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THE NON-COPRIME GRAPHS OF UPPER UNITRIANGULAR MATRIX GROUPS OVER THE RING OF INTEGER MODULO WITH PRIME ORDER AND THEIR TOPOLOGICAL INDICES

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ABSTRACT

In its application graph theory is widely applied in various fields of science, including scheduling, transportation, industry, and structural chemistry, such as topological indexes. The study of graph theory is also widely applied as a form of representation of algebraic structures, including groups. One form of graph representation that has been studied is non-coprime graphs. The upper unitriangular matrix group is a form of group that can be represented in graph form. This group consists of upper unitriangular matrices, which are a special form of upper triangular matrix with entries in a ring R and all main diagonal entries have a value of one. In this research, we look for the form of a non-coprime graph from the upper unitriangular matrix group over a ring of prime modulo integers and several topological indexes, namely the Harmonic index, Wiener index, Harary index, and First Zagreb index. The findings of this research indicate that the structure of the graph and the general formula for the Harmonic index, Wiener index, Harary index, and First Zagreb index were successfully obtained.

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1. INTRODUCTION

Graph theory is an interesting topic in mathematics to study. Graph theory studies a number of objects and the relationship between these objects. In its application, graph theory is widely applied in various fields of science, scheduling, computers, industry, biological systems, global economic-financial, and chemistry [1], [2]. The molecular graph-based molecular structure descriptors, also known as topological indices, are often used to model the physicochemical properties of chemical compounds [3], [4], [5]. The topological indexes on graphs can be used to describe the shape of a molecule, where it can represent the connectivity of atomic bonds [6]. The use of topological index-derived chemical information has proven to be valuable in chemical documentation, distinguishing between isomers, and establishing correlations between structure and properties [7]. The topological index can also be used as a measure of the characteristics of a graph such as distance, density, or other special properties [8].

A lot of research on graphs is related to abstract algebra theory. The study of graph theory is widely applied as a form of representation of algebraic structures, including groups. There is a lot of previous research on graph representation in a group. Some graph representations of a group are identity graph, inverse graph, commuting, non-commuting graph, coprime graph, non-coprime graph, and others [9]. In 2021, a study on the characteristics of coprime graphs of dihedral groups is given by Syarifudin and team [10]. Not only coprime graphs, one form of graph representation that has been studied is non-coprime graphs.

Non-coprime graphs were introduced by Mansoori and team, they examined several characteristics of non-coprime graphs of a finite group. Some of the graph characteristics obtained are diameter, girth, dominating number, and chromatic number [11]. Then, research by Masriani et al. (2020) also researched non-coprime graphs of groups of integers modulo n for n which are prime [12]. Misuki et al. (2021) researched non-coprime graphs of dihedral groups D_{2n} [13]. Then research by Aulia et al. (2023) also researched non-coprime graphs of dihedral groups with regular composite orders [14]. Apart from the groups described previously, matrix groups can also be objects in non-coprime graph representations.

A matrix is a rectangular arrangement consisting of rows and columns. One special form of a matrix is an upper triangular matrix with all entries on the main diagonal having a value of one, known as an upper unitriangular matrix. Gupta et al. (2016) conducted research on nilpotent graphs from upper unitriangular matrix multiplication groups [15]. Gupta's research also succeeded in finding several topological indexes of the same graph. Since no research has been conducted on non-coprime graphs in groups, specifically in upper unitriangular matrix groups, we have undertaken a study on this topic.

2. RESEARCH METHODS

In this study, the method employed involves collecting several cases based on order, and from these case patterns, forming a conjecture. Once this conjecture is proven, it is stated as a theorem. If the proof fails, the process of collecting cases based on order is repeated with a different pattern

3. RESULTS AND DISCUSSION

In this study, we give the non-coprime graph form of the upper unitriangular matrix group over the ring of modulo prime integers and its four topological indexes, namely the Harmonic index, Wiener index, Harary index, and First Zagreb index. First, we will define the upper unitriangular 3×3 matrix multiplication group over the modulo integer ring.

Definition 1. Let n be an integer with $n > 1$. A set of 3×3 upper unitriangular matrices over $\mathbb{Z}_n$ is defined as follows.

$$G_n = \left\{ \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \middle| \bar{0}, \bar{1}, \bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{Z}_n \right\}.$$

Then it will be shown that G_n is a group of matrix multiplication operations via the following **Theorem 1**.

Theorem 1. *Let n be an integer with $n > 1$. Then the set G_n is a group over the matrix multiplication operation.*

Proof. G_n is closed to the matrix multiplication operation

Let $\Delta, \Lambda \in G_n$, with $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \bar{1} & \bar{\delta} & \bar{\epsilon} \\ \bar{0} & \bar{1} & \bar{\lambda} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, then we have

$$\Delta\Lambda = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{\delta} & \bar{\epsilon} \\ \bar{0} & \bar{1} & \bar{\lambda} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \overline{\delta+\alpha} & \overline{\epsilon+\alpha\lambda+\beta} \\ \bar{0} & \bar{1} & \overline{\lambda+\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \in G_n$$

It is proven that G_n is closed to the matrix multiplication operation.

Multiplication on G_n is associative

Let $\Delta, \Lambda, \Gamma \in G_n$, with $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, $\Lambda = \begin{pmatrix} \bar{1} & \bar{\delta} & \bar{\epsilon} \\ \bar{0} & \bar{1} & \bar{\lambda} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, and $\Gamma = \begin{pmatrix} \bar{1} & \bar{\theta} & \bar{\mu} \\ \bar{0} & \bar{1} & \bar{\sigma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, So

$$\Delta(\Lambda\Gamma) = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\left(\begin{pmatrix} \bar{1} & \bar{\delta} & \bar{\epsilon} \\ \bar{0} & \bar{1} & \bar{\lambda} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{\theta} & \bar{\mu} \\ \bar{0} & \bar{1} & \bar{\sigma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right)$$
$$= \begin{pmatrix} \bar{1} & \overline{\alpha+\delta+\theta} & \overline{\beta+\epsilon+\delta\sigma+\alpha\lambda+\alpha\sigma+\mu} \\ \bar{0} & \bar{1} & \overline{\gamma+\lambda+\sigma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

$$(\Delta\Lambda)\Gamma = \left(\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{\delta} & \bar{\epsilon} \\ \bar{0} & \bar{1} & \bar{\lambda} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right)\begin{pmatrix} \bar{1} & \bar{\theta} & \bar{\mu} \\ \bar{0} & \bar{1} & \bar{\sigma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{1} & \overline{\alpha+\delta+\theta} & \overline{\beta+\epsilon+\delta\sigma+\alpha\lambda+\alpha\sigma+\mu} \\ \bar{0} & \bar{1} & \overline{\gamma+\lambda+\sigma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

Because $\Delta(\Lambda\Gamma) = (\Delta\Lambda)\Gamma$, it is proven that G_n is associative to the multiplication operation.

G_n has an identity element

Let $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \in G_n$. It will be proven that $I = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$ is the identity element of G_n.

$$\Delta I = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = \Delta$$

$$I\Delta = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = \Delta$$

Because $\Delta I = I\Delta = \Delta$. Then it is proven that I is the identity element of G_n.

Each element in G_n has an inverse element

Let $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \in G_n$,Then there exists $\Delta^{-1} = \begin{pmatrix} \bar{1} & \overline{n-\alpha} & \overline{n-\beta+\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{n-\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, such that it holds

$$\Delta\Delta^{-1} = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \overline{n-\alpha} & \overline{n-\beta+\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{n-\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = I$$
$$\Delta^{-1}\Delta = \begin{pmatrix} \bar{1} & \overline{n-\alpha} & \overline{n-\beta+\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{n-\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} = I$$

Since $\Delta\Delta^{-1} = \Delta^{-1}\Delta = I$, it is proven that Δ^{-1} is the inverse of Δ in G_n.

So, it is proven that G_n is a group with respect to the matrix multiplication operation. ■

Next Lemma is important to prove the Theorem in this study. We will explain the power properties of the 3×3 upper unitriangular matrix over $\mathbb{Z}_n$ as follows.

Lemma 1. *Let Δ be a 3×3 upper unitriangular matrix with* $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$, $\bar{0}, \bar{1}, \alpha, \bar{\bar{\beta}}, \bar{\gamma} \in \mathbb{Z}_n$ *and* $\eta \in \mathbb{N}$, *then it holds* $\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{\eta} = \begin{pmatrix} \bar{1} & \overline{\eta\alpha} & \overline{\eta\beta + \frac{\eta(\eta-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{\eta\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$.

Proof. Lemma 1 above will be proven by mathematical induction.

First, it will be shown to be true for $\eta = 1$

$$\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{1} = \begin{pmatrix} \bar{1} & \overline{1\cdot\alpha} & \overline{1\cdot\beta + \frac{1(1-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{1\cdot\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

Then assume true for $\eta = k$,

$$\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{k} = \begin{pmatrix} \bar{1} & \overline{k\alpha} & \overline{k\beta + \frac{k(k-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{k\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

Then we will show that it is true for $\eta = k + 1$,

$$\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{k+1} = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{k}\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{1}$$
$$= \begin{pmatrix} \bar{1} & \overline{(k+1)\alpha} & \overline{\overline{(k+1)\beta + \left(\frac{k(k+1)}{2}\right)\alpha\gamma}} \\ \bar{0} & \bar{1} & \overline{(k+1)\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$$

Hence, $\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{\eta} = \begin{pmatrix} \bar{1} & \overline{\eta\alpha} & \overline{\eta\beta + \frac{\eta(\eta-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{\eta\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$. ■

Lemma 1 can make it easier to search for the order of elements in G_p. In this research, we will study non-coprime graphs from the group G_n with n being prime and symbolized by $\bar{\Gamma}_{G_p}$. First, we will consider the graph $\bar{\Gamma}_{G_2}$.

Example 1. Let G_2 be a 3×3 upper unitriangular matrix group over the ring $\mathbb{Z}_2$ with $G_2 = \left\{ \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \middle| \bar{0}, \bar{1}, \bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{Z}_2 \right\}$. It is easy to see that the number of members of G_2 is 8. The order of each element other than the identity in G_2 is $\left|\begin{pmatrix} \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = \left|\begin{pmatrix} \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = \left|\begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = \left|\begin{pmatrix} \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = \left|\begin{pmatrix} \bar{1} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = 2$, $\left|\begin{pmatrix} \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = \left|\begin{pmatrix} \bar{1} & \bar{1} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right| = 4$.

As a result, for every $\Delta, \Lambda \in G_2 - e$, $(|\Delta|, |\Lambda|) \neq 1$ holds so that Δ and Λ are neighbors in the graph $\bar{\Gamma}_{G_2}$. Therefore, the non-coprime graph form of G_2 is the complete graph K_7 or the complete graph K_{2^3-1}.

Then we will consider the non-coprime graph of G_p for $p \geq 3$. To make it easier to find the order of each element in G_p for $p \geq 3$, consider the following **Lemma 2**.

Lemma 2. *Let p be prime, $p \geq 3$ and define the set G_p with $G_p = \left\{ \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \middle| \bar{0}, \bar{1}, \bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{Z}_p \right\}$. Then the order of each element in $G_p - \{e\}$ is p.*

Proof. Take any $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \in G_p - \left\{ \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix} \right\}$. Then the order of A is the smallest positive integer n such that $\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{n} = \begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$. Based on **Lemma 1**, we obtain $\begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}^{n} = \begin{pmatrix} \bar{1} & \overline{n\alpha} & \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{n\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$. For the expression $\begin{pmatrix} \bar{1} & \overline{n\alpha} & \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} \\ \bar{0} & \bar{1} & \overline{n\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$ to be an identity element, it must be $\overline{n\alpha} = \overline{n\gamma} = \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \bar{0}$.

Case 1 ($\bar{\alpha}, \bar{\gamma} \neq \bar{0}$). Since $\bar{\alpha}$ and $\bar{\gamma}$ are not $\bar{0}$, in order for $\overline{n\alpha} = \overline{n\gamma} = \bar{0}$, then n must be κp for some $\kappa \in \mathbb{N}$. However, because n is the smallest positive integer, it must be $\kappa = 1$, resulting in $n = p$. Then it will also be shown that $n = p$ results in the expression $\overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \bar{0}$. Note that for $n = p$, the expression $\overline{n\beta} = \bar{0}$. Then for the expression $\overline{\frac{n(n-1)}{2}\alpha\gamma}$, because $p \geq 3$, we obtain $\overline{\frac{p(p-1)}{2}\alpha\gamma} = \bar{0}$. Then it is proven that in case 1, in order for $\overline{n\alpha} = \overline{n\gamma} = \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \bar{0}$, it must be $n = p$.

Case 2 ($\bar{\alpha}, \bar{\gamma}$ one of which is $\bar{0}$). Without reducing generality, let's say $\bar{\alpha} = \bar{0}$. Then it is clear that $\overline{n\alpha} = \bar{0}$ and $\overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \overline{n\beta}$. In order for $\overline{n\gamma} = \bar{0}$, then n must be κp for some $\kappa \in \mathbb{N}$. However, because n is the smallest positive integer, it must be $\kappa = 1$, resulting in $n = p$. Also, note that $n = p$ results in $\overline{n\beta} = \bar{0}$. Then it is proven that in case 2, for $\overline{n\alpha} = \overline{n\gamma} = \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \bar{0}$, it must be $n = p$.

Case 3 ($\bar{\alpha}, \bar{\gamma}$ both have the value $\bar{0}$). In case 3, it is clear that $\overline{n\alpha} = \overline{n\gamma} = \bar{0}$ and $\overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \overline{n\beta}$ with $\bar{\beta} \neq \bar{0}$ So that $\overline{n\beta} = \bar{0}$, then n must have the value κp for some $\kappa \in \mathbb{N}$. However, because n is the smallest positive integer, it must be $\kappa = 1$, resulting in $n = p$. Then it is proven that in case 3, in order for $\overline{n\alpha} = \overline{n\gamma} = \overline{n\beta + \frac{n(n-1)}{2}\alpha\gamma} = \bar{0}$, it must be $n = p$.

So from the three cases above, it can be concluded that the order of $\Delta = \begin{pmatrix} \bar{1} & \bar{\alpha} & \bar{\beta} \\ \bar{0} & \bar{1} & \bar{\gamma} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}$ is p with p prime and $p \geq 3$. Because A is any element of $G_p - \left\{\begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right\}$, it follows that every element in the group $G_p - \left\{\begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \end{pmatrix}\right\}$ is of order p. ■

Once the order of each element in the group G_p is known for $p \geq 3$, we can look for the non-coprime graph $\bar{\Gamma}_{G_p}$. The non-coprime graph of the group G_p is given in the following **Theorem 2**.

Theorem 2. *Let G_p be a group of 3×3 upper unitriangular matrices over $\mathbb{Z}_p$ with p prime. Then the non-coprime graph of G_p denoted by $\bar{\Gamma}_{G_p}$ is the complete graph K_{p^3-1}.*

Proof. Let $G_p$be a group of 3×3 upper unitriangular matrices over $\mathbb{Z}_p$ with p prime. Based on **Example 1**, it is clear that the graph $\bar{\Gamma}_{G_2}$ is the complete graph $K_7 = K_{p^3-1.}$ Then we will show the form of the graph $\bar{\Gamma}_{G_p}$for $p \geq 3$. It is easy to see that many elements of $G_p - e$ are $p^3 - 1$. Let Δ, Λ be any different elements in $G_p - e$. Based on **Lemma 2**, the order of each element in $G_p - e$ is p, so for every two different elements Δ, Λ in $G_p - e$ it becomes $(|\Delta|, |\Lambda|) = p \neq 1$. As a result, every two different points on the graph $\bar{\Gamma}_{G_p}$ are adjacent to each other. Then we get that the graph $\bar{\Gamma}_{G_p}$is a complete graph $K_{p^3-1.}$ ■

Based on **Theorem 2**, we will give the general formula of the topological indices. First, the Harmonic index is defined as follows.

Definition 2. **[16]** Let G represent a graph. The Harmonic index of the graph G denoted by $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{\deg(u) + \deg(v)}$$

Where $deg\ (u)$ is the degree of node u, namely the number of nodes $v \neq u$ that are neighbors of u.

The Harmonic index of the graph $\bar{\Gamma}_{G_p}$ is explained through the following theorem.

Theorem 3. *Suppose $\bar{\Gamma}_{G_p}$ is a non-coprime graph of the group G_p, where p is a prime number, then the Harmonic index of $\bar{\Gamma}_{G_p}$ is*

$$H\left(\bar{\Gamma}_{G_p}\right) = \frac{p^3 - 1}{2}$$

Proof. Based on **Theorem 2**, every vertex in the graph $\bar{\Gamma}_{G_p}$has the same degree, which is $p^3 - 2$. Furthermore, because the graph $\bar{\Gamma}_{G_p}$ is a complete graph $K_{p^3-1.}$ then every different point in $\bar{\Gamma}_{G_p}$ is adjacent to the resulting number of edges is $\frac{(p^3-1)(p^3-2)}{2}$. So the Harmonic index of the graph $\bar{\Gamma}_{G_p}$ is

$$H(G) = \sum_{uv \in E(G)} \frac{2}{\deg(u) + \deg(v)}$$

$$H\left(\bar{\Gamma}_{G_p}\right) = \sum_{uv \in E(\bar{\Gamma}_{G_p})} \frac{2}{(p^3 - 2) + (p^3 - 2)}$$

$$H\left(\bar{\Gamma}_{G_p}\right) = \sum_{i=1}^{\frac{(p^3-1)(p^3-2)}{2}} \frac{2}{(p^3-2)+(p^3-2)}$$
$$H\left(\bar{\Gamma}_{G_p}\right) = \sum_{i=1}^{\frac{(p^3-1)(p^3-2)}{2}} \frac{2}{2p^3-4}$$
$$H\left(\bar{\Gamma}_{G_p}\right) = \sum_{i=1}^{\frac{(p^3-1)(p^3-2)}{2}} \frac{2}{2(p^3-2)}$$
$$H\left(\bar{\Gamma}_{G_p}\right) = \sum_{i=1}^{\frac{(p^3-1)(p^3-2)}{2}} \frac{1}{p^3-2}$$
$$H\left(\bar{\Gamma}_{G_p}\right) = \left(\frac{(p^3-1)(p^3-2)}{2}\right)\left(\frac{1}{p^3-2}\right)$$
$$H\left(\bar{\Gamma}_{G_p}\right) = \frac{p^3-1}{2}. \blacksquare$$

Next, we will give the general formula of the Wiener index. The sum of distances between all unordered pairs of vertices in a graph defines the Wiener index, making it one of the most widely used molecular descriptors [17]. The definition of the Wiener index is given in the following definition.

Definition 3. [18] Let G represent a simple connected graph. The Wiener Index of G symbolized by $W(G)$ can be defined as

$$W(G) = \frac{1}{2}\left(\sum_{u,v \in V(G)} d(u,v)\right)$$

Where $d(u,v)$ is the distance from u to v and $d(u,v) \neq d(v,u)$.

The general formula of the Wiener Index of the graph is given by the following theorem.

Theorem 4. *Let $\bar{\Gamma}_{G_p}$ be a non-coprime graph of the group G_p, where p is a prime number, then the Wiener index of $\bar{\Gamma}_{G_p}$ is*

$$W\left(\bar{\Gamma}_{G_p}\right) = \frac{(p^3-1)(p^3-2)}{2}$$

Proof. Based on Theorem 2, the distance of every two vertices in $\bar{\Gamma}_{G_p}$ have a distance of 1. And, because every two different vertices in $\bar{\Gamma}_{G_p}$ are neighbors, the number of ordered pairs of distinct points in the graph $\bar{\Gamma}_{G_p}$ is $P_2^{p^3-1} = (p^3-1)(p^3-2)$. So the Wiener index of the graph $\bar{\Gamma}_{G_p}$ is

$$W(G) = \frac{1}{2}\left(\sum_{u,v \in V(G)} d(u,v)\right)$$
$$W\left(\bar{\Gamma}_{G_2}\right) = \frac{1}{2}\left(\sum_{i=1}^{(p^3-1)(p^3-2)} 1\right)$$
$$W\left(\bar{\Gamma}_{G_p}\right) = \frac{1}{2}((p^3-1)(p^3-2))$$
$$W\left(\bar{\Gamma}_{G_p}\right) = \frac{(p^3-1)(p^3-2)}{2}. \blacksquare$$

Next, we will give the general formula of the Harary index. The Harary index is another measure of graphs that is closely associated with the Wiener index [19].The definition of the index Harary is given in the following definition.

Definition 4. [7] Suppose G is connected and nontrivial, the Harary Index of G symbolized by $\mathcal{H}(G)$ is defined as

$$\mathcal{H}(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}$$

Where $d(u,v)$ is the distance from u to v and $d(u,v) = d(v,u)$

And the general formula of the index Harary is given by the following theorem.

Theorem 5. *Suppose $\bar{\Gamma}_{G_p}$ is a non-coprime graph of the group G_p, where p is a prime number, then the Harary index of $\bar{\Gamma}_{G_p}$ is*

$$\mathcal{H}\left(\bar{\Gamma}_{G_p}\right) = \frac{(p^3-1)(p^3-2)}{2}$$

Proof. Based on **Theorem 2**, every two different vertices in $\bar{\Gamma}_{G_p}$ have a distance of 1. Because every two different vertices in $\bar{\Gamma}_{G_p}$ are neighbors, the number of distinct pairs of points on the graph $\bar{\Gamma}_{G_p}$ is $C_2^{p^3-1} = \frac{(p^3-1)(p^3-2)}{2}$. So the Harary index of the graph $\bar{\Gamma}_{G_p}$ is

$$\mathcal{H}(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}$$

$$\mathcal{H}\left(\bar{\Gamma}_{G_p}\right) = \sum_{u,v \in V(\bar{\Gamma}_{G_p})} \frac{1}{1}$$

$$\mathcal{H}\left(\bar{\Gamma}_{G_p}\right) = \sum_{i=1}^{\frac{(p^3-1)(p^3-2)}{2}} 1$$

$$\mathcal{H}\left(\bar{\Gamma}_{G_p}\right) = \frac{(p^3-1)(p^3-2)}{2}. \blacksquare$$

Next, we will give the general formula of the first Zagreb index. The first Zagreb index is one of the oldest vertex–degree–based molecular structure descriptors [20]. The First Zagreb index of the graph is given in the following definition.

Definition 5. [21] Let G be a nontrivially connected graph. The first Zagreb eccentricity index of G symbolized by $Z_1(G)$ can be defined as follows

$$Z_1(G) = \sum_{v \in V(G)} (deg(v))^2$$

Example 2. We will find the first Zagreb index of the graph $\bar{\Gamma}_{G_2}$. Based on **Example 1**, the graph $\bar{\Gamma}_{G_2}$ is a complete graph. As a result, the graph $\bar{\Gamma}_{G_2}$ has 7 vertices with the degree of each vertex being 6. So by **Definition 5**, the first Zagreb index of the graph $\bar{\Gamma}_{G_2}$ is

$$Z_1(\bar{\Gamma}_{G_2}) = \sum_{v \in V(\bar{\Gamma}_{G_2})} (deg(v))^2$$

$$Z_1\left(\bar{\Gamma}_{G_2}\right) = (6)^2 + (6)^2 + (6)^2 + (6)^2 + (6)^2 + (6)^2 + (6)^2$$

$$Z_1\left(\bar{\Gamma}_{G_2}\right) = 36 + 36 + 36 + 36 + 36 + 36 + 36$$

$$Z_1\left(\bar{\Gamma}_{G_2}\right) = 252$$

The First Zagreb Index of the graph $\bar{\Gamma}_{G_p}$ is given by the following theorem.

Theorem 6. *Let $\bar{\Gamma}_{G_p}$ be a non-coprime graph of the group G_p, where p is a prime number, then the first Zagreb index of $\bar{\Gamma}_{G_p}$ is*

$$Z_1\left(\bar{\Gamma}_{G_p}\right) = (p^3-1)\,(p^3-2)^2$$

Proof. Based on Theorem 2, then the degree of each vertex in $\overline{\Gamma}_{G_p}$ is all same $p^3 - 2$. So the First Zagreb index of the graph $\overline{\Gamma}_{G_p}$ is

$$Z_1(G) = \sum_{v \in V(G)} (deg(v))^2$$

$$Z_1(\overline{\Gamma}_{G_p}) = \sum_{v \in V(\overline{\Gamma}_{G_p})} (deg(v))^2$$

$$Z_1(\overline{\Gamma}_{G_p}) = \sum_{i=1}^{p^3-1} (deg(v_i))^2$$

$$Z_1(\overline{\Gamma}_{G_p}) = \sum_{i=1}^{p^3-1} (p^3 - 2)^2$$

$$Z_1\left(\overline{\Gamma}_{G_p}\right) = (p^3 - 1)(p^3 - 2)^2. \blacksquare$$

Next, using Theorem 6, we can also calculate the First Zagreb index from the graph $\overline{\Gamma}_{G_2}$ as in **Example 3** below.

Example 3. Using Theorem 6, the First Zagreb index of the graph $\overline{\Gamma}_{G_2}$ where $p = 2$ is

$$Z_1(\overline{\Gamma}_{G_p}) = (p^3 - 1)(p^3 - 2)^2$$

$$Z_1(\overline{\Gamma}_{G_2}) = (2^3 - 1)(2^3 - 2)^2$$

$$Z_1(\overline{\Gamma}_{G_2}) = (7)(6)^2$$

$$Z_1(\overline{\Gamma}_{G_2}) = 252$$

Based on **Example 3**, using Theorem 6 can give the same results as **Example 2** above.

4. CONCLUSIONS

Based on the study, the structure of the non-coprime graph form of the 3×3 upper unitriangular matrix group over the ring of integers modulo is a complete graph whenever the order is prime. The general formulas for the topological indices, such as the Harmonic index, the Wiener index, the Harary index, and the First Zagreb index of the graph are also obtained.

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