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FRACTIONAL-ORDER MODEL OF THE DRUG USER TRANSMISSION

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ABSTRACT

Drug abuse poses significant challenges to public health and socio-economic stability worldwide. Narcotics, which are psychotropic compounds, are typically used for treating specific medical conditions. Currently, many individuals abuse drugs outside of the function of treatment. This misuse leads to central nervous system disorders, resulting in significant mental and behavioral health issues. In this article, we discuss a fractional-order mathematical model for the transmission of drug users with fractional-order $\alpha \in (0,1]$. We employ fractional-order differential equations using the Caputo derivative approach to model the transmission dynamics. We analyze the local stability of drug-free and endemic equilibrium points and calculate the basic reproduction number ($\mathcal{R}_0$). Our analysis indicates that the drug-free equilibrium is locally asymptotically stable when $\mathcal{R}_0 < 1$, while the endemic equilibrium is stable when $\mathcal{R}_0 > 1$. We implement a numerical scheme to simulate the fractional-order model, illustrating the theoretical findings.

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1. INTRODUCTION

Drug abuse is a pervasive global issue that significantly impacts public health and socio-economic stability [1]. Over the past decade, the incidence of drug abuse has risen dramatically, exacerbating public health crises and economic burdens. The proliferation of illicit drug trafficking has permeated all societal levels, including vulnerable populations such as teenagers, posing severe long-term consequences for national health and stability. This condition is very detrimental and has a big impact on the life of the nation and state in the future [2]. The term 'narcotics' encompasses various dangerous drugs and substances that pose a high risk of addiction [3]. Narcotics, such as opioids and stimulants, are psychotropic compounds used in the treatment of specific medical conditions. However, currently, many individuals who abuse drugs are not in the function of treatment, which disrupts the central nervous system, causing these individuals to suffer from mental and behavioral disorders [4].

In 2022, the National Narcotics Agency and Police of the Republic of Indonesia reported 43,099 cases of drug abuse, with 55,452 individuals identified as suspects [5]. Globally, an estimated 275 million individuals used drugs in 2020, a number that increased to approximately 296 million in 2021 [7]. Drug abuse has induced the deaths of half a million people worldwide, especially for injecting drug users who suffer from HIV or Hepatitis C [6].

Mathematical modeling provides a valuable tool for understanding the complex behavioral dynamics of drug users. Researchers have proposed to describe mathematical models to study the dynamics of the transmission of drug abuse. Liu et al. [8] studied a mathematical model of drug distribution by taking into account the history of drug abuse. Mushanyu and Nyabazda [9] described the transmission of drug abuse with attention to the level of high and low risk in drug abuse. Furthermore, Batista et al. [10] analyzed the drug abuse model by considering individual drug addicts due to abuse and individual drug addicts due to undergoing treatment using prescribed drugs. Ginoux et al. [11] constructed a mathematical model of drug use in the classic predator-prey model with the assumption that drug users are predators, while non-drug users are prey. Zhang et al. [13] investigated the transmission of drug use by adding a time delay factor. Hafiruddin et al. [12] developed a mathematical model for the dynamics of the spread of drug addicts by adding criminal penalties aspect and implementing optimal control strategies to reduce the number of drug addicts. Akanni et al. [14] constructed the dynamics of drug use based on mental health factors. Akanni [15] also developed a mathematical model of drug users by noticing its consequences on the dynamics of the spread of terrorism.

Fractional differential equations can be used to understand the dynamics of real-life situations. Fractional differential equations contain derivatives of unknown functions of fractional order. In this case, it can provide better and more detailed information in modeling a problem so that it can be explained better. [16]. For example, Zafar et al. [17] discussed White and Comiskey's mathematical model, which describes drug abuse using fractional order derivatives. Achar et al. [18] expanded the model for the spread of drug use using a predator-prey model approach with fractional order derivatives and incorporating a Holling type-III functional response. Furthermore, [19] represented a fractional-order mathematical model of the dynamics of drug distribution and money laundering.

This study proposes a fractional-order mathematical model for drug user transmission, building on the integer model developed by Akanni et al. [14]. We also converted the basic model in [14] by combining the populations of light, heavy, and mentally ill drug users into one population and proposed a fractional order differential equation model to obtain a more accurate picture of the transmission dynamics of drug users.

2. RESEARCH METHODS

This section outlines the theorems, definitions, and procedural steps used to analyze the fractional-order model of drug user transmission.

2.1 Caputo Fractional Derivative

Calculus is a mathematical discipline that focuses on rates of change and accumulation of quantities, using integer-order derivatives and integrals. Derivatives indicate how a function changes (like speed as the

derivative of distance), and integrals measure the accumulation of quantity while integrals represent accumulated quantities (like area under a curve).

Fractional calculus extends these concepts by allowing the order of derivatives and integrals to be fractional, rather than just whole numbers. For example, while you might be familiar with the first derivative (denoted as $f'(x)$) or the second derivative (denoted as $f''(x)$), fractional calculus allows us to talk about derivatives like $f^{\frac{1}{2}}(x)$ or $f^{\frac{3}{2}}(x)$. This innovative approach provides a broader framework for understanding dynamic systems and can model complex phenomena more effectively than traditional calculus.

The Caputo fractional derivative is a fundamental concept in fractional calculus, a field that generalizes traditional calculus by allowing derivatives of non-integer orders. Developed by the mathematician Francesco Caputo in the 1960s, this type of derivative is particularly significant for its applications in various scientific and engineering fields, especially when modeling processes with memory or hereditary properties.

Definition 1. *Let $f(t)$ is a function that is differentiable to the m^{th} derivative. The Caputo derivative of $f(t)$ of order $\alpha \in (0,1]$ can be defined as follows:*

$$D^{\alpha}f(t) = \frac{d^{\alpha}f(t)}{dt^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, & m-1<\alpha<m \\ f^{(m)}(t), & \alpha = m \end{cases}, \tag{1}$$

where $t > 0$ and $m \in \mathbb{N}$.

Caputo fractional derivative of a constant C is zero, namely $\frac{d^{\alpha}C}{dt^{\alpha}} = 0$ and holds:

$$\frac{d^{\alpha}t^{m}}{dt^{\alpha}} = \begin{cases} 0, & m \le \alpha - 1 \\ \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}, & m > \alpha - 1 \end{cases}.$$

If $\alpha = 1$, then the Caputo fractional derivative is an ordinary derivative. Thus, the Caputo fractional derivative is a generalization of the ordinary derivative [20].

2.2 Stability of The Fractional-order Systems

The stability theorem is a fundamental concept in the dynamical system that provides conditions under which a dynamic system's equilibrium point is stable. In simple terms, a system is considered stable if, after a small disturbance, it tends to return to its equilibrium state rather than diverging away from it.

Theorem 1. *Consider a system of fractional-order nonlinear differential equations using the Caputo approach as follows:*

$$D^{\alpha}x(t) = f(x), \tag{2}$$

where $x \in \mathbb{R}^n$ and $\alpha \in (0,1]$. The equilibrium point of **Equation (2)** *are solution to $f(x) = 0$. An equilibrium point is locally asymptotically stable in all eigenvalues (λ_i) of the Jacobian matrix $J = \partial f/\partial x$ evaluated at the equilibrium point satisfy $|arg\,(\lambda_i)| > \alpha\pi/2$* [20].

2.3 Research Procedures

a. Literature Study

We conduct a literature study through journals, reference books, and scientific articles related to drugs, the mathematical theory used, and its application to the fractional-order model of the transmission of drug users.

b. Construct the Fractional-Order Model

The fractional-order mathematical model for drug user transmission, based on [14], originally includes six compartments: vulnerable individuals using drugs $\boldsymbol{S(t)}$, light users $\boldsymbol{L(t)}$, heavy users $\boldsymbol{H(t)}$, individuals with mental illness $\boldsymbol{M(t)}$, rehabilitated individuals $\boldsymbol{T(t)}$, and those who quit

$\boldsymbol{Q(t)}$. It is modified to four compartments: vulnerable individuals $\boldsymbol{S(t)}$, drug users $\boldsymbol{I(t)}$, rehabilitated individuals $\boldsymbol{T(t)}$, and those who quit $\boldsymbol{Q(t)}$. This model applies the Caputo fractional order derivative to analyze drug user dynamics more effectively.

c. Analyze the Local Stability Equilibrium Points

The steps used are as follows:

i. Determine the equilibrium points in the fractional-order model of the drug user transmission.

ii. Linearize the model using the Jacobian matrix.

iii. Substitute the equilibrium points into the Jacobian matrix to obtain a linear system.

iv. Test the local stability of each equilibrium point by calculating the eigenvalues from the linearization results.

d. Simulate and Interpret the Results

We conduct a numerical simulation of the fractional-order model of the transmission of drug users using MATLAB software and conclude the previous steps to determine the dynamics of the transmission of drug users for different values of the fractional-order $(\boldsymbol{\alpha})$.

e. Conclude

We summarize the findings, emphasizing how fractional-order derivatives after the understanding of drug transmission dynamics compared to traditional integer-order models. Discuss implications for public health interventions and potential areas for further research.

3. RESULTS AND DISCUSSION

This section presents the results of the fractional-order model for drug user transmission, including model formulation, stability analysis, numerical simulations, and sensitivity analysis.

3.1 Model Formulation

In the fractional-order model of the transmission of drug users, there are several assumptions used as follows:

a. The rate of individual recruitment is constant.

This assumption simplifies population dynamics by keeping the number of new individuals stable over time, allowing for predictable growth patterns and clearer analysis of other variables' effects.

b. Rehabilitation is isolated so that individuals undergoing rehabilitation cannot influence vulnerable individuals.

Isolating rehabilitation effects enables focused study on its efficacy without interference from interactions with individuals still using drugs, thus minimizing external influences.

c. Populations of individuals stop using drugs after undergoing rehabilitation.

This assumption suggests that rehabilitation effectively eliminates drug use, providing a clear outcome that facilitates the examination of rehabilitation's impact on overall population dynamics.

d. The population of individuals who stop using drugs can return to individuals who relapse into using drugs.

Acknowledging the possibility of relapse enhances the model's realism, as recovery from substance use disorder is often non-linear, reflecting actual recovery scenarios.

e. All parameter values have positive values.

Ensuring all parameters remain positive is crucial for maintaining the model's mathematical validity, preventing nonsensical interpretations and unstable population behaviors.

The following is a definition of each variable and parameter used in the fractional order mathematical model of the distribution of drug use which is listed in Tabel 1 and Tabel 2.

Table 1. Definition and Description of Variables

Variable	Description
$S(t)$	The population of individuals is vulnerable to using drugs at this time t
$I(t)$	The Population of individual drug users at the time t
$T\ (t)$	The population of individuals rehabilitated at the time t
$Q(t)$	The population of individuals who have stopped using drugs at the time t

Table 2. Definition and Description of Parameters

Parameter	Description
Π	Recruitment rate of individuals vulnerable to using drugs
β	The rate at which vulnerable individuals are influenced to use drugs due to contact with individuals who use drugs
γ	Rate of rehabilitation/treatment of drug users
ρ	The rate at which rehabilitated individuals stop using drugs
α	The rate at which individuals who stop using drugs relapse
μ	Natural death rate
δ	Death rate due to drug use

Based on the assumptions and definitions of variables and parameters previously explained, a transmission diagram for a fractional-order model of the transmission of drug users can be formed in Figure 1 as follows:

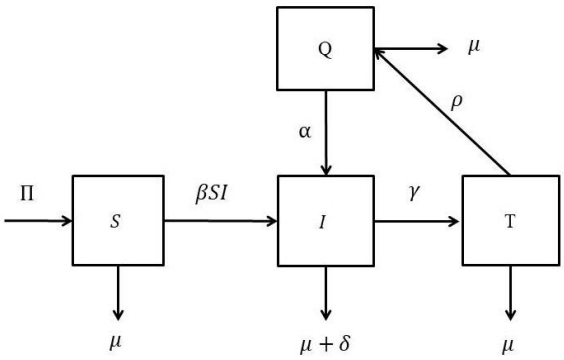


Figure 1. Compartment Diagram of a Fractional Order Model of The Transmission of Drug User

Based on the compartment diagram in Figure 1, the change in the susceptible population over time is influenced by a constant recruitment rate (Π), the rate at which vulnerable individuals are influenced to use drugs (β), and the natural death rate of susceptible individuals (μ). This illustrates a new individual who is vulnerable to drug users. Next, the number of drug users increases as the number of drug users increases from vulnerable individuals who use drugs (β), The rate at which individuals who stop using drugs relapse (α) and decreases due to the rate of rehabilitation (γ), natural death (μ), and death rate due to drug use (δ). This indicates the dynamics of transmission and rehabilitation for drug users.

The rehabilitated population increases as individuals recover from drug (γ) and decreases as rehabilitated individuals stop using drugs and natural death (μ) This illustrates the success of the rehabilitation program so that rehabilitated individuals stop using drug (ρ). Next, the number of individuals who stop using drugs increases due to the rate at which individuals stop using drugs (ρ) and decreases at which individuals who stop using drugs relapse (α) and due to individuals returning to using drugs and natural deaths (μ). This indicates that there are individuals who have successfully stopped using drugs after undergoing rehabilitation. Thus, the fractional-order model of the transmission of drug users can be expressed as follows:

$$\begin{aligned}\frac{dS}{dt} &= \Pi - \beta SI - \mu S \\ \frac{dI}{dt} &= \beta SI + \alpha Q - (\gamma + \mu + \delta)I \\ \frac{dT}{dt} &= \gamma I - (\rho + \mu)T \\ \frac{dQ}{dt} &= \rho T - (\alpha + \mu)Q\end{aligned} \tag{3}$$

.

The solution of **Equation (3)** is defined for every $t \geq 0$ in the set of positive invariants, with:

$$\Omega = \left\{(S, I, T, Q) \in \mathbb{R}_+^4 \,\middle|\, S, I, T, Q \geq 0, 0 \leq N \leq \frac{\Pi}{\mu}\right\}. \tag{4}$$

Next, we consider a fractional-order model of the **Equation (5)**. The fractional-order model of the transmission of drug users is as follows:

$$\begin{aligned}\frac{d^\alpha S}{dt^\alpha} &= \Pi - \beta SI - \mu S, \\ \frac{d^\alpha I}{dt^\alpha} &= \beta SI + \alpha Q - (\gamma + \mu + \delta)I, \\ \frac{d^\alpha T}{dt^\alpha} &= \gamma I - (\rho + \mu)T, \\ \frac{d^\alpha Q}{dt^\alpha} &= \rho T - (\alpha + \mu)Q.\end{aligned} \tag{5}$$

where $\alpha \in (0, 1]$ is the order of the fractional derivative. The fractional derivative used in **Equation (3)** is the Caputo approach. The Caputo fractional derivative is defined as follows **Theorem 1.**

3.2 Analysis Model

In this sub-section, we study the stability of the equilibrium in **Equation (3)**. We will follow several intermediate steps to clarify the process. This will involve determining the equilibrium points of the system, linearizing the model around these equilibrium points, and then analyzing the stability of these points using eigenvalues. This process helps identify how the system will respond to changes in parameters and initial conditions, providing insights into the effectiveness of interventions aimed at controlling drug use and promoting recovery.

3.2.1 Drug-free Equilibrium

Based on **Theorem 1**, the equilibria are satisfied by setting the equation in **Equation (3)** to be constant with time or:

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dT}{dt} = \frac{dQ}{dt} = 0. \tag{6}$$

The drug-free equilibrium is denoted by $E_0 = (S_0, I_0, T_0, Q_0) = \left(\frac{\Pi}{\mu}, 0,0,0\right)$.

3.2.2 Basic Reproduction Number

Next, the value of the basic reproduction number $(\mathcal{R}_0)$ which will be used to determine the threshold for a condition that indicates the transmission of drug users in non-endemic or endemic conditions. Determine endemic criteria in a population $(\mathcal{R}_0)$ is obtained by building a matrix that generates the number of newly infected individuals, namely by using the NGM (*Next Generation Matrix*) method **[21]**, we take the population I_C, I_A and T. The Jacobian matrix $\mathbb{F}$ and $\mathbb{Z}$ for the new infection acquired from susceptible individuals and the rate of individual movement from one subpopulation to another subpopulation are given by:

$$\mathbb{F} = \begin{bmatrix} \beta S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\mathbb{Z} = \begin{bmatrix} m_1 & 0 & -\alpha \\ -\gamma & m_2 & 0 \\ 0 & -\rho & m_3 \end{bmatrix}$$

The basic reproduction number of the **Equation (3)** is obtained by determining the spectral radius of the matrix $\mathbb{F}\mathbb{Z}^{-1}$, so that it is obtained:

$$\mathcal{R}_0 = \frac{\beta\Pi(\rho+\mu)(\alpha+\mu)}{\mu(\gamma+\mu+\delta)(\rho+\alpha+\mu)+\alpha\rho(\mu+\delta)} \tag{8}$$

3.2.3 Stability of Drug-Free Equilibrium

Analysis of the stability of the system's equilibrium point is carried out by first determining the eigenvalues of the matrix A. The following theorem provides the local stability of the drug-free equilibrium.

Theorem 2. *The drug-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. The Jacobian matrix of the **Equation (3)** around the drug-free equilibrium E_0 is given by:

$$J(E_0) = \begin{bmatrix} -\beta I - \mu & \frac{-\beta\Pi}{\mu} & 0 & 0 \\ \beta I & \frac{\beta\Pi}{\mu} - (\gamma+\mu+\delta) & 0 & \alpha \\ 0 & \gamma & -(\rho+\mu) & 0 \\ 0 & 0 & \rho & -(\alpha+\mu) \end{bmatrix}. \tag{9}$$

The eigenvalues of the matrix $J(E_0)$ are obtained, namely $\lambda_1 = -\mu$, and the roots of the following equation:

$$(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0. \tag{10}$$

with $a_1 = \frac{\mu(m_1+m_2m_3)-\beta\pi}{\mu}$, $a_2 = \frac{\mu(m_2m_3+m_1m_2+m_1m_3)-\beta\Pi(m_2+m_3)}{\mu}$, $a_3 = \frac{\mu(m_1m_2m_3-\alpha\rho\gamma)-\beta\Pi m_2m_3}{\mu}$, thus we have $|\arg(\lambda_1)| = |\arg(\lambda_2)| = |\arg(\lambda_3)| = |\arg(\lambda_4)| = \pi > \frac{\alpha\pi}{2}$. Next, we will examine the roots of **Equation (10)** using the Routh-Hurwitz criterion, $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$ are necessary and sufficient condition to satisfy $|\arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}$ and $0 < \alpha < 1$. It is clear that for all negative eigenvalues $\left(|\arg(\lambda_j)| = \pi > \frac{\alpha\pi}{2},\ \text{for } j = 1,2,3,4\right)$ if $R_0 < 1$. Therefore, the drug-free equilibrium E_0 is asymptotically stable for every $\alpha \in (0,1]$ if $\mathcal{R}_0 < 1$.■

This represents that if $\mathcal{R}_0 < 1$ there would be no transmission of drug users in the population.

3.2.4 Stability of Endemic Equilibrium

Furthermore, we calculated the endemic equilibrium of **Equation (3)** namely (E_1). This is a situation when there are individuals using drugs. In other words, the number of individuals using drugs are not zero or can be stated individually $(I \neq 0)$. The endemic equilibrium E_1 of the **Equation (3)** is given by:

$$E_1 = (S^*, I^*, T^*, Q^*), \tag{11}$$

with,

$$S^* = \frac{\Pi}{(\beta I + \mu)},$$

$$I^* = \frac{\mu(\mathcal{R}_0 - 1)}{\beta},$$

$$T^* = \frac{\gamma}{m_2} I^*,$$

$$Q^* = \frac{\rho\gamma}{m_2 m_3} I^*.$$

The endemic equilibrium E_1 exists if $\mathcal{R}_0 > 1$. Next, we will discuss the stability analysis of the obtained equilibrium. The stability of the endemic equilibrium (E_1) is difficult to prove analytically because it involves a quartic equation that depends on $S^*, I^*$and is of order 4. Thus, the local stability of the endemic

equilibrium will be analyzed through numerical simulations using the phase field. Numerical simulations show that the endemic equilibrium tends to be locally asymptotically stable if $\mathcal{R}_0 > 1$. This can be seen in **Figure 2**.

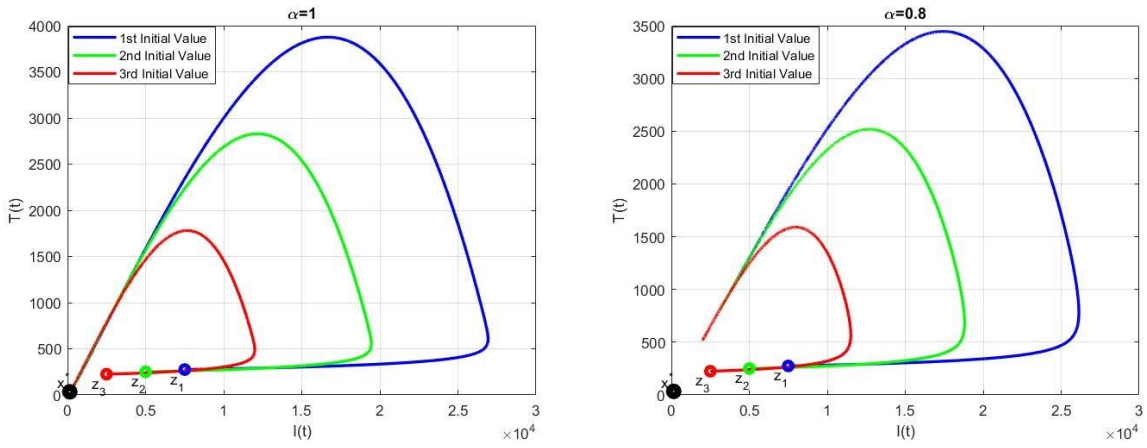
For the simulation, three different sets of initial conditions are chosen to assess the convergence behavior of the system and represent different starting points for the populations in the model, which could include varying numbers of susceptible, infected, treatment, and quitted individuals. As the simulation progresses, the orbits corresponding to these initial conditions converge to the same point, demonstrating the stability of the endemic equilibrium. Numerical simulations to determine the stability of the endemic equilibrium E_1 are carried out by taking two orders α values, namely $\alpha = 0.8$ and $\alpha = 1$. The choice of these orders allows for the exploration of how fractional calculus influences the dynamics of the model. The initial values and parameter values used in this numerical simulation are given in **Table 3** and **Table 4** as follows:

Table 3. Initial Values in Phase Portrait Simulation

Initial Values	S	I	T	Q
z_1	5000	2000	20	10
z_2	6000	1000	10	15
z_3	7000	3000	60	20

Table 4. Parameter Values in the Fractional-Order Model of The Transmission of Drug User

Parameter	Value	Reference
Π	10	Assumption
β	7.61×10^{-4}	Assumption
γ	0.0489	[9]
ρ	0.2	[9]
α	0.002	[9]
μ	$\frac{1}{70}$	Assumption
δ	0.00213	[9]



(a) **(b)**

Figure 2. Phase Portrait of Equation (3) for $\mathcal{R}_0 > 1$, (a) $\alpha = 1$, (b) $\alpha = 0.8$

Based on the numerical simulation results in **Figure 2**, the solution of **Equation (3)** with three different initial values given converges to the endemic equilibrium point E_1, where $\mathcal{R}_0 = 8.8038 > 1$. Thus, it is proven that the endemic equilibrium E_1 is locally asymptotically stable for $\alpha \in (0, 1]$ if $\mathcal{R}_0 > 1$.

3.3 Sensitivity Analysis

In this section, we present the sensitivity analysis of the reproduction number $(\mathcal{R}_0)$ to the parameters in the **Equation (3)**. Sensitivity analysis aims to determine parameters that have a large influence on a model and measure the parameters that have the most effect on $\mathcal{R}_0$ **[22]**. The index of each parameter involved can be formulated as follows:

$$e_m^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial m}\frac{m}{\mathcal{R}_0}, \tag{12}$$

where e_m is the sensitivity index of the parameter m and m is the parameter to be computed for the remaining parameters using parameter values in **Table 4**. The results of calculating the sensitivity index on the parameters of the fractional-order model of the transmission of drug users can be seen in **Table 5** as follows:

Table 5. Parameter Sensitivity Index Calculation Results

Parameter	Sensitivity index
Π	1
β	1
γ	-0.7287
μ	-1.32514
δ	-0.0352
α	0.082752
ρ	0.006289

Based on **Table 5**, the sensitivity index can be analyzed as follows. The positive sensitivity index shows that an increase in the parameters will lead to an increase in the basic reproduction number, while a negative sensitivity index means that an increase in the parameter will lead to a decrease in the basic reproduction number. For example, for $\beta = 1$, increasing the value β by 10% increases the reproduction number $\mathcal{R}_0$ by 10%. Thus increasing the natural death rate γ by 10% decreases R_0 by 7.287%.

Based on the sensitivity analysis results, the parameters that have a significant impact on the model are β, μ, γ, and Π. This is because the sensitivity index values have the highest absolute values compared to other parameters. The recruitment rate of individuals vulnerable to using drugs (Π) and the natural death rate (μ) cannot be controlled, the influencing parameters are β and γ. So, underscore the importance of carefully managing the parameters influencing drug use transmission dynamics to effectively control the epidemic and determine effective prevention strategies.

We also show sensitivity simulation to verify our sensitivity analysis. The parameter values used in the simulations are given in **Table 4**. In **Figure 3**, we can see that, the parameter chosen with distinct values of β, respectively, $\mathcal{R}_0$ increases monotonically. These results indicate that increasing β will increase the basic reproduction number $\mathcal{R}_0$.

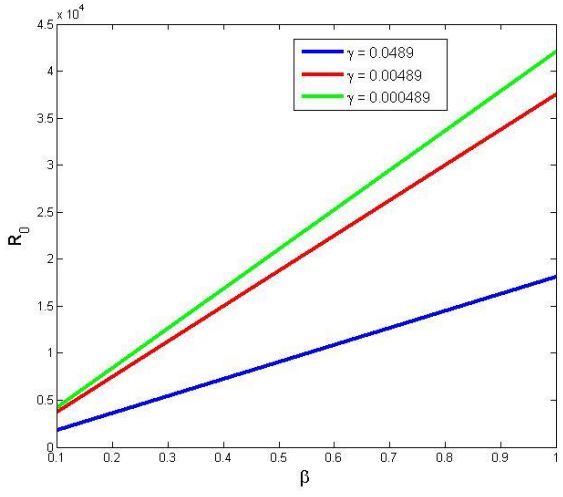
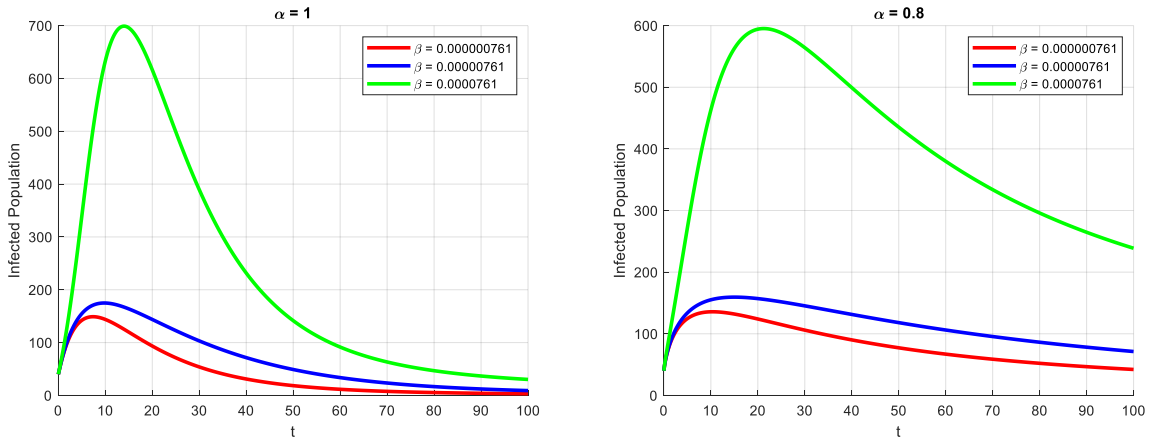
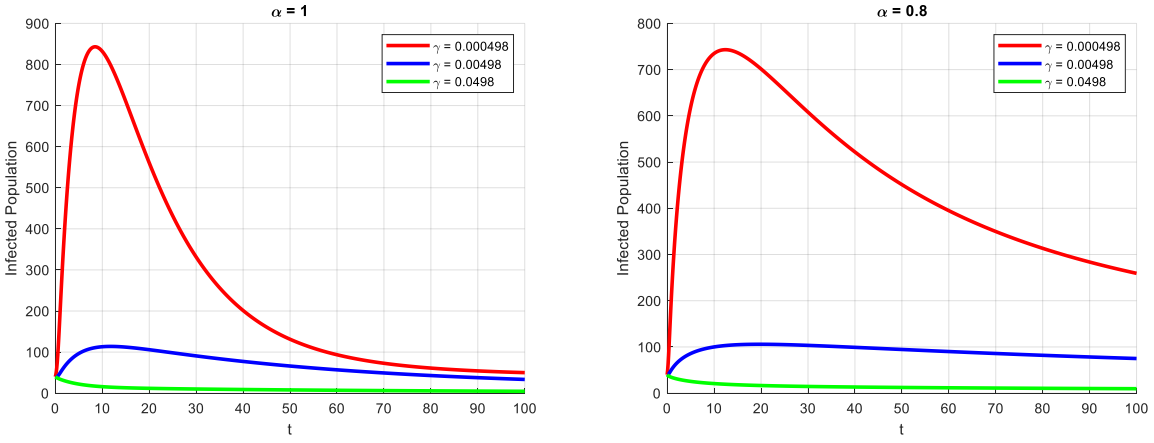


Figure 3. Sensitivity of R_0 with Respect to β for Different Values of γ



(a) (b)

Figure 4. Sensitivity of β of Equation (3) in I plane, (a) $\alpha = 1$, (b) $\alpha = 0.8$



(a) (b)

Figure 5. Sensitivity of γ of Equation (3) in I plane, (a) $\alpha = 1$, (b) $\alpha = 0.8$

Based on Figure 4, it can be seen that the number of infected individuals will increase if the rate at which vulnerable individuals are influenced to use drugs due to contact with individuals who use drugs increases, and the number of infected individuals will decrease if the rate of rehabilitation/treatment of drug users decrease can be seen at Figure 5.

3.4 Numerical Simulation

In this section, a numerical simulation will be carried out on a fractional-order model of the transmission of drug users. We will perform a numerical simulation for the drug-free equilibrium E_0. The parameter values used in this numerical simulation are $\beta = 5 \times 10^{-5}$ and other parameter values are taken in Table 4. The initial value used is $(S(0), I(0), T(0), Q(0) = (5000;\ 1000;\ 40;\ 15)$. The simulation uses several fractional-order values $\alpha \in (0.4, 1]$. These values effectively capture the complexities of drug use behaviors, allowing for the representation of memory effects and the non-linear dynamics associated with addiction recovery and relapse [23].

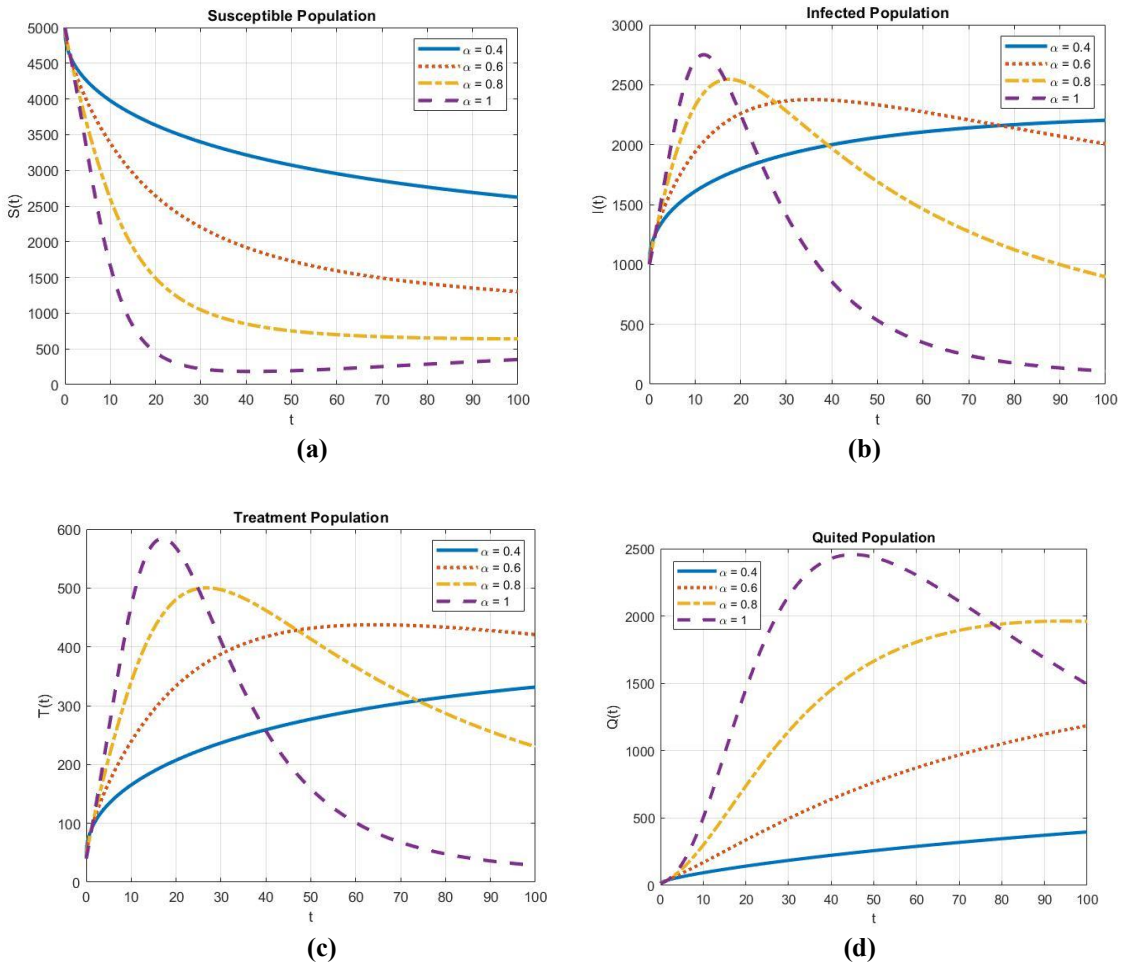


Figure 6. Numerical Solution of Equation (3) when $\mathcal{R}_0 < 1$,
(a) Susceptible Population, (b) Infected Population, (c) Treatment Population, (d) Quitted Population

Based on the numerical simulation results in **Figure 6**, the value of $\mathcal{R}_0$ is $R_0 = 0.088 < 1$, which means there is no spread of drug users in a population. We observe that solution **Equation (3)** converges to drug-free equilibrium when $R_0 < 1$. Apart from that, numerical simulation with high-order α have faster convergence speeds compared to smaller α.

Next, we will perform a numerical simulation for the endemic equilibrium E_1. The parameter values used in this numerical simulation are taken in **Table 4**. The initial value used is $\big(S(0), I(0), T(0), Q(0)\big) = (5000; 1000; 40; 15)$. The simulation uses several fractional-order values $\alpha \in (0.4, 1]$.

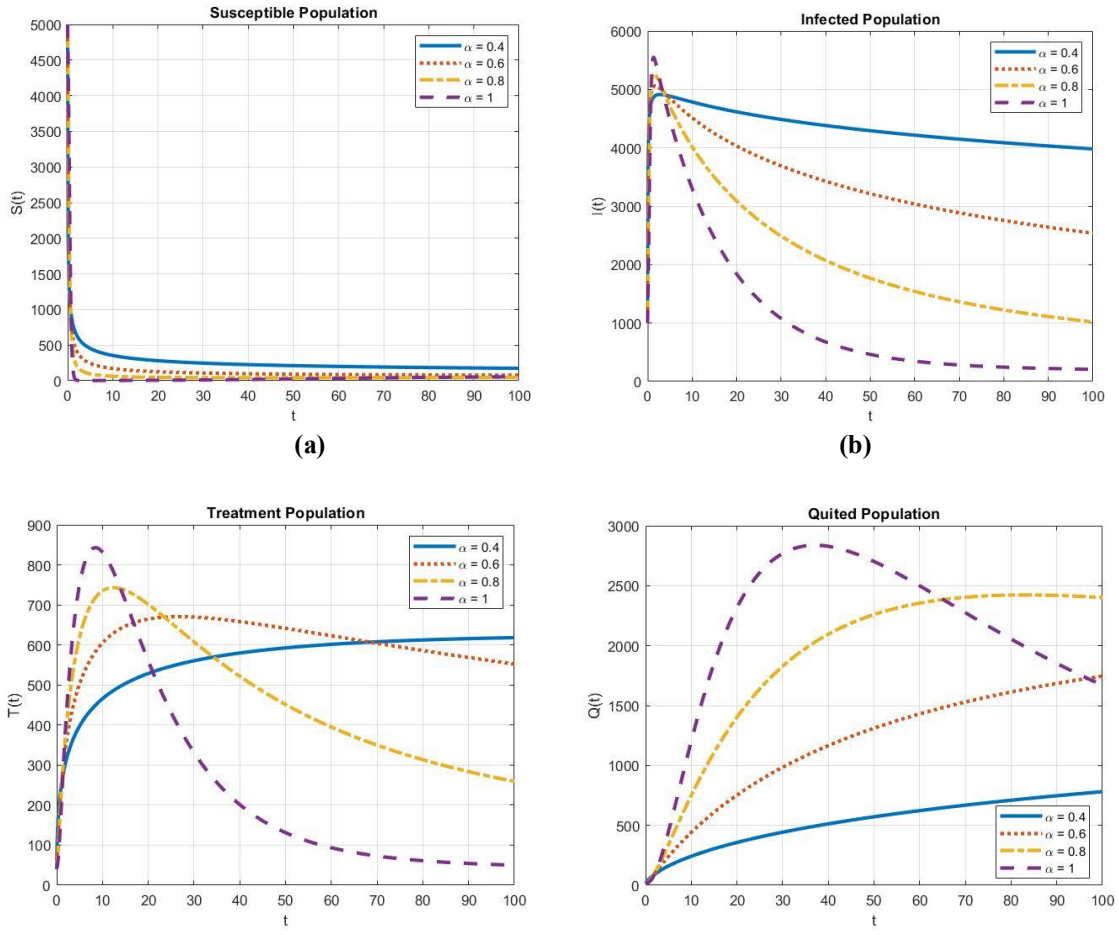


Figure 7. Numerical Solution of Equation (3) when $\mathcal{R}_0 > 1$,
(a) Susceptible Population, (b) Infected Population, (c) Treatment Population, (d) Quitted Population

Based on the numerical simulation results in Figure 7, the value of $\mathcal{R}_0 < 1$ is $\mathcal{R}_0 = 6.4632 > 1$, which means there is a transmission of drug users in a population. We observe that the solution of Equation (3) converges to endemic equilibrium when $\mathcal{R}_0 > 1$. Apart from that, numerical simulation with high-order α have faster convergence speeds compared to smaller α.

4. CONCLUSIONS

This paper presents a fractional-order differential equation model using the Caputo approach to study the transmission dynamics of drug users. The model identifies two types of equilibrium: the drug-free equilibrium and the endemic equilibrium. We analyzed the local stability of both the drug-free and endemic equilibrium points. The drug-free equilibrium is locally asymptotically stable when the basic reproduction number $(\mathcal{R}_0)$ is less than one, while the endemic equilibrium is stable when $(\mathcal{R}_0)$ is greater than one. We conducted both analytical and numerical sensitivity analyses to identify parameters that significantly impact the basic reproduction number. Our sensitivity simulations indicate that the number of infected individuals increases with higher values of parameter (β) and decreases with higher values of parameter (γ). Finally, our numerical simulations for different fractional-order values demonstrate that higher-order derivatives result in faster convergence speeds compared to lower-order derivatives. In the future, this work could be used to study different fractional operators and fractal-fractional parameter estimation with real data.

REFERENCES

[1] Wang, X., Cui, J. J., Zhuo, Y., Shen, B. H., Zhang, S. J., Liu, W., Shen, M., and Xiang, P., "A retrospective of prevalence of drugs of abuse by hair analysis in Shanghai using LC-MS-MS," *Journal of Analytical Toxicology*, vol. 44, no. 5, pp. 482-489, 2020.

[2] National Narcotics Agency of Gorontalo Province, "Drugs are taking away the future of the younger generation," 2020. [Online]. Available: https://gorontalo.bnn.go.id/narkoba-merenggut-masa-depan-generasi-muda-2/. [Accessed: 21-Apr-2024].

[3] Ministry of Health of the Republic of Indonesia, *General Overview of Drug Abuse in Indonesia*, Jakarta: Ministry of Health RI, 2014.

[4] Sasmita, F., *Narcotics, Drugs and Drugs*, Yogyakarta: Media Education Center, 2018.

[5] National Narcotics Agency, *Indonesia Drug Report 2023*, National Narcotics Agency Research, Data and Information Center, Jakarta, 2023.

[6] United Nations Office on Drugs and Crime (UNODC), *World Drug Report 2021*, United Nations Publication, Sales No. E.20.XI.6, Austria, 2021.

[7] United Nations Office on Drugs and Crime (UNODC), *World Drug Report 2023*, United Nations Publication, Austria, 2023.

[8] Liu, P., Zhang, L., and Xing, Y., "Modelling and stability of a synthetic drug transmission model with relapse and treatment," *Journal of Applied Mathematics and Computing*, vol. 60, no. 1-2, pp. 465-484, 2018.

[9] Mushanyu, J., and Nyabadza, F., "A Risk-Structured Model for Understanding the Spread of Drug Abuse," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 60, pp. 1-16, 2018.

[10] Battista, N. A., Pearcy, L. B., and Strickland, W. C., "Modeling the prescription opioid epidemic," *Bulletin of Mathematical Biology*, vol. 81, no. 6, pp. 2258-2289, 2019.

[11] Ginoux, J. M., Naeck, R., Ruhomally, Y. B., Dauhoo, M. Z., and Perc, M., "Chaos in a predator-prey-based mathematical model for illicit drug consumption," *Applied Mathematics and Computation*, vol. 347, pp. 502-513, 2019.

[12] Hafiruddin, M., Fatmawati, F., and Miswanto, M., "Mathematical model analysis of a drug transmission with criminal law and its optimal control," in *AIP Conference Proceedings*, AIP Publishing, 2019.

[13] Zhang, Z., Yang, F., and Xia, W., "Hopf bifurcation analysis of a synthetic drug transmission model with time delays," *Complexity*, vol. 2019, pp. 1-17, 2019.

[14] Akanni, J. O., Olaniyi, S., and Akinpelu, F. O., "Global asymptotic dynamics of a nonlinear illicit drug use system," *Journal of Applied Mathematics and Computing*, vol. 66, no. 1-2, pp. 39-60, 2021.

[15] Akanni, J. O., "Mathematical assessment of the role of illicit drug use on terrorism spread dynamics," *Journal of Applied Mathematics and Computing*, vol. 68, no. 6, pp. 3873-3900, 2022.

[16] Barros, L. C. D., Lopes, M. M., Pedro, F. S., Esmi, E., Santos, J. P. C. D., and Sánchez, D. E., "The memory effect on fractional calculus: an application in the spread of COVID-19," *Computational and Applied Mathematics*, vol. 40, no. 1, pp. 1-21, 2021.

[17] Zafar, Z. U. A., Rezazadeh, H., Inc, M., Nisar, K. S., Sulaiman, T. A., and Yusuf, A., "Fractional order heroin epidemic dynamics," *Alexandria Engineering Journal*, vol. 60, no. 6, pp. 5157-5165, 2019.

[18] Achar, S. J., and Baishya, C., "Dynamics of modified fractional illicit drug consumption model," *Palestine Journal of Mathematics*, vol. 11, no. 3, pp. 112-126, 2022.

[19] Nwajeri, U. K., Fadugba, S. E., Ohaeri, E. O., Oshinubi, K. I., Ogunrinde, R. R., and Ogunrinde, R. B., "Co-dynamic Model of Drug Trafficking and Money Laundering Coupled with Fractional Derivative," *International Journal of Applied and Computational Mathematics*, vol. 9, no. 5, pp. 1-17, 2023.

[20] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer Verlag, New York, 2010.

[21] Brauer, F., and Chavez, C. C., *Mathematical Models in Population Biology and Epidemiology*, 2nd ed. New York: Springer-Verlag, 2012.

[22] Hussain, T., Ozair, M., Ali, F., ur Rehman, S., Assiri, T. A., and Mahmoud, E. E., "Sensitivity analysis and optimal control of COVID-19 dynamics based on SEIQR model," *Results in Physics*, vol. 22, Article 103956, 2021.

[23] Khalique, C. M., et al., "Dynamics of fractional-order models of drug addiction," *Mathematical Modelling and Analysis*, vol. 21, no. 3, pp. 375-388, 2016.