future inventory management, thereby requiring data for the current year. Meanwhile, future data could be a demanding forecast information. Forecasting is a method for quantitatively estimating events expected to occur in the future, based on related and relevant historical data from previous periods [14]. An example is the time series, which is divided into several methods, such as smoothing, used to adjust past data according to the seasonality. The process is carried out by averaging a series of data, ensuring it has a relatively balanced distance and equivalent amount. The smoothing method comprised both the Simple and Weighted Moving Average. Meanwhile, the Single and Double Exponential Smoothing had two parameters, namely Brown's Linear and Holt-Winter's methods [15]. Holt-Winter's method is divided into two, namely Additive and Multiplicative based on usage [16]. Another method is the Autoregressive Integrated Moving Average (ARIMA) model [17]-[18].

Previous research compared time series forecasting methods, including [19] Holt-Winter's Exponential Smoothing and Exponential Smoothing Event Based and concluded that the best was Holt-Winter's Exponential Smoothing with the smallest Mean Absolute Percentage Error (MAPE) value. This method is also more accurate than Moving Average, Recurrent Neural Network Model, ARIMA, the *R* forecasting, and *V*-forecasting. *R*-forecasting and *V*-forecasting are two methods used in Single Spectrum Analysis (SSA) for time series forecasting. These methods focus on extending patterns from historical data into future predictions [20]–[23]. Therefore, in this research, Holt-Winter's model used the demand forecast for liquid aluminium sulfate.

Research in PDAM on Q probabilistic inventory modeling generally assumes that the demand level parameters in the model have a normal probabilistic distribution, and the demand data is forecasted using simple linear regression analysis, such as in research [7][8][24]. In this context, Q stands for Quantity, representing the optimal number of units that should be ordered each time to balance costs. The results of demand forecasting data using simple linear regression analysis in PDAMs tend to be straight lines with a downward or upward trend without any data volatility. In this study, in PDAM Tirta Musi, the demand data is obtained from the forecast results using the Holt-Winter method so that the volatility of the data is represented. In the previous description, it has also been said that the Holt Winter Method is more accurate than the Moving Average, Recurrent Neural Network Model, ARIMA, R-forecasting, and V-forecasting [20]–[23]. This research also tests the erlang probability distribution assumption on the demand forecasting data using the Kolmogorov Smirnov method, then mathematically develops a Q probabilistic inventory model for erlang probability distributed demand. In addition, the inventory policy results of the Q erlang probabilistic inventory model will be compared with the inventory policy results of the normal Q probabilistic inventory model.

The results obtained are expected to serve as input in implementing optimal inventory policies useful in PDAM in terms of controlling the amount of liquid aluminium sulfate raw materials, as well as avoiding the risk of excess or shortage when needed. This research was conducted in early 2023. Therefore, the main objective is to predict the demand for liquid aluminium sulfate chemicals in PDAM from 2023 to 2024. This also includes implementing a more optimal policy, between the normal Q and erlang Q probabilistic models on the inventory of liquid aluminium sulfate chemicals in PDAM.

2. RESEARCH METHODS

The research method was carried out through several stages: obtaining research data, forecasting with demand data using the Holt-Winter method, developing a probabilistic inventory model with demand level having an Erlang probabilistic distribution, and obtaining an optimal solution to the inventory model.

2.1 Research Data

Observation data on liquid aluminium sulfate chemical demand was obtained from PDAM Tirta Musi. The data is in the form of monthly time series data totaling 84 data entries, from January 2016 to December 2022. Other information, in the form of cost data, include purchase, and storage costs, as well as costs and shortages of chemicals.

2.2 Forecasting with Holt-Winter Method

The procedure for generating a time series model and performing forecasting was stated as follows:

1. The time series data obtained from January 2016 to December 2022, was described in the form of graphs.
2. The initial value was determined using the first year of data, namely from January 2016 to December 2017. However, in this method, time series was obtained using a set of data realized over a certain period [25]. [26] stated that the initial value or initialization was obtained using the following equations.

$$S_l = \frac{1}{l}(X_1 + X_2 + \cdots + X_l) \tag{1}$$

$$b_l = \frac{1}{l}\left(\frac{(X_{l+1}-X_1)}{l} + \frac{(X_{l+2}-X_2)}{l} + \cdots + \frac{(X_{l+k}-X_k)}{l}\right) \tag{2}$$

$$I_{M_k} = \frac{X_k}{S_l} \tag{3}$$

$$I_{A_k} = X_k - S_l \tag{4}$$

   with S_l: the initial value of level smoothing, X_l: the l-th data, b_l: the initial value of trend smoothing, X_k: the k-th data, I_{A_k}: the initial value of the additive seasonal smoothing of the k-th year, and I_{M_k}: the initial value of the multiplicative seasonal smoothing of the k-th year..

3. Certain data refinement was determined using information from January 2017 to December 2022. Furthermore, Holt-Winter's additive or multiplicative method was based on three smoothing equations to obtain levels, trends, and seasonality in the data [16], [27]. The formulations are,

$$S_{At} = \delta\,(X_t - I_{A(t-l)}) + (1-\delta)\big(S_{A(t-1)} + b_{(t-1)}\big) \tag{5}$$

$$b_{At} = \beta\big(S_{At} - S_{A(t-1)}\big) + (1-\beta) b_{A(t-1)} \tag{6}$$

$$I_{At} = \gamma(X_t - S_{At}) + (1-\gamma) I_{A(t-l)} \tag{7}$$

$$S_{Mt} = \delta\,\frac{X_t}{I_{M(t-l)}} + (1-\delta)\big(S_{M(t-1)} + b_{M(t-1)}\big) \tag{8}$$

$$b_{Mt} = \beta\big(S_{Mt} - S_{M(t-1)}\big) + (1-\beta) b_{M(t-1)} \tag{9}$$

$$I_{Mt} = \gamma\,\frac{X_t}{S_{Mt}} + (1-\gamma) I_{M(t-l)} \tag{10}$$

with S_{At}: Level of the additive smoothing in year t, S_{Mt}: Level of the multiplicative smoothing in year t, b_{At}: Trend additive smoothing in year t, b_{Mt}: Trend multiplicative smoothing in year t, X_t: Data t-th, δ: Exponential smoothing weight constant ($0 < \delta < 1$), I_{At}: Additive seasonal smoothing in year t, I_{Mt}: Multiplicative seasonal smoothing in year t, and l: Length of season. The forecast and initial values of the m-th and t-th years for the additive and multiplicative model were calculated using **Equation (11)** or **Equations (12)**:

$$F_{A(t+m)} = S_{At} + b_{At} m + I_{A(t-l+m)} \tag{11}$$

$$F_{M(t+m)} = (S_{At} + b_{At} m) I_{A(t-l+m)} \tag{12}$$

with $F_{A(t+m)}$: The additive forecast for m for the future period from the year t-th, $F_{M(t+m)}$: The multiplicative forecast for m for the future period from the year t-th, and m: The period to be predicted.

4. Calculating the difference between the forecast value and the actual data is known as the forecast error. Additionally, the MAPE was used to measure the accuracy of forecasting models, by showing the extent of error compared to the actual value [16].

2.3 Back Order Probabilistic Inventory Modelling

After forecasting monthly data from January 2023 to December 2024, it was then used as annual data. This comprised all original and forecasted data from January 2016 to December 2022, and January 2013 to December 2024, respectively. The application of the annual data in Q model's Back Order probabilistic inventory model, comprised several stages, namely:

1. The Kolmogorov-Smirnov (KS) method was applied to test the probability distribution assumption on the liquid aluminium sulfate demand data. Equation (13) is used to test the hypotheses H_0: Residuals follow a certain distribution, and its opposite H_1: Residuals do not follow a certain distribution. In **Equation (13)**, X_i is the residual data sorted from smallest to largest. The rejection region for H_0 is defined by $KS\_\text{value} > KS_{\alpha,n}$ or $KS\_P\text{value} < \alpha$. The KS_value is given as:

$$KS\_\text{value} = \sup_x |F_n(x) - F_0(x)| \tag{13}$$

   where $F_n(x)$ is the cumulative probability of a specific distribution and $F_0(x)$ is the empirical cumulative probability of the tested data [28].

2. Formulating a probabilistic inventory model with normal and erlang probability distributed liquid aluminium sulfate chemical demand, obtained from the optimization of the total cost, using the following equation:

$$Tc = Pc + O_C + Sc + SHc \tag{14}$$

   with Tc: total cost, Pc: purchase cost, O_C: order cost, Sc: storage cost, and SHc: shortage cost. **Equation (15)** is stated as follows:

$$Tc = DP + \frac{BD}{Q} + H\left(\frac{1}{2}Q + r - D_L\right) + \frac{S_h D}{Q}\int_r^{\infty}(x-r) f(x)\,dx \tag{15}$$

   with D: demand rate (kg), P: price of goods (IDR), B: order cost (IDR), Q: number of goods once ordered (kg), r: number of goods stored when placing an order (kg), D_L: demand at *lead-time*, H: storage cost, S_h: shortage cost, x: random variable of demand, $f(x)$: demand probability density function.

3. Optimizing the formulation of the inventory model to obtain the appropriate solution using the exact method. In addition, this led to the realization of the optimal r and Q values. The exact method was determined by ensuring the first and second derivatives of Tc against r and Q, were equal to, and greater than zero. From $\frac{\partial Tc}{\partial r} = 0$ and $\frac{\partial Tc}{\partial Q} = 0$ obtained:

$$\int_r^{\infty} f(x)\,dx = \frac{HQ}{S_h D} = \alpha \tag{16}$$

and

$$Q = \sqrt{\frac{2D[B+S_h N]}{H}} \tag{17}$$

The optimal r and Q values were obtained through several iterations until it became convectional, by having a single solution. N is the sum of the defects, namely $N = \int_r^{\infty}(x-r)f(x)\,dx$.

4. Assuming the demand for liquid aluminium sulfate had a normal probability distribution, then the optimal solution can be obtained using **Equation (18)** and **Equation (19)**:

$$r = D_L + Z_\alpha S\sqrt{L} \tag{18}$$

and

$$N = \int_r^{\infty}(x-r)f(x)\,dx = S_L[f(Z_\alpha) - Z_\alpha \Psi(Z_\alpha)] \tag{19}$$

where Z_α: the value of the random variable Z at the critical value α, S: standard deviation, L: lead-time, S_L: standard deviation at the time of *lead-time*. $f(Z_\alpha)$: a function of the probabilistic concentration of the random variable Z_α, $\Psi(Z_\alpha)$: the partial expected value of the random variable Z_α.

5. The optimum solution can be determined when the demand for liquid aluminium sulfate has erlang distribution. Furthermore, **Equation (16)** and **Equation (17)** were used to obtain formulation r for erlang probability distributed demand as follows:

$$\int_r^{\infty} f(x)\,dx = \alpha$$

$$\Leftrightarrow 1 - \int_0^{r} f(x)\,dx = \alpha$$

$$\Leftrightarrow 1 - \frac{\gamma(k,\lambda r)}{(k-1)!} = \alpha$$

$$\Leftrightarrow \frac{\gamma(k,\lambda r)}{\Gamma(k)} = \frac{(1-\alpha)(k-1)!}{\Gamma(k)}$$

$$\Leftrightarrow r = \left[Sc.Gammaincinv\left(k, \frac{(1-\alpha)}{\Gamma(k)}\right)\right]/(\lambda) \tag{20}$$

with k: shape parameters and λ: rate parameters. There is also a scaler parameter, namely β, where the relationship with the rate is $\lambda = 1/\beta$. N formulation with erlang probability distributed requests was obtained using the following equation:

$$N = \int_r^{\infty}(x-r)f(x)\,dx$$

$$\Leftrightarrow N = \int_r^{\infty}(x-r)\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}\,dx \tag{21}$$

6. Compare the results of the optimum solution or policy obtained from the two probabilistic inventory models, with normal and erlang probability distribution demand. Additionally, the calculations were performed with the Minitab and Python software **[29]**.

A flowchart of the research method that specifically illustrates the use of the Holt-Winters forecasting method and the Q probabilistic inventory model can be seen in **Figure 1**.

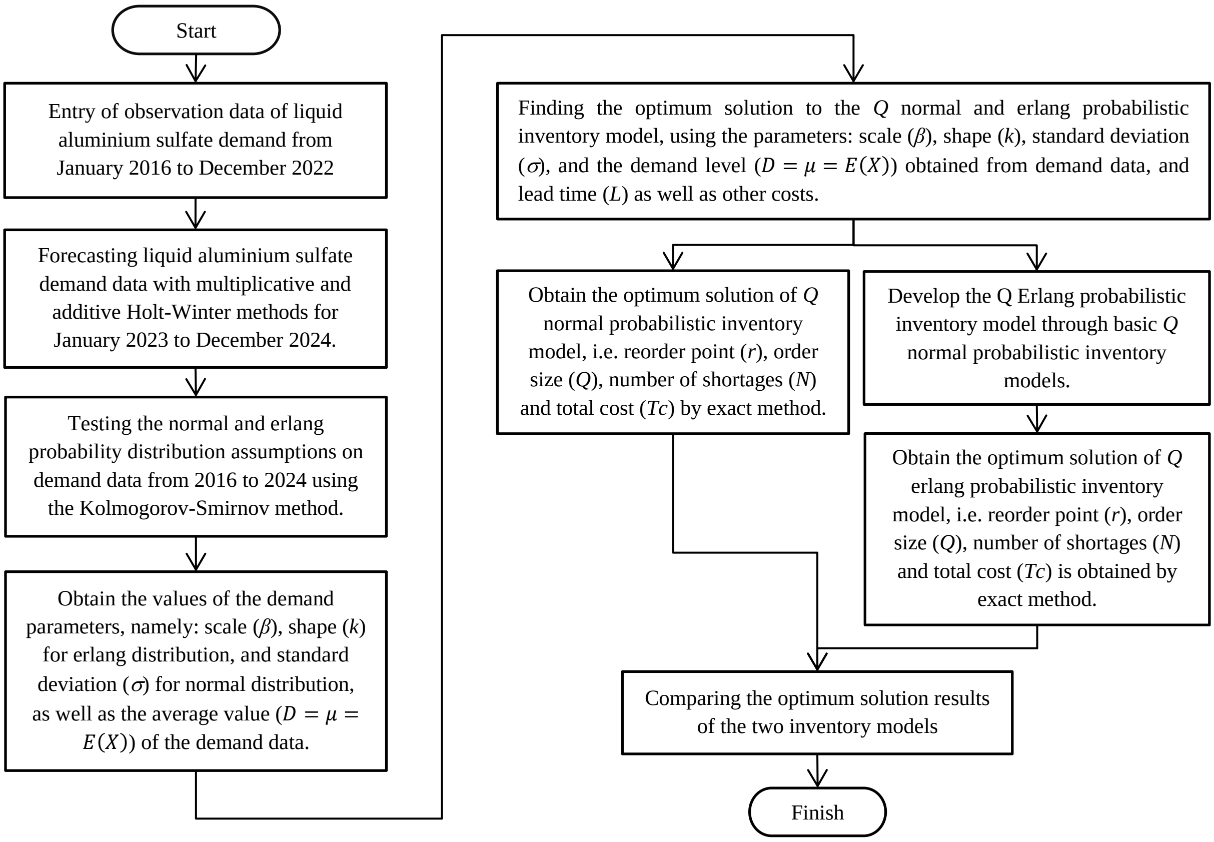


Figure 1. Flowchart of Research Method

3. RESULTS AND DISCUSSION

The results obtained were stated as follows.

3.1 Results of Forecasting Demand for Liquid Aluminium Sulfate Chemicals

The results or data description of the demand level for liquid aluminium sulfate from January 2016 to December 2022 are shown in Table 1.

Table 1. Demand Rate of Liquid Aluminium sulfate from January 2016 to December 2022

Years	Month	Demand Level for Liquid Aluminium sulfate (kilogram)
2016	January	541748.31
2016	February	525464.68
2016	March	476638.14
2016	April	519478.38
2016	May	496264.23
⋮	⋮	⋮
2022	November	658124.30
2022	December	636048.01

Data source*: PDAM Tirta Musi Palembang*

Initial values were calculated using Equation (1) to Equation (4). Meanwhile, level smoothing was determined as stated in Equations (1).

$$S_{12} = \frac{1}{12}\,(X_1 + X_2 + \cdots + X_L)$$

$$\Leftrightarrow S_{12} = \frac{1}{12}(541748.31 + 525464.68 + 476638.14 + \cdots + 475060.79)$$

$$\Leftrightarrow S_{12} = 504621.78 \tag{22}$$

The results of *trend* smoothing obtained using **Equations (2)** are stated as follows:

$$b_{12} = \frac{1}{12}\left(\frac{(X_{12+1} - X_1)}{12} + \frac{(X_{12+2} - X_2)}{12} + \cdots + \frac{(X_{12+12} - X_{12})}{12}\right)$$

$$\Leftrightarrow b_{12} = \frac{1}{12}\left(\frac{(494658.66 - 541748.31)}{12} + \cdots + \frac{(520983.5 - 475060.79)}{12}\right)$$

$$\Leftrightarrow b_{12} = -146.57 \tag{23}$$

The results of seasonal smoothing in the additive and multiplicative model used in **Equations (3)** and **Equations (4)** from I_1 to I_{12} and forms were stated as follows:

$$I_{A1} = X_1 - S_{12} = 541.748,31 - 504.621,78 = 37.126,53, \dots, I_{A12} = X_{12} - S_{12} = 475.060,79 - 504.621,78 = -29.560,99. \tag{24}$$

$$I_{M1} = \frac{X_1}{S_{12}} = \frac{541748.31}{504621.78} = 1.07, \dots, I_{M12} = \frac{X_{12}}{S_{12}} = \frac{475060.79}{504621.78} = 0.94 \tag{25}$$

The use of **Equation (5)** and **Equation (8)** to obtain data from January 2017 to December 2022, and the initial values determined with **Equations (22)** to **Equation (25)**, produced a forecast for January 2023 to December 2024. The results from smoothing the level from January 2017 to December 2022 using **Equation (5)** and **Equation (8)** are stated as follows $t = 13$ to $= 84$, or S_{13} to S_{84}. The values of the parameters δ, β, and γ were between zero and one. If we use the values of δ, β, and γ of 0.34, 0 and 0.53, respectively, we get

$$S_{A13} = 0.34\,(475060.79 - 37126.53) + (1 - 0.34)\big(504621.78 + (-146.57)\big) = 488514.56, \dots, S_{M84} = 636921.84 \tag{26}$$

$$S_{M13} = 0.34\,\frac{475060.79}{1.07} + (1 - 0.34)\big(504621.78 + (-146.57)\big) = 489611.77, \dots, S_{M84} = 637683.86. \tag{27}$$

Furthermore, smoothing the *trend* using **Equation (6)** and **Equation (9)** for b_{13} to b_{84} is stated as follows:

$$b_{A13} = 0(488514.56 - 504621.78) + (1 - 0) - 146.57 = -146.57, \cdots, b_{M84} = -146.57. \tag{28}$$

$$b_{M13} = 0(489611.77 - 504621.78) + (1 - 0)(-146.57) = -146.57, \dots, b_{M84} = -146.57. \tag{29}$$

Seasonal smoothing using **Equation (7)** and **Equation (10)** from I_{13} to I_{84} is stated as follows:

$$I_{A13} = 0.53(494658.66 - 488514.56) + (1 - 0,53)(-29.560,99) = 20705.84, \dots, I_{A84} = -5043.70 \tag{30}$$

$$I_{M13} = 0,53\,\frac{494658.66}{489611.77} + (1 - 0,53)(0.9)(1.07) = 1.04, \dots, I_{M84} = 0.99. \tag{31}$$

The forecast for January 2023 to December 2024 or 24 months, using **Equation (11)** and **Equation (12)**, with *m* values starting from 1 to 24, namely F_{85} to F_{108} is stated as follows:

$$F_{A85} = 636921.84 + (-146.57)(1) + 16484.83 = 653349.24, \dots, F_{A108} = 628507.00 \tag{32}$$

$$F_{M85} = (637683.86 - 146.57(1))1.01 = 645897.60, \dots, F_{M108} = 629208.34. \tag{33}$$

Meanwhile, *Minitab Software* was used to obtain the smallest MAPE value. Several tests were conducted to get a relatively small MAPE as shown in **Table 2**. The values *of* δ, β, and $\gamma = 0.34$, 0, and 0.53, respectively, with the holt-winter multiplicative method giving the smallest MAPE value of 5.35%. Then, the forecasting results with the Holt-winter multiplicative method will be used for the next stage.

Table 2. Holt-Winter's δ, β, and γ Exponential Smoothing Experiments using Microsoft Excel

δ	β	γ	*MAPE*	
			Additive Holt-Winter	**Multiplicative Holt-Winter**
0.34	0	0.53	5.4%	5.35%

δ	β	γ	MAPE	
			Additive Holt-Winter	Multiplicative Holt-Winter
0.55	0	0.51	5.43%	5.5%
0.65	0.1	0.43	5.87%	6.03%
0.85	0.44	0.23	7.48%	7.5%
0.17	0.52	0.33	8.17%	8.12%

***Data source**: Minitab Software Processing*

The plot of the forecast for 24 months is shown in Figure 2.

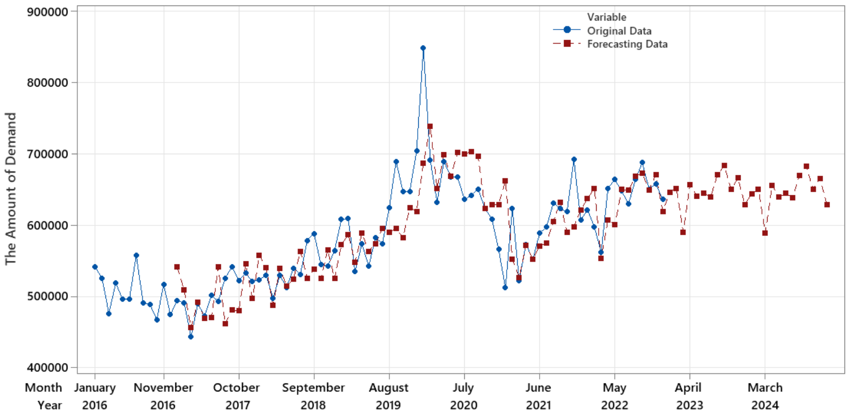


Figure 2. Time Series Graph of Liquid Aluminium sulfate for 2016-2024

Figure 2 described the time series data pattern of observation liquid aluminium sulfate demand data from January 2016 to December 2022 (blue line), along with 72 months of prediction data from January 2017 to December 2022 (red line), and 24 months of forecast data from January 2023 to December 2024 (red line too). From the diagram in Figure 1, the actual data has an up and down pattern, which indicates volatility. This pattern can be followed by the predicted data very well, and gives a MAPE value = 5.35%, meaning that the Holt-Winter Multiplicative Method with parameter values δ, β, and $\gamma = 0.34$, 0, and 0.53, respectively, has excellent criteria in forecasting data. The forecast results by this Holt-Winter Multiplicative method for 24 months from January 2023 to December 2024 are 645897.60 kg, ..., 629208.34 kg. In Figure 1, the graph of the 24-month forecast data from January 2023 to December 2024 appears to be constant at first glance, but it is not constant, there are increases and decreases. Volatility patterns cannot be shown when using simple linear regression analysis as in **[8][24]**.

3.2 Probabilistic Inventory Model Optimization Results

The KS test results on liquid aluminium sulfate demand data using **Equation (13)** obtained a KS_value of 0.2207 and KS_pvalue of 0.695 > α = 0.05 for normal probabilistic distribution testing, and KS_value of 0.2208, and KS_pvalue of 0.694 > α = 0.05 for Erlang distribution testing. Based on the KS test results, it is concluded that the liquid aluminium sulfate data met normal probabilistic distribution with an average parameter and Standard Deviation (S) of 7,132,585 kg and 754,130 kg, respectively. It also met the two-parameter erlang probabilistic distribution assumption, with a scale β, shape k, and average value of 73,809.45, 96.64, and 7,132,585.42 kg, respectively. Lead-Time (L) is defined as the waiting time when goods are shipped until it arrives at the company. In this case, PDAM company estimated that the goods would arrive in 2 days, equivalent to 0.005 years.

The value of S_L is equivalent to multiplying the standard deviation by the root of *lead time* to obtain 55.823. The purchase price per kilogram of chemicals (P) is the cost incurred to buy a kilogram of the chemical. In this case, PDAM provided an estimate for purchasing per kilogram including 11% VAT of IDR 1,736.17. Ordering costs (B) are the costs incurred whenever chemicals are being ordered. Regarding this situation, PDAM provided an estimated price of IDR 5,000 per order. Storage costs (H) were calculated from 10% of the unit price of goods (P) set by the company. The holding fee was IDR 173,617, while the cost of inventory shortage (S_h) was 105% of the chemical unit price (P), equivalent to IDR 1,822.97.

Research in PDAMs on the probabilistic Q inventory modeling problem generally assumes the demand level is standard normal probability distributed, such as in research at PDAMs in Nganjuk regency or PDAM Tirta Mayang in Jambi city [7][8][24]. This can be due to the ease of the inventory model used in obtaining optimum values. The cumulative probability values in the model can also be easily obtained through standard normal tables that are widely available.

For demand with a normal probabilistic distribution, such as in the [7][8][24] studies, the first step entailed calculating the initial Q (Q_0) value:

$$Q_0 = \sqrt{\frac{2AD}{h}} = \sqrt{\frac{2(5.000)(7.132.585)}{173,617}} = 20,268.7 \text{ kg} \tag{34}$$

Furthermore, the Q_0 was substituted in Equation (16) to obtain the α and r values as follows:

$$\alpha = \frac{HQ}{S_h D} = \frac{173,617\ (20.269\ )}{(1.822,97)\ (7,132,585)} = 0.00027 \tag{35}$$

$$r = D_L + Z_\alpha S\sqrt{L} = \left(\frac{2}{365} \times 7,132,585\right) + (3.5)\left(\sqrt{\frac{2}{365}} \times 754,130\right) = 199,289.48 \text{ kg} \tag{36}$$

Additionally, N value was calculated using Equation (19):

$$N = S_L[f(Z_\alpha) - Z_\alpha \Psi(Z_\alpha)] = 55,823[0.009 - 3.5(0.00006)] = 490.6 \text{ kg} \tag{37}$$

The values of α, r, and N were then substituted into Equation (17), as follows:

$$Q = \sqrt{\frac{2D[B+S_h N]}{H}} = \sqrt{\frac{2(7.132.585)[5.000+1.822,97(491)]}{173,617}} = 271,945.88 \text{ kg} \tag{38}$$

Iteration was carried out to obtain the optimal r value. This is applicable only when the r value obtained in the iteration is relatively the same. The optimum r and Q values obtained after the 4th iteration were equal to:

$$r = 187,014,\ Q = 270,842,\ ss = z_\alpha S\sqrt{L} = 147,931.49 \text{ kg}, \tag{39}$$

and service levels of

$$\eta = 1 - \frac{N}{D_L}(100\%) = 1 - \frac{487}{\frac{2}{365}(7,132,585)}(100\%) = 98.75\%. \tag{40}$$

The optimum total cost expectation obtained using Equation (15) is Tc = IDR 12,507,086,283.87.

Using the same procedure as in the normal probabilistic distribution, the optimum value obtained after the 5th iteration in erlang probabilistic distribution is $\alpha = 0.0017$, while r and N were obtained using Equations (20) and (21) as follows:

$$r = \frac{\left[Sc.Gammaincinv\left(k, \frac{(1-\alpha)}{\Gamma(k)}\right)\right]}{(\lambda)} = \frac{\left[Sc.Gammaincinv\left((96.64), \frac{(1-0.0017)}{\Gamma(96.64)}\right)\right]}{(1/73.809.45)} = 28597.09 \text{ kg} \tag{41}$$

$$N = \int_r^\infty (x - r)\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}dx$$

$$\Leftrightarrow N = \int_{28597.09}^{\infty} (x - 28597.09)\frac{\left(\frac{1}{73.809.45}\right)^{96.64} x^{96.64-1} e^{-\left(\frac{1}{73.809.45}\right)x}}{(96.64-1)!}dx$$

$$\Leftrightarrow N = 109.63 \text{ kg} \tag{42}$$

The optimum Q value of 129739.58 kg was obtained using Equation (17), with an $ss = 0$ kg depicting it does not require a safety stock. Furthermore, the service level value was $\eta = 99.72\%$, with the total cost obtained being $Tc = IDR$ 12,455,085,352.0051. The total cost of erlang Q model is smaller than the normal Q model. Meanwhile, the service level of erlang Q model is 0.97% higher. The calculations were performed using the Python software in the *math* and *scipy libraries*. The comparison results of the two Q models, namely the normally distributed demand inventory and erlang, are shown in Figure 3 and Figure 4.

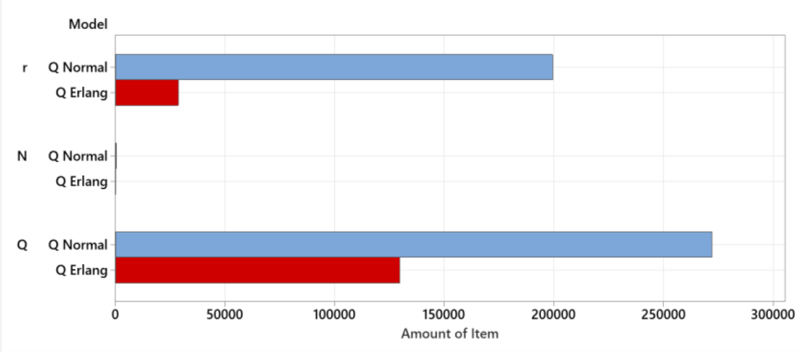
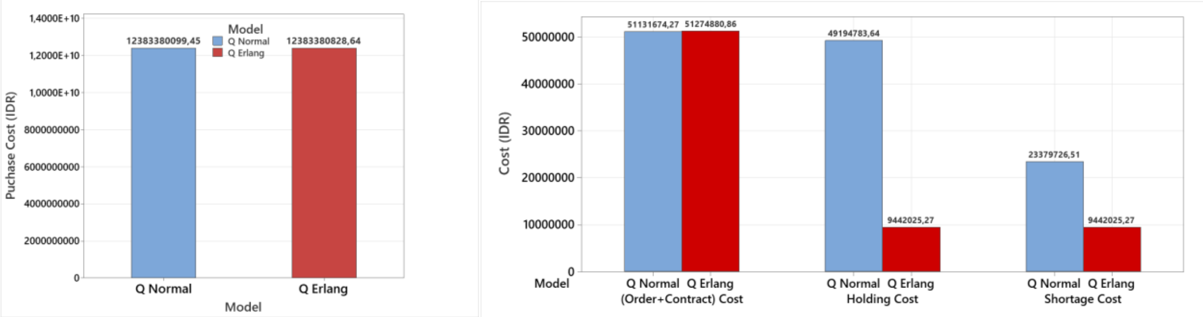


Figure 3. Comparison of Reorder Points, Number of Material Shortages, and Purchase Quantities



(a) **(b)**

Figure 4. The Amount of Costs, (a) Purchasing, (b) Ordering, Storing and Shortages

In **Figure 3**, the reorder point, number of shortages, and amount per purchase for erlang *Q* model were less than the normal *Q* model. In respect to the cost comparison shown in **Figure 4**, both orders, and holding costs, including shortages for erlang *Q* model were less than the normal *Q* model. However, the purchase costs for both models were relatively the same.

Based on the results of this study, which developed a probabilistic *Q* inventory model with the demand level having an Erlang distribution, it is expected to provide a choice of probabilistic *Q* inventory model because it can provide more optimal results, especially for PDAM companies.

4. CONCLUSIONS

This research provides the following conclusions: Forecasting liquid aluminium sulfate demand data at PDAM Tirta Musi can be approached using additive and multiplicative Holt-Winter methods with excellent criteria. The forecasting demand level data results with the Holt-Winter multiplicative method provide the smallest MAPE value, so the forecasting results are used to form a probabilistic inventory model Q. The development of the Q probabilistic inventory model with uncertain demand is done mathematically from the basic Q probabilistic inventory model with normal probability distributed demand. The developed model is assumed to have an erlang distribution and has fulfilled the assumption test using the KS method. A comparison of both normal and erlang probabilistic inventory models shows that the erlang probabilistic inventory model provides a more optimal policy solution than the normal probabilistic inventory model, with a minimum total cost and a higher service level.

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