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THE PARTITION DIMENSION OF CYCLE BOOKS GRAPH $B_{m,n}$ WITH A COMMON PATH P_2

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ABSTRACT

Suppose G is a connected graph with v elements of a set of vertices G denoted by $V(G)$ and S a subset of $V(G)$. The distance between v and S is the shortest distance v to every vertex x in S. Let ζ be a partition of $V(G)$, where each subset S belongs to ζ. The representation of a vertex v with respect to ζ is defined as the set of distances from v to each vertex in S. If each representation of each vertex of $V(G)$ is different, then the partition ζ is called the resolving partition of G, and the partition dimension $pd(G)$ is the smallest integer k such that G has a resolving partition with k members. In this research, we show the partition dimensions of the cycle books graph $B_{m,n}$. Cycle books graph $B_{m,n}$ is a graph consisting of m copies of a cycle with a common path P_2. The partition dimension of the cycle books graph $B_{m,n}$ for $m \geq 6$ and $n \geq 2$ is shown.

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1. INTRODUCTION

Partition dimension in a graph is an exciting topic for researchers in the field of discrete mathematics, especially in graph theory. Several researchers have examined partition dimensions in a graph. Chartrand et al. [1] introduced the concept of partition dimension of graph G. Let G be a graph with $v \in V(G)$ and $S \subset V(G)$. The distance between v and S, denoted by $d(v,S)$, is the minimum of $d(v,x)$ where $x \in S$ and $d(v,x)$ is the distance between v and x. Given $\zeta = \{S_1, S_2, S_3, \dots, S_k\}$ is a partition of $V(G)$, the representation of the vertices of v with respect to ζ is $r(v|\zeta) = (d(v,S_1), d(v,S_2), \dots, d(v,S_k))$. If for any two vertices $u, v \in V(G)$ we have $r(u|\zeta) \neq r(v|\zeta)$, then it is called the differentiating partition of $V(G)$. The differentiating partition with the minimum cardinality is the minimum differentiating partition of G called the partition dimension, which is denoted by $pd(G)$.

Some of the research that underlies determining partition dimensions in several graphs are as follows. Chartrand *et al.* [1] have shown that a connected graph G of order $n \geq 2$ has partition dimensions $pd(G) = 2$ if and only if $G = P_n$ and $pd(G) = n$ if and only if $G = K_n$. Vertana et al. [2] determine the partition dimension of a $C_m + P_n$ graph for $m \geq 3$ and $n \geq 2$. Amrullah et al. [3] have studied the topic of partition dimensions of a subdivision of a complete graph. Haryeni et al. [4] have shown that the partition dimension of disconnected graphs. Yero [5] introduced the concepts of strong resolving partition and strong partition dimension. Darmaji and Alfarisi [6] has shown that the partition dimension of the comb product of the path and complete graph, Asim *et al.* [7] determined the 2-partition dimension of the circulant graphs. Khabyah [8] has shown the partition dimension of COVID antiviral drug structures. Nadeem et al. [9] introduced a 2-partition dimension of rotationally symmetric graphs, and Hasmawati et al. [10] have shown that the partition dimension of the vertex amalgamation of some cycles. The study by Jia et al. in [11] explored bounds on the partition dimension of convex polytopes. Partition dimensions of fullerene graphs, certain classes of series-parallel graphs, certain honeycomb networks, grid graphs, rainbow vertex connection, chain graph and circulant graphs have been discussed in [12], [13], [14], [15], [16], [17] and [18]. The following are some of the latest studies related to graph.

The cycle books graph $B_{m,n}$, as defined by Santoso and Darmaji in [19],[20] is a graph consisting of n copies of the cycle C_m with a common path P_2. Several studies have investigated various properties of cycle books graphs. Swita [21] examined the edge-magic total labeling of cycle books. Simanihuruk et al. [22] studied a conjecture on the super edge-magic total labeling of 4-cycle books. Furthermore, Santoso and Darmaji [19] determined the partition dimensions of the cycle books graph for $B_{C_3,n}$, $B_{C_4,n}$ and $B_{C_5,n}$.

Based on the existing research, this study aims to establish the partition dimension of the cycle books graph $B_{m,n}$ for $n \geq 6$, the result has contribution but definitely not significant, since it just part of class of graphs.

2. RESEARCH METHODS

This research used a literature review from papers and known research on partition dimension on graphs as the research method. The following provides the basic concepts needed in the construction of the concept of the ideal. The cycle books graph is a graph consisting of m copies of the cycle with a common path P_2. The partition dimensions of the cycle books graph for $B_{C_3,m}$, $B_{C_4,m}$, and $B_{C_5,m}$, which have been proven in previous research on [19] are shown in the following theorem.

Theorem 1. [19] *Let G be the cycle books graph $B_{C_3,m}$ for $m = 2,3,4,\dots$ Then, the partition dimension of G is given by*

$$pd(G) = \begin{cases} 3 & for\ m = 2,3; \\ m & for\ m \geq 4. \end{cases}$$

Theorem 2. [19] *Let G be the cycle books graph $B_{C_4,m}$ for $m = 2,3,4,\dots$ Then, the partition dimension of G is given by*

$$pd(G) = \begin{cases} 3+2k & for\ m = 3k+2; \\ 4+2(k-1) & for\ m = 3k+1; \\ 3+2(k-1) & for\ m = 3k. \end{cases}$$

Theorem 3. [19] *Let G be the cycle books graph $B_{C_5,m}$ for $m = 2,3,4,\ldots$. Then, the partition dimension of G is given by $pd(G) = m+1$.*

In this research, the following theorem is used to determine the lower bound of the metric dimension of a particular graph.

Theorem 4. [1] *Let G be a connected graph of order $n \geq 2$. Then*

i) $pd(G) = 2$ *if and only if* $G = P_n$,

ii) $pd(G) = n$ *if and only if* $G = K_n$.

3. RESULTS AND DISCUSSION

In this section, we will show the general form of partition dimensions in cycle book graphs. Before we prove the graph's partition dimensions, we will show examples of partition dimensions in cycle graphs.

Example 1. Given a cycle books graph $B_{4,5}$ shown in **Figure 1**, we will determine the partition dimensions of $B_{4,5}$. The cycle books graph $B_{4,5}$ has vertices $V(G) = \{v_{i,j}; i = 1,2,3,4,5; j = 1,2\} \cup \{v_{c_1}, v_{c_2}\}$. Let $\zeta = \{S_1, S_2, S_3, S_4\}$ with $S_1 = \{v_{c_1}, v_{1,1}, v_{2,1}, v_{3,1}\}$, $S_2 = \{v_{c_2}, v_{1,2}, v_{5,2}\}$, $S_3 = \{v_{2,2}, v_{4,1}, v_{5,1}\}$ and $S_4 = \{v_{3,2}, v_{4,2}\}$. The representation of all vertices with respect to ζ are as follows.

$$\begin{array}{lll} r(v_{c_1}|\zeta) = (0,1,1,2) & r(v_{2,1}|\zeta) = (0,2,1,3) & r(v_{4,1}|\zeta) = (1,2,0,1) \\ r(v_{c_2}|\zeta) = (1,0,1,1) & r(v_{2,2}|\zeta) = (1,1,0,2) & r(v_{4,2}|\zeta) = (2,1,1,0) \\ r(v_{1,1}|\zeta) = (0,1,2,3) & r(v_{3,1}|\zeta) = (0,2,2,1) & r(v_{5,1}|\zeta) = (1,1,0,3) \\ r(v_{1,2}|\zeta) = (1,0,2,2) & r(v_{3,2}|\zeta) = (1,1,2,0) & r(v_{5,2}|\zeta) = (2,0,1,2) \end{array}$$

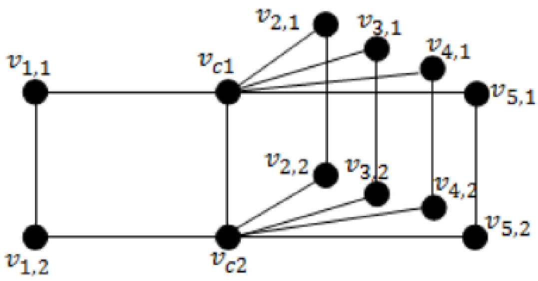

Figure 1. Cycle Books Graph $B_{4,5}$

From the above representation, it is obtained that ζ is resolving the partition of $B_{4,5}$. Therefore, the upper bound of the $B_{4,5}$ is $pd(G) \leq 4$. To determine the lower bound of the partition dimension of $B_{4,5}$, take a differentiating partition with a cardinality smaller than 4. Let $\zeta = \{S_1, S_2, S_3\}$ with $S_1 = \{v_{c_1}, v_{1,1}, v_{2,1}, v_{3,1}\}$, $S_2 = \{v_{c_2}, v_{1,2}, v_{5,2}\}$ and $S_3 = \{v_{2,2}, v_{3,2}, v_{4,1}, v_{4,2}, v_{5,1}\}$. There is the same representation of the vertex $V(G)$ with respect to ζ, $r(v_{2,2}|\zeta) = r(v_{5,1}) = (1,1,0)$ so that $\zeta = \{S_1, S_2, S_3\}$ is not a resolving the partition of $V(G)$ then $pd(G) \geq 4$. So, it can be concluded that the partition dimension of $B_{4,5}$ is $pd(B_{4,5}) = 4$.

Next, we will discuss the general form of the partition dimensions of $B_{m,n}$ in the following theorem. This result serves as a direct continuation of **Theorem 1** to **Theorem 3**, refining the partition dimension constraints established earlier.

Theorem 5. *Let $B_{m,n}$ be the cycle books graph, where $m \geq 6$ and $n \geq 2$. If $k \in \mathbb{N}$, then the partition dimension of $B_{m,n}$ is given by*

$$pd(B_{m,n}) = \begin{cases} 3 & if \quad n = 2,3; \\ n & if \quad n = 4; \\ k+3 & if \quad n = 2k+3 \; and \; n = 2k+4. \end{cases}$$

Proof. To prove this theorem, we will divide into six cases. For $m \geq 6, n = 2,3$; $m \geq 6, n = 4$; m is odd and $n = 2k+3$; m is odd and $n = 2k+4$; m is even and $n = 2k+3$; and for m is even and $n = 2k+4$.

Case 1. For $m \geq 6$ and $n = 2,3$. We will prove that $pd(G) = 3$. For $n = 2$ by choosing $\zeta = \{S_1, S_2, S_3\}$ and $S_1 = \{v_{c_1}, v_{2,1}\} \cup \{v_{i,j}; 1 \leq j \leq m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,j}; j = m-2\}$; and $S_3 = \{v_{2,j}; 2 \leq j \leq m-2\}$, then representation of all vertices $v \in V(G)$ with respect to ζ are as follows: $r(v_{c_1}|\zeta) = (0,1,2)$, $r(v_{c_2}|\zeta) = (1,0,1)$, the representation of each vertex for odd m is

$r(v_{1,j}|\zeta) = (0, j+1, j+2); 1 \leq j \leq \frac{m-1}{2} - 1$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j); \frac{m-1}{2} \leq j \leq m-3$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0); 2 \leq j \leq \frac{m-1}{2} - 1$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0); \frac{m-1}{2} + 1 \leq j \leq m-2$

$r(v_{1,j}|\zeta) = (1,0,m-j); j = m-2$

$r(v_{2,j}|\zeta) = (0, j+1, j); j = 1$

$r(v_{2,j}|\zeta) = (j-1, j, 0); j = \frac{m-1}{2}$

the representation of each vertex for even m is

$r(v_{1,j}|\zeta) = (0, j+1, j+2); 1 \leq j \leq \frac{m}{2} - 2$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j); \frac{m}{2} \leq j \leq m-3$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0); 2 \leq j \leq \frac{m}{2} - 1$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0); \frac{m}{2} + 1 \leq j \leq m-2$

$r(v_{1,j}|\zeta) = (0, j, j+2); j = \frac{m}{2} - 1$

$r(v_{1,j}|\zeta) = (1,0,m-j); j = m-2$

$r(v_{2,j}|\zeta) = (0, j+1, j); j = 1$

$r(v_{2,j}|\zeta) = (j-1, j-1, 0); j = \frac{m}{2}$

For $n = 3$ by choosing $\zeta = \{S_1, S_2, S_3\}$ and $S_1 = \{v_{c_1}, v_{2,1}\} \cup \{v_{i,j}; 1 \leq j \leq m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,j}; j = m-2, v_{3,j}; 2 \leq j \leq m-2\}$; and $S_3 = \{v_{2,j}; 2 \leq j \leq m-2\}$, then representation of all vertices $v \in V(G)$ with respect to Z are follows, $r(v_{c_1}|\zeta) = (0,1,1)$, $r(v_{c_2}|\zeta) = (1,0,1)$, the representation of each vertex for odd m is

$r(v_{1,j}|\zeta) = (0, j+1, j+1); 1 \leq j \leq \frac{m-1}{2} - 1$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j); \frac{m-1}{2} \leq j \leq m-3$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0); 2 \leq j \leq \frac{m-1}{2} - 1$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0); \frac{m-1}{2} + 1 \leq j \leq m-2$

$r(v_{3,j}|\zeta) = (m-j, 0, m-j); \frac{m-1}{2} + 1 \leq j \leq m-2$

$r(v_{1,j}|\zeta) = (1,0,m-j); j = m-2$

$r(v_{2,j}|\zeta) = (0, j+1, j); j = 1$

$r(v_{2,j}|\zeta) = (j-1, j, 0); j = \frac{m-1}{2}$

$r(v_{3,j}|\zeta) = (j, j, 0); j = 1$

$r(v_{3,j}|\zeta) = (j, 0, j-1); 2 \leq j \leq \frac{m-1}{2}$

the representation of each vertex for even m is

$r(v_{1,j}|\zeta) = (0, j+1, j+2); 1 \leq j \leq \frac{m}{2} - 2$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j); \frac{m}{2} \leq j \leq m-3$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0); 2 \leq j \leq \frac{m}{2} - 1$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0); \frac{m}{2} + 1 \leq j \leq m-2$

$r(v_{1,j}|\zeta) = (0, j, j+1); \frac{m}{2} - 1$

$r(v_{1,j}|\zeta) = (1,0,m-j); j = m-2$

$r(v_{2,j}|\zeta) = (0, j+1, j); j = 1$

$r(v_{2,j}|\zeta) = (j-1, j-1, 0); \; j = \frac{m}{2}$

$r(v_{3,j}|\zeta) = (j, 0, j-1); 2 \le j \le \frac{m}{2}$ $\quad r(v_{3,j}|\zeta) = (j, j, 0);\ j = 1$

$r(v_{3,j}|\zeta) = (m-j, 0, m-j); \frac{m}{2}+1 \le j \le m-2.$

Since all vertices of $B_{m,n}$ have distinct representations, so ζ is the resolving partition of $V(B_{m,n})$ and refers to [1] that the partition dimension of a graph is two if and only if it is a path. Then **case 1** proves that $pd(B_{m,n}) = 3$ for $m \ge 6, n = 2,3$.

Case 2. For $m \ge 6$ and $n = 4$. To determine the upper bounds of the partition dimension select $\zeta = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{v_{c_1}\} \cup \{v_{1,j}; 1 \le j \le m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,m-2}, v_{4,j}; 2 \le j \le m-2\}$, $S_3 = \{v_{2,j}; 2 \le j \le m-2, v_{3,j}; 1 \le j \le m-3, v_{4,1}\}$ and $S_4 = \{v_{3,m-2}\}$, the representation vertices $v \in V(G)$ with respect to ζ are as follows: $r(v_{c_1}|\zeta) = (0,1,1,2)$, $r(v_{c_2}|\zeta) = (1,0,1,1)$, the representation of each vertex for odd m is

$r(v_{1,j}|\zeta) = (0, j+1, j+1, j+2); 1 \le j \le \frac{m-1}{2} - 1$ $\quad r(v_{1,j}|\zeta) = (1, 0, m-j, m-j); j = m-2$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j, m-j); \frac{m-1}{2} \le j \le m-3$ $\quad r(v_{2,j}|\zeta) = (0, j+1, j+2); j = 1$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0, j+2); 2 \le j \le \frac{m-1}{2} - 1$ $\quad r(v_{2,j}|\zeta) = (j-1, j, 0, j+1); j = \frac{m-1}{2}$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0, m-j);\ \frac{m-1}{2}+1 \le j \le m-2$

$r(v_{3,j}|\zeta) = (j, j+1, 0, j+2); 1 \le j \le \frac{m-1}{2} - 2$

$r(v_{3,j}|\zeta) = (j, j+1, 0, j+1); j = \frac{m-1}{2} - 1$ $\quad r(v_{3,j}|\zeta) = (j, j, 0, j-1); j = \frac{m-1}{2}$

$r(v_{3,j}|\zeta) = (m-j, m-j-1, 0, m-j-2);\ \frac{m-1}{2}+1 \le j \le m-3$

$r(v_{3,j}|\zeta) = (m-j, m-j-1, m-j-1, 0); j = m-2$ $\quad r(v_{4,j}|\zeta) = (j, j, 0, j+2); j = 1$

$r(v_{4,j}|\zeta) = (j, 0, j-1, j+2); 2 \le j \le \frac{m-1}{2} - 1$ $\quad r(v_{4,j}|\zeta) = (j, 0, j-1, j+1);\ j = \frac{m-1}{2}$

$r(v_{4,j}|\zeta) = (m-j, 0, m-j, m-j); \frac{m-1}{2}+1 \le j \le m-2$

the representation of each vertex for even m is

$r(v_{1,j}|Z) = (0, j+1, j+1, j+2); 1 \le j \le \frac{m}{2} - 2$ $\quad r(v_{1,j}|\zeta) = (0, j, j+1, j+2); j = \frac{m}{2} - 1$

$r(v_{1,j}|\zeta) = (0, m-j-2, m-j, m-j); \frac{m}{2} \le j \le m-3$ $\quad r(v_{1,j}|\zeta) = (1, 0, m-j, m-j); j = m-2$

$r(v_{2,j}|\zeta) = (j-1, j+1, 0, j+2); 2 \le j \le \frac{m}{2} - 1$ $\quad r(v_{2,j}|\zeta) = (0, j+1, j, j+2); j = 1$

$r(v_{2,j}|\zeta) = (m-j, m-j-1, 0, m-j); \frac{m}{2}+1 \le j \le m-2$

$r(v_{2,j}|\zeta) = (j-1, j-1, 0, j+1);\ j = \frac{m}{2}$

$r(v_{3,j}|\zeta) = (j, j+1, 0, j+2); 1 \le j \le \frac{m}{2} - 2$ $\quad r(v_{3,j}|\zeta) = (j, j+1, 0, j); j = \frac{m}{2} - 1$

$r(v_{3,j}|\zeta) = (m-j, m-j-1, 0, m-j-2);\ \frac{m}{2} \le j \le m-3$

$r(v_{3,j}|\zeta) = (m-j, m-j-1, m-j-1, 0); j = m-2$ $\quad r(v_{4,j}|\zeta) = (j, j, 0, j+2); j = 1$

$r(v_{4,j}|\zeta) = (j, 0, j-1, j+2); 2 \le j \le \frac{m}{2} - 1$ $\quad r(v_{4,j}|\zeta) = (j, 0, j-1, j);\ j = \frac{m}{2}$

$r(v_{4,j}|\zeta) = (m-j, 0, m-j, m-j); \frac{m}{2}+1 \le j \le m-2.$

Thus, ζ is the resolving partition of $B_{m,n}$ since $r(v|\zeta)$ is unique for each $v \in B_{m,n}$. Consequently, the upper bound of the partition dimension of the $B_{m,n}$ is $pd(B_{m,n}) \le 4$. The same method as in the previous case is used to determine the lower bound. So, we get $pd(B_{m,n}) = 4$.

Case 3. For m is odd and $n = 2k + 3$. To determine the upper bounds of the partition dimension select $\zeta = \{S_1, S_2, S_3, \dots, S_{3+k}\}$ where $S_1 = \{v_{c_1}\} \cup \{v_{1,j}, v_{2,j}; 1 \le j \le m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,m-2}, v_{n-1,m-2}, v_{n,j}; 2 \le j \le m-2\}$, $S_3 = \{v_{2,j}; 2 \le j \le m-2, v_{3,j}; 1 \le j \le m-3, v_{4,1}\}$, $S_4 = \{v_{6,1}\} \cup \{v_{3,m-2}, v_{4,j}; 2 \le j \le m-2, v_{5,j}; 1 \le j \le m-3\}$, $S_5 = \{v_{5,m-2}, v_{6,j}; 2 \le j \le m-2, v_{7,j}; 1 \le j \le m-3, v_{8,1}\}$, $S_6 = \{v_{7,m-2}, v_{8,j}; 2 \le j \le m-2, v_{9,j}; 1 \le j \le m-3, v_{10,1}\}$,…,$S_{3+k} = \{v_{n-2,m-2}, v_{m-1,j}; 2 \le j \le m-3, v_{m,1}\}$. The representation vertices $v \in V(B_{m,n})$ with respect to ζ are follows: $r(v_{c_1}|\zeta) = (0, \underbrace{1, \dots, 1}_{k+2})$, $r(v_{c_2}|\zeta) = (1, 0, \underbrace{1, \dots, 1}_{k+1})$

$$r(v_{1,j}|\zeta) = \left(0, \underbrace{j+1, \dots, j+1}_{k+2}\right); 1 \le j \le \frac{m-1}{2} - 1 \qquad r(v_{1,j}|\zeta) = \left(1, j-1, \underbrace{j+1, \dots, j+1}_{k+1}\right); j = \frac{m-1}{2}$$

$$r(v_{1,j}|\zeta) = \left(0, m-j-2, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m-1}{2} \le j \le m-3$$

$$r(v_{1,j}|\zeta) = \left(1, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); j = m-2 \qquad r(v_{2,j}|\zeta) = \left(0, 2, 1, \underbrace{j+1, \dots, j+1}_{k}\right); j = 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j+1, 0, \underbrace{j+1, \dots, j+1}_{k}\right); 2 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j, 0, \underbrace{j+1, \dots, j+1}_{k}\right); j = \frac{m-1}{2}$$

$$r(v_{2,j}|\zeta) = \left(j-1, j-2, 0, \underbrace{j-1, \dots, j-1}_{k}\right); j = \frac{m-1}{2} + 1$$

$$r(v_{2,j}|\zeta) = \left(m-j, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k}\right); \frac{m-1}{2} + 2 \le j \le m-2$$

the representation of each vertex for $3 \le i \le m-2$ and i is odd follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{\frac{i-3}{2}}, 0, j-1, \underbrace{j+1, \dots, j+1}_{k-1-\frac{i-3}{2}}\right); j = \frac{m-1}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, 0, m-j-2, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); \frac{m-1}{2} + 1 \le j \le m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); j = m-2$$

the representation of each vertex for $3 \le i \le m-2$ and i is even follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, 0, 1, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); j = 1$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); 2 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{\frac{i-4}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-1-\frac{i-4}{2}}\right); j = \frac{m-1}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-\frac{i-2}{2}}\right); \frac{m-1}{2} + 1 \le j \le m-3$$

the representation of each vertex for $3 \le i \le m-2$ and i is odd follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{\frac{i-3}{2}}, 0, j-1, \underbrace{j+1, \dots, j+1}_{k-1-\frac{i-3}{2}}\right); j = \frac{m-1}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, 0, m-j-2, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); \frac{m-1}{2} + 1 \le j \le m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-2}{2}}\right); j = m-2$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{k}, 0, 1\right); \; j = 1$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{k}, j-1, 0\right); 2 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{n-1,j}|\zeta) = \left(\frac{m-1}{2}, \frac{m-1}{1} - 1, \underbrace{m-j, \dots, m-j}_{k-1}, j-1, 0\right); \; j = \frac{m-1}{2}$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, m-j-2, \underbrace{m-j, \dots, m-j}_{k}, 0\right); \; \frac{m-1}{2} + 1 \le j \le m-3$$

$$r(v_{n-1,j}|\zeta) = \left(j-1, 0, \underbrace{m-j, \dots, m-j}_{k}, 1\right); \; j = m-2$$

$$r(v_{n,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k}, 0\right); \; j = 1$$

$$r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j+1, \dots, j+1}_{k}, j-1\right); 2 \le j \le \frac{m-1}{2}$$

$$r(v_{n,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m-1}{2} + 1 \le j \le m-2.$$

So ζ is the resolving partition of $B_{m,n}$ because $r(v|\zeta)$ is different for each $v \in B_{m,n}$. Therefore, the upper bound of the partition dimension of the $B_{m,n}$ is $pd(B_{m,n}) \le k+3$. To determine the lower bound of the partition dimension, take a differentiating partition with cardinality smaller than $k+3$, $\zeta = \{S_1, S_2, S_3, \dots, S_{2+k}\}$. Then, there is the same representation of vertices $v \in B_{m,n}$ with results $pd(B_{m,n}) \geq k+3$. So, we get $pd(B_{m,n}) = k+3$.

Case 4. For m is odd and $n = 2k+4$. To determine the upper bounds of the partition dimension, select $\zeta = \{S_1, S_2, S_3, \dots, S_{3+k}\}$ where $S_1 = \{v_{c_1}, v_{2,1}\} \cup \{v_{1,j}; 1 \leq j \leq m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,m-2}, v_{n-1,m-2}, v_{n,j}; 2 \leq j \leq m-2\}$, $S_3 = \{v_{4,1}\} \cup \{v_{2,j}; 2 \leq j \leq m-2, v_{3,j}; 1 \leq j \leq m-3\}$, $S_4 = \{v_{6,1}\} \cup \{v_{3,m-2}, v_{4,j}; 2 \leq j \leq m-2, v_{5,j}; 1 \leq j \leq m-3\}$, $S_5 = \{v_{5,m-2}, v_{6,j}; 2 \leq j \leq m-2, v_{7,j}; 1 \leq j \leq m-3, v_{8,1}\}$, $S_6 = \{v_{10,1}\} \cup \{v_{7,m-2}, v_{8,j}; 2 \leq j \leq m-2, v_{9,j}; 1 \leq j \leq m-3\}$,…,$S_{3+k} = \{v_{n-3,m-2}, v_{m-2,j}; 2 \leq j \leq m-2, v_{m-1,j}; 1 \leq j \leq m-3, v_{m,1}\}$. The representation vertices $v \in V(B_{m,n})$ with respect to ζ are as follows:

$r(v_{c_1}|\zeta) = (0, \underbrace{1, \dots, 1}_{k+2})$ $\qquad r(v_{c_2}|\zeta) = (1, 0, \underbrace{1, \dots, 1}_{k+1})$

$$r(v_{1,j}|\zeta) = \left(0, \underbrace{j+1, \dots, j+1}_{k+2}\right); 1 \leq j \leq \frac{m-1}{2} - 1 \qquad r(v_{1,j}|\zeta) = \left(0, j-1, \underbrace{j+1, \dots, j+1}_{k+1}\right); j = \frac{m-1}{2}$$

$$r(v_{1,j}|\zeta) = \left(0, m-j-2, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m-1}{2} + 1 \leq j \leq m-3$$

$$r(v_{1,j}|\zeta) = \left(1, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); j = m-2 \qquad r(v_{2,j}|\zeta) = \left(0, 2, 1, \underbrace{j+1, \dots, j+1}_{k}\right); j = 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j+1, 0, \underbrace{j+1, \dots, j+1}_{k}\right); 2 \leq j \leq \frac{m-1}{2} - 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j, 0, \underbrace{j+1, \dots, j+1}_{k}\right); j = \frac{m-1}{2}$$

$$r(v_{2,j}|\zeta) = \left(j-1, j-2, 0, \underbrace{j-1, \dots, j-1}_{k}\right); j = \frac{m-1}{2} + 1$$

$$r(v_{2,j}|\zeta) = \left(m-j, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k}\right); \frac{m-1}{2} + 2 \leq j \leq m-2$$

the representation of each vertex for $3 \leq i \leq m-2$ and i is odd are as follows.

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \leq j \leq \frac{m-1}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{\frac{i-3}{2}}, 0, j-1, \underbrace{j+1, \dots, j+1}_{k-1-\frac{i-3}{2}}\right); j = \frac{m-1}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, 0, m-j-2, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); \frac{m-1}{2} + 1 \leq j \leq m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); j = m-2$$

the representation of each vertex for $3 \leq i \leq m-2$ and i is even are as follows.

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, 0, 1, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}} \right); j = 1$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}} \right); 2 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{\frac{i-4}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-1-\frac{i-4}{2}} \right); j = \frac{m-1}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-\frac{i-2}{2}} \right); \frac{m-1}{2} + 1 \le j \le m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-\frac{i-2}{2}} \right); j = m-2$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{k+1}, 0 \right); 1 \le j \le \frac{m-1}{2} - 1$$

$$r(v_{n-1,j}|\zeta) = \left(\frac{m-1}{2}, \frac{m-1}{1} - 1, \underbrace{m-j, \dots, m-j}_{k}, 0 \right);\ j = \frac{m-1}{2}$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, m-j-2, \underbrace{m-j, \dots, m-j}_{k}, 0 \right);\ \frac{m-1}{2} + 1 \le j \le m-3$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k}, 1 \right);\ j = m-2$$

$$r(v_{n,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k}, 0 \right);\ j = 1$$

$$r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j+1, \dots, j+1}_{k}, j-1 \right); 2 \le j \le \frac{m-1}{2}$$

$$r(v_{n,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k+1} \right); \frac{m-1}{2} + 1 \le j \le m-2.$$

So ζ is the resolving partition of $B_{m,n}$ because $r(v|\zeta)$ is different for each $v \in B_{m,n}$. Therefore, the upper bound of the partition dimension of the $B_{m,n}$ is $pd(B_{m,n}) \le k+3$. The same method as in the previous case is used to determine the lower bound. So, we get $pd(B_{m,n}) = k+3$.

Case 5. For m is even and $n = 2k+3$. To determine the upper bounds of the partition dimension select $\zeta = \{S_1, S_2, S_3, \dots, S_{3+k}\}$ where $S_1 = \{v_{c_1}, v_{2,1}\} \cup \{v_{1,j}; 1 \le j \le m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,m-2}, v_{n-1,m-2}, v_{n,j}; 2 \le j \le m-2\}$, $S_3 = \{v_{4,1}\} \cup \{v_{2,j}; 2 \le j \le m-2, v_{3,j}; 1 \le j \le m-3\}$, $S_4 = \{v_{6,1}\} \cup \{v_{3,m-2}, v_{4,j}; 2 \le j \le m-2, v_{5,j}; 1 \le j \le m-3\}, S_5 = \{v_{8,1}\} \cup \{v_{5,m-2}, v_{6,j}; 2 \le j \le m-2, v_{7,j}; 1 \le j \le m-3\}, S_6 = \{v_{10,1}\} \cup \{v_{7,m-2}, v_{8,j}; 2 \le j \le m-2, v_{9,j}; 1 \le j \le m-3\}, \dots, S_{3+k} = \{v_{n-3,m-2}, v_{m-1,j}; 2 \le j \le m-3, v_{m,1}\}$. The representation vertices $v \in V(B_{m,n})$ with respect to ζ are as follows: $r(v_{c_1}|\zeta) = (0, \underbrace{1, \dots, 1}_{k+2})$, $r(v_{c_2}|\zeta) = (1, 0, \underbrace{1, \dots, 1}_{k+1})$

$$r(v_{1,j}|\zeta) = \left(0, \underbrace{j+1, \dots, j+1}_{k+2} \right); 1 \le j \le \frac{m}{2} - 2 \qquad r(v_{1,j}|\zeta) = \left(0, j, \underbrace{j+1, \dots, j+1}_{k+1} \right); j = \frac{m}{2} - 1$$

$$r(v_{1,j}|\zeta) = \left(0, m-j-2, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m}{2}+1 \le j \le m-3$$

$$r(v_{1,j}|\zeta) = \left(1,0, \underbrace{m-j, \dots, m-j}_{k+1}\right); j = m-2 \qquad r(v_{2,j}|\zeta) = \left(0, j+1, j, \underbrace{j+1, \dots, j+1}_{k}\right); j = 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j+1, 0, \underbrace{j+1, \dots, j+1}_{k}\right); 2 \le j \le \frac{m}{2}-1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j-1, 0, \underbrace{j, \dots, j}_{k}\right); j = \frac{m}{2}$$

$$r(v_{2,j}|\zeta) = \left(m-j, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k}\right); \frac{m}{2}+2 \le j \le m-2$$

the representation of each vertex for $3 \le i \le m-1$ and i is odd follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \le j \le \frac{m}{2}-2$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, j, \underbrace{j+1, \dots, j+1}_{k-\frac{i-1}{2}}\right); j = \frac{m}{2}-1$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, 0, m-j-2, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); \frac{m}{2} \le j \le m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); j = m-2$$

the representation of each vertex for $3 \le i \le m-2$ and i is even follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, 0, j, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); j = 1$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); 2 \le j \le \frac{m}{2}-1$$

$$r(v_{i,j}|\zeta) = \left(j, j-1, \underbrace{j, \dots, j}_{\frac{i-4}{2}}, j-1, 0, \underbrace{j, \dots, j}_{k-1-\frac{i-4}{2}}\right); j = \frac{m}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-\frac{i-2}{2}}\right); \frac{m}{2}+1 \le j \le m-2$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, 0, j, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); j = 1$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{k}, j-1, 0\right); 2 \le j \le \frac{m}{2} - 2$$

$$r(v_{n-1,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k-1}, j-1, 0\right);\ j = \frac{m}{2} - 1$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, m-j-2, \underbrace{m-j, \dots, m-j}_{k}, 0\right);\ \frac{m}{2} \le j \le m-3$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k}, 1\right);\ j = m-2$$

$$r(v_{n,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k}, 0\right);\ j = 1$$

$$r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j+1, \dots, j+1}_{k}, j-1\right); 2 \le j \le \frac{m}{2} - 1$$

$$r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j, \dots, j}_{k}, j-1\right);\ j = \frac{m}{2}$$

$$r(v_{m,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m}{2} + 1 \le j \le m-2$$

So ζ is the resolving partition of $B_{m,n}$ because $r(v|\zeta)$ is different for each $v \in B_{m,n}$. The same method as in the previous case is used to determine the lower bound. We get $pd(B_{m,n}) = k+3$.

Case 6. For m is even and $n = 2k+4$. To determine the upper bounds of the partition dimension, select $\zeta = \{S_1, S_2, S_3, \dots, S_{3+k}\}$ where $S_1 = \{v_{c_1}, v_{2,1}\} \cup \{v_{1,j}; 1 \le j \le m-3\}$, $S_2 = \{v_{c_2}\} \cup \{v_{1,m-2}, v_{n-1,m-2}, v_{n,j}; 2 \le j \le m-2\}$, $S_3 = \{v_{4,1}\} \cup \{v_{2,j}; 2 \le j \le m-2, v_{3,j}; 1 \le j \le m-3\}$, $S_4 = \{v_{6,1}\} \cup \{v_{3,m-2}, v_{4,j}; 2 \le j \le m-2, v_{5,j}; 1 \le j \le m-3\}, S_5 = \{v_{8,1}\} \cup \{v_{5,m-2}, v_{6,j}; 2 \le j \le m-2, v_{7,j}; 1 \le j \le m-3\}, S_6 = \{v_{10,1}\} \cup \{v_{7,m-2}, v_{8,j}; 2 \le j \le m-2, v_{9,j}; 1 \le j \le m-3\}, \dots, S_{3+k} = \{v_{n-3,m-2}, v_{n-1,j}; 2 \le j \le m-3, v_{n-1,j}; 1 \le j \le m-3, v_{m,1}\}$. The representation vertices $v \in V(B_{m,n})$ with respect to ζ are as follows: $r(v_{c_1}|Z) = (0, \underbrace{1, \dots, 1}_{k+2})$, $r(v_{c_2}|Z) = (1, 0, \underbrace{1, \dots, 1}_{k+1})$

$$r(v_{1,j}|\zeta) = \left(0, \underbrace{j+1, \dots, j+1}_{k+2}\right); 1 \le j \le \frac{m}{2} - 2 \qquad r(v_{1,j}|\zeta) = \left(0, j, \underbrace{j+1, \dots, j+1}_{k+1}\right); j = \frac{m}{2} - 1$$

$$r(v_{1,j}|\zeta) = \left(0, m-j-2, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m}{2} \le j \le m-3$$

$$r(v_{1,j}|\zeta) = \left(1, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); j = m-2 \qquad r(v_{2,j}|\zeta) = \left(0, j+1, j, \underbrace{j+1, \dots, j+1}_{k}\right); j = 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j+1, 0, \underbrace{j+1, \dots, j+1}_{k}\right); 2 \le j \le \frac{m}{2} - 1$$

$$r(v_{2,j}|\zeta) = \left(j-1, j-1, 0, \underbrace{j, \dots, j}_{k}\right); j = \frac{m}{2}$$

$$r(v_{2,j}|\zeta) = \left(m-j, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k}\right);\ \frac{m}{2} + 2 \le j \le m-2$$

the representation of each vertex for $3 \le i \le m-2$ and i is odd follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \le j \le \frac{m}{2} - 2$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, j, \underbrace{j+1, \dots, j+1}_{k-\frac{i-1}{2}}\right); j = \frac{m}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, 0, m-j-2, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); \frac{m}{2} \le j \le m-3$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-3}{2}}, m-j-1, 0, \underbrace{m-j, \dots, m-j}_{k-1-\frac{i-3}{2}}\right); j = m-2$$

the representation of each vertex for $3 \le i \le m-1$ and i is even follows

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, 0, j, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); j = 1$$

$$r(v_{i,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-2}{2}}, j-1, 0, \underbrace{j+1, \dots, j+1}_{k-\frac{i-2}{2}}\right); 2 \le j \le \frac{m}{2} - 1$$

$$r(v_{i,j}|\zeta) = \left(j, j-1, \underbrace{j, \dots, j}_{\frac{i-4}{2}}, j-1, 0, \underbrace{j, \dots, j}_{k-1-\frac{i-4}{2}}\right); j = \frac{m}{2}$$

$$r(v_{i,j}|\zeta) = \left(m-j, m-j-1, \underbrace{m-j, \dots, m-j}_{\frac{i-2}{2}}, 0, \underbrace{m-j, \dots, m-j}_{k-\frac{i-2}{2}}\right); \frac{m}{2} + 1 \le j \le m-2$$

$$r(v_{n-1,j}|\zeta) = \left(j, \underbrace{j+1, \dots, j+1}_{\frac{i-1}{2}}, 0, \underbrace{j+1, \dots, j+1}_{k+1-\frac{i-1}{2}}\right); 1 \le j \le \frac{m}{2} - 2$$

$$r(v_{n-1,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k}, 0\right);\ j = \frac{m}{2} - 1$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, m-j-2, \underbrace{m-j, \dots, m-j}_{k}, 0\right);\ \frac{m}{2} \le j \le m-3$$

$$r(v_{n-1,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k}, m-j-1\right);\ j = m-2$$

$$r(v_{n,j}|\zeta) = \left(j, j, \underbrace{j+1, \dots, j+1}_{k}, 0\right);\ j = 1 \qquad r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j+1, \dots, j+1}_{k}, j-1\right); 2 \le j \le \frac{m}{2} - 1$$

$$r(v_{n,j}|\zeta) = \left(j, 0, \underbrace{j, \dots, j}_{k}, j-1\right);\ j = \frac{m}{2} \qquad r(v_{n,j}|\zeta) = \left(m-j, 0, \underbrace{m-j, \dots, m-j}_{k+1}\right); \frac{m}{2} + 1 \le j \le m-2$$

So ζ is the resolving partition of $B_{m,n}$ because $r(v|\zeta)$ is different for each $v \in B_{m,n}$. Therefore, the upper bound of the partition dimension of the $B_{m,n}$ is $pd(B_{m,n}) \leq k+3$. To determine the lower bound of the partition dimension, take a differentiating partition with cardinality smaller than $k+3$, $\zeta = \{S_1, S_2, S_3, \dots, S_{2+k}\}$. Then, there is the same representation of vertices $v \in B_{m,n}$ with results $pd(B_{m,n}) \geq k+3$. So, we get $pd(B_{m,n}) = k+3$. The proof of **Theorem 4** is established through the proof of the six cases described above. ■

Example 2. Given a cycle books graph $B_{m,n}$ for $m = 6$ and $n = 4$ shown in **Figure 2**, we will determine the partition dimensions of $B_{6,4}$. To determine the partition dimension on this graph, we can refer to **Theorem 5** (**case 2**): For $m \geq 6$ and $n = 4$, then $pd(G) = n$. Let $\zeta = \{S_1, S_2, S_3, S_4\}$ with $S_1 = \{v_{c_1}, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}\}$, $S_2 = \{v_{c_2}, v_{1,4}, v_{4,2}, v_{4,3}, v_{4,4}\}$, $S_3 = \{v_{2,2}, v_{2,3}, v_{2,4}, v_{3,1}, v_{3,2}, v_{3,3}, v_{4,1}\}$ and $S_4 = \{v_{3,4}\}$. The representation of all vertices with respect to ζ are as follows.

$$r(v_{c_1}|\zeta) = (0,1,1,2) \quad r(v_{2,1}|\zeta) = (0,2,1,3) \quad r(v_{3,3}|\zeta) = (3,2,0,1)$$
$$r(v_{c_2}|\zeta) = (1,0,1,1) \quad r(v_{2,2}|\zeta) = (1,3,0,4) \quad r(v_{3,4}|\zeta) = (2,1,1,0)$$
$$r(v_{1,1}|\zeta) = (0,2,2,3) \quad r(v_{2,3}|\zeta) = (2,2,0,4) \quad r(v_{4,1}|\zeta) = (1,1,0,3)$$
$$r(v_{1,2}|\zeta) = (0,2,3,4) \quad r(v_{2,4}|\zeta) = (2,1,0,2) \quad r(v_{4,2}|\zeta) = (2,0,1,4)$$
$$r(v_{1,3}|\zeta) = (0,1,3,3) \quad r(v_{3,1}|\zeta) = (1,2,0,3) \quad r(v_{4,3}|\zeta) = (3,0,2,3)$$
$$r(v_{1,4}|\zeta) = (1,0,2,2) \quad r(v_{3,2}|\zeta) = (2,3,0,2) \quad r(v_{4,4}|\zeta) = (2,0,2,2)$$

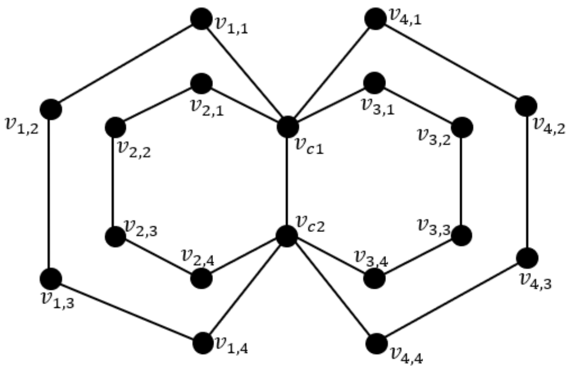

Figure 2. $B_{6,4}$ Cycle Books Graph

From the above representation, it is obtained that ζ resolves the partition of $B_{6,4}$. So, based on **Theorem 5** (**case 2**), it can be concluded that the partition dimension of graph $B_{6,4}$ is 4, which is denoted by $pd(B_{6,4}) = 4$.

4. CONCLUSIONS

As we can see from our main results section, the partition dimension of cycle books graph $B_{m,n}$ with a common path P_2 is as follows. The partition dimension of cycle books graph $B_{m,n}$ denoted by $pd(B_{m,n})$ is $pd(B_{m,n}) = 3$ for $m \geq 6$ *and* $n = 2,3$, $pd(B_{m,n}) = n$ for $m \geq 6$ *and* $n = 4$, and $pd(B_{m,n}) = 3 + k$ for $k \in \mathbb{N}$ and m, n otherwise. This result extends the findings of **Theorem 1** to **Theorem 3** and provides a more complete characterization of the partition dimension for this class of graphs.

Our findings contribute to the broader study of partition dimensions in graph theory. Future research may explore the partition dimension of generalized cycle book graphs or consider variations under different constraints.

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