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CRAMER'S RULE IN INTERVAL MIN-PLUS ALGEBRA

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ABSTRACT

A min-plus algebra is a set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of all real numbers, equipped with the minimum ($\oplus'$) and addition ($\otimes$) operations. The system of linear equations $A \otimes x = b$ in min-plus algebra can be solved using Cramer's rule. Interval min-plus algebra is an extension of min-plus algebra, with the elements in it being closed intervals. The set is denoted by $I(\mathbb{R})_{\varepsilon'}$ equipped with two binary operations, namely minimum ($\overline{\oplus}'$) and addition ($\overline{\otimes}$). The matrix with notation $I(\mathbb{R})_{\varepsilon'}^{m\times n}$ is a matrix over interval min-plus algebra with size $m \times n$. Since the structure of min-plus algebra and interval min-plus algebra are analogous, the system of linear equations $A \overline{\otimes} x = b$ in interval min-plus algebra can be solved using Cramer's rule. Based on the research results, the sufficient conditions of Cramer's rule in interval min-plus algebra are $sign(\underline{A}_i) = sign(\underline{A}), sign(\overline{A}_i) = sign(\overline{A})$ for $1 \le i \le n$, and $dom(A) = [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$. The Cramer rule is $x_i \overline{\otimes}\, dom(A) = dom(A_i)$.

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1. INTRODUCTION

In algebra, there are various structures, and one of them is conventional algebra. Conventional algebra is the set of all real numbers equipped with addition $(+)$ and multiplication $(\times)$ operations [1]. In conventional algebra, a system of linear equations is expressed as $Ax = b$. As long as $\det(A) \neq 0$, Cramer's rule can be used to solve the system of linear equations $Ax = b$ [2][3].

There is another structure, namely max-plus algebra [4]. In max-plus algebra, the maximum operation does not have an inverse so the determinant in max-plus algebra is not defined similarly to conventional algebra. There are two methods for representing determinants in max-plus algebra, permanent and dominant. [5][6]. The system of linear equations $A \otimes x = b$ in max-plus algebra can be solved using Cramer's rule with dominant values [7][8][9]. Then, interval max-plus algebra is created by extending max-plus algebra [10][11][12]. The system of linear equations $A \overline{\otimes} x = b$ in interval max-plus algebra can also be solved using Cramer's rule with dominant value [13][14].

A min-plus algebra is the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of all real numbers, equipped with the minimum $(\oplus')$ and addition $(\otimes)$ operations. Therefore, the operation $a \oplus' b = \min(a, b)$ and $a \otimes b = a + b$ applies. Min-plus algebra is a semiring denoted by $\mathbb{R}_{min} = (\mathbb{R}_{\varepsilon'}, \oplus', \otimes)$ and has a neutral element $\varepsilon' = +\infty$ for the minimum operation and a neutral element $e = 0$ for the addition operation [15][16]. Min-plus algebra can be formed into a set of matrices of size $m \times n$, m and n are positive integers, with the entries being elements of $\mathbb{R}_{\varepsilon'}$ and denoted as $\mathbb{R}_{\varepsilon'}^{n \times n}$ [17][18][19]. The determinant in min-plus algebra is represented the same as in max-plus algebra, which is permanent and dominant [17]. The system of linear equations $A \otimes x = b$ in min-plus algebra always has the smallest sub-solution. Cramer's rule in min-plus algebra is $x_i \otimes dom(A) = dom(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n), i = 1,2, \dots, n$ with a_j is the entry in the j-th column of the matrix A. The solution of the system of linear equations $A \otimes x = b$ does not necessarily exist in min-plus algebra, and if it does, it is not always singular, much as in conventional algebra. As a result, an additional condition is needed so that the system of linear equations can be solved using Cramer's rule, namely $sign(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) = sign(A)$ for $i = 1,2, \dots, n$ with a_j is the entry in the j-th column of the matrix A [20].

Interval min-plus algebra is an extension of min-plus algebra, with elements in it being closed intervals. The set is denoted by $I(\mathbb{R})_{\varepsilon'}$ equipped with two binary operations, namely minimum $(\overline{\oplus}')$ and addition $(\overline{\otimes})$. The structure of interval min-plus algebra can be written as $I(\mathbb{R})_{min} = (I(\mathbb{R})_{\varepsilon'}, \overline{\oplus}', \overline{\otimes})$ [21]. The matrix with the notation $I(\mathbb{R})_{\varepsilon'}^{m \times n}$ is a matrix over interval min-plus algebra with size $m \times n$. If $m = n$, a set of square matrices is obtained, namely $I(\mathbb{R})_{\varepsilon'}^{n \times n}$ [22][23]. Similar to min-plus algebra, the determinant in an interval min-plus algebra is defined with two approaches, $perm(A) = [perm(\underline{A}), perm(\overline{A})]$ with $perm(\underline{A}) = \oplus'_{\sigma \in P_n} \otimes_{i=1}^{n} (\underline{a}_{i\sigma(i)})$ and $perm(\overline{A}) = \oplus'_{\sigma \in P_n} \otimes_{i=1}^{n} (\overline{a}_{i\sigma(i)})$ and dominant i.e. $dom(A) = \left[\min\left(dom(\underline{A}), dom(\overline{A})\right), dom(\overline{A})\right]$ with $dom(\underline{A}) = \begin{cases} lowest\ exponent\ in \det(z^{\underline{A}}), & if\ \det(z^{\underline{A}}) \neq 0 \\ \varepsilon', if \det(z^{\underline{A}}) = 0 \end{cases}$ and $dom(\overline{A}) = \begin{cases} lowest\ exponent\ in \det\left(z^{\overline{A}}\right), if \det(z^{\overline{A}}) \neq 0 \\ \varepsilon', if \det\left(z^{\overline{A}}\right) = 0 \end{cases}$ [24]. In this article, we will discuss the sufficient conditions for a system of linear equations $A \overline{\otimes} x = b$ in interval min-plus algebra to be solved using Cramer's rule and its solution.

2. RESEARCH METHODS

The research method used in writing this article is a literature study by using references to books, articles from [20][24] or writings on interval min-plus algebra, matrices over interval min-plus algebra, and systems of linear equations over interval min-plus algebra. In addition, it also uses references that discuss min-plus algebra and Cramer's rule in min-plus algebra.

In this study, three steps were taken which are described as follows.

1. Determine Cramer's rule in interval min-plus algebra using matrix dominants based on analogies corresponding to min-plus algebra.
2. Determine the definition of $sign$ matrix over interval min-plus algebra.
3. Determine the sufficient condition that the system of linear equations $A \overline{\otimes} x = b$ in interval min-plus algebra can be solved using Cramer's rule.

3. RESULTS AND DISCUSSION

Suppose $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_{\varepsilon'}^{n\times n}, x \approx [\underline{x}, \overline{x}] \in I(\mathbb{R})_{\varepsilon'}^{n\times 1}$, and $b \approx [\underline{b}, \overline{b}] \in I(\mathbb{R})_{\varepsilon'}^{n\times 1}$, the system of linear equations $A \overline{\otimes} x = b$ in interval min-plus algebra always has a smallest sub-solution. The smallest sub-solution is not always the solution of a system of linear equations.

Determinants in interval min-plus algebra are represented with two approaches, namely permanent and dominant. By the same analogy with min-plus algebra, the system of linear equations $A \overline{\otimes} x = b$ with $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_{\varepsilon'}^{n\times n}, x \approx [\underline{x}, \overline{x}] \in I(\mathbb{R})_{\varepsilon'}^{n\times 1}$, and $b \approx [\underline{b}, \overline{b}] \in I(\mathbb{R})_{\varepsilon'}^{n\times 1}$ in interval min-plus algebra always has the smallest subsolution. The smallest sub-solution is not always the solution of a system of linear equations. The solution of a system of linear equations in interval min-plus algebra is presented in **Lemma 1**.

Lemma 1. *Suppose $A \approx [\underline{A}, \overline{A}], x \approx [\underline{x}, \overline{x}]$, and $b \approx [\underline{b}, \overline{b}]$. If the linear equations $\underline{A} \otimes \underline{x} = \underline{b}$ and $\overline{A} \otimes \overline{x} = \overline{b}$ each has a single solution $\underline{\check{x}} = -\left(\underline{A}^T \otimes (-\underline{b})\right)$ and $\overline{\check{x}} = -\left(\overline{A}^T \otimes (-\overline{b})\right)$ with $\underline{\check{x}} \leq \overline{\check{x}}$ then the system of linear equations $A \overline{\otimes} x = b$ has a single solution $x \approx [\underline{\check{x}}, \overline{\check{x}}]$.*

Proof. Suppose $A \approx [\underline{A}, \overline{A}], x \approx [\underline{x}, \overline{x}]$, and $b \approx [\underline{b}, \overline{b}]$. According to the matrix operations in the min-plus algebra of intervals $A \overline{\otimes} x \approx [\underline{A}, \overline{A}] \overline{\otimes} [\underline{x}, \overline{x}] = [\underline{A} \otimes \underline{x}, \overline{A} \otimes \overline{x}]$. This means that $\underline{A} \otimes \underline{x} = \underline{b}$ and $\overline{A} \otimes \overline{x} = \overline{b}$. Since the linear equation systems $\underline{A} \otimes \underline{x} = \underline{b}$ and $\overline{A} \otimes \overline{x} = \overline{b}$ each has a single solution $\underline{\check{x}} = -\left(\underline{A}^T \otimes (-\underline{b})\right)$ and $\overline{\check{x}} = -\left(\overline{A}^T \otimes (-\overline{b})\right)$ with $\underline{\check{x}} \leq \overline{\check{x}}$ then the system of linear equations $A \overline{\otimes} x = b$ has a single solution $x \approx [\underline{\check{x}}, \overline{\check{x}}]$.

Cramer's rule in interval min-plus algebra is written as,

$$x_i \overline{\otimes} dom(A) = dom(A_i), i = 1,2, \dots, n, \tag{1}$$

with $A_i = a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n$ and $A_i \approx [\underline{A}_i, \overline{A}_i]$ then $\underline{A}_i = \underline{a}_1, \dots, \underline{a}_{i-1}, \underline{b}, \underline{a}_{i+1}, \dots, \underline{a}_n$ and $\overline{A}_i = \overline{a}_1, \dots, \overline{a}_{i-1}, \overline{b}, \overline{a}_{i+1}, \dots, \overline{a}_n$.

Furthermore, if $dom(A) \approx [dom(\underline{A}), dom(\overline{A})]$ and $dom(A_i) \approx [dom(\underline{A}_i), dom(\overline{A}_i)]$ then it means $\underline{x}_i \leq \overline{x}_i$. Hence it holds

$$x_i \overline{\otimes} dom(A) = dom(A_i) \Leftrightarrow \underline{x}_i \otimes dom(\underline{A}) = dom(\underline{A}_i) \text{ and } \overline{x}_i \otimes dom(\overline{A}) = dom(\overline{A}_i) \tag{2}$$

Analogous to Cramer's rule in min-plus algebra, $dom(A) \approx [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$ is not enough to show that the system of linear equations has a solution. There is an additional condition for the system of linear equations $A \overline{\otimes} x = b$ to have a solution, namely

$$sign(\underline{A}_i) = sign(\underline{A}) \text{ and } sign(\overline{A}_i) = sign(\overline{A}), \tag{3}$$

for $1 \leq i \leq n$. The following is a description of the notations used to define the $sign$ of a matrix.

Definition 1. Let P_n be the set of all permutations of $\{1,2, \dots, n\}$ and $t_1, t_2, \dots, t_L$ are all possible values such that $\underline{t}_j = \otimes_{j=1}^{n} (\underline{a}_{j\sigma(j)})$ and $\overline{t}_j = \otimes_{j=1}^{n} (\overline{a}_{j\sigma(j)})$ with dengan $A = [a_{ij}] \in I(\mathbb{R})_{\varepsilon'}^{n\times n}, A \approx [\underline{A}, \overline{A}]$ for all permutations $\sigma \in P_n$. Given $\underline{S}_j$ and $\overline{S}_j$ suppose that

$$\underline{S}_j = \{\sigma \in P_n | \underline{t}_j = \otimes_{j=1}^{n} (\underline{a}_{j\sigma(j)})\},$$

$$\underline{S}_{je} = \{\sigma \in \underline{S}_j | \sigma \in P_n^e\},$$
$$\underline{S}_{jo} = \{\sigma \in \underline{S}_j | \sigma \in P_n^o\},$$
$$\underline{k}_{je} = |\underline{S}_{je}|, and\ \underline{k}_{jo} = |\underline{S}_{jo}|,$$

and

$$\overline{S}_j = \{\sigma \in P_n | \overline{t}_j = \otimes_{j=1}^{n} (\overline{a}_{j\sigma(j)})\},$$
$$\overline{S}_{je} = \{\sigma \in \overline{S}_j | \sigma \in P_n^e\},$$
$$\overline{S}_{jo} = \{\sigma \in \overline{S}_j | \sigma \in P_n^o\},$$
$$\overline{k}_{je} = |\overline{S}_{je}|, and\ \overline{k}_{jo} = |\overline{S}_{jo}|.$$

With P_n^e and P_n^o being the set of all even and odd permutations of P_n, respectively.

Using **Definition 1**, the following definition of the sign is obtained.

Definition 2. Given $A = [a_{ij}] \in I(\mathbb{R})_{\varepsilon'}^{n\times n}$ with $A \approx [\underline{A}, \overline{A}]$. For $\underline{t}_j = dom(\underline{A}), \overline{t}_j = dom(\overline{A})$, and $1 \leq j \leq L$, $sign(\underline{A}) = \begin{cases} 1, if\ \underline{k}_{je} - \underline{k}_{jo} > 0 \\ -1, if\ \underline{k}_{je} - \underline{k}_{jo} < 0 \end{cases}$ and $sign(\overline{A}) = \begin{cases} 1, if\ \overline{k}_{je} - \overline{k}_{jo} > 0 \\ -1, if\ \overline{k}_{je} - \overline{k}_{jo} < 0 \end{cases}$. If $dom(\underline{A}) = \varepsilon'$ and $dom(\overline{A}) = \varepsilon'$ then $sign(\underline{A}) = \varepsilon'$ and $sign(\overline{A}) = \varepsilon'$.

Example 1. Given matrix $A \in I(\mathbb{R})_{\varepsilon'}^{3\times 3}$

$$A = \begin{pmatrix} [2,3] & [2,5] & [0,2] \\ [3,5] & [-2,-1] & [-1,0] \\ [-1,2] & [2,3] & [1,3] \end{pmatrix}$$

Interval matrix A can be written as matrix interval $[\underline{A}.\overline{A}]$ i.e.

$$A \approx [\underline{A}, \overline{A}] = \left[\begin{pmatrix} 2 & 2 & 0 \\ 3 & -2 & -1 \\ 5 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 2 \\ 5 & -1 & 0 \\ 2 & 3 & 3 \end{pmatrix} \right]$$

Matrix A is a matrix with $n = 3$ so we get $3! = 3 \times 2 \times 1 = 6$. This means there are six permutations of $\{1, 2, 3\}$ with each result of the permutation of $\underline{t}_j$ and $\overline{t}_j$ shown in **Table 1**.

Table 1. Result of Permutation $n = 3$ of $\underline{t}_j$ and $\overline{t}_j$

$\sigma_j = 1, 2, 3$	$\underline{t}_j = \otimes_{i=1}^{3} (\underline{a}_{i\sigma_j(i)})$	$\overline{t}_j = \otimes_{i=1}^{3} (\overline{a}_{i\sigma_j(i)})$
$\sigma_1 = 1,2,3$	$\underline{t}_1 = 2 \otimes -1 \otimes 1 = 2$	$\overline{t}_1 = 3 \otimes -1 \otimes 3 = 5$
$\sigma_2 = 1,3,2$	$\underline{t}_2 = 2 \otimes -1 \otimes 2 = 3$	$\overline{t}_2 = 3 \otimes 0 \otimes 3 = 6$
$\sigma_3 = 2,1,3$	$\underline{t}_3 = 2 \otimes 3 \otimes 1 = 6$	$\overline{t}_3 = 5 \otimes 5 \otimes 3 = 13$
$\sigma_4 = 2,3,1$	$\underline{t}_4 = 2 \otimes -1 \otimes -1 = 0$	$\overline{t}_4 = 5 \otimes 0 \otimes 2 = 7$
$\sigma_5 = 3,1,2$	$\underline{t}_5 = 0 \otimes 3 \otimes 2 = 5$	$\overline{t}_5 = 2 \otimes 5 \otimes 3 = 10$
$\sigma_6 = 3,2,1$	$\underline{t}_6 = 0 \otimes -2 \otimes -1 = -3$	$\overline{t}_6 = 2 \otimes -1 \otimes 2 = 3$

Then calculate the dominant value of the matrix $\underline{A}$ using reference [24],

$$z^{\underline{A}} = \begin{pmatrix} z^2 & z^2 & z^0 \\ z^3 & z^{-2} & z^{-1} \\ z^{-1} & z^2 & z^1 \end{pmatrix},$$

because $det(z^{\underline{A}}) = z^1 - z^3 - z^6 + z^0 + z^5 - z^{-3} \neq 0$ results in $dom(\underline{A}) = -3$. Based on **Table 1.** $dom(\underline{A}) = -3 = \underline{t}_6$. Subsequently obtained $\underline{S}_6, \underline{S}_{6e}, \underline{S}_{6o}, \underline{k}_{6e}$ and $\underline{k}_{6o}$ which are written as

$$\underline{S}_6 = \{\sigma \in P_3 | \underline{t}_3 = \otimes_{j=1}^{n} (\underline{a}_{j\sigma(j)})\} = \{\sigma_6\},$$
$$\underline{S}_{6e} = \{\sigma \in \underline{S}_6 | \sigma \in P_3^e\} = \{\},$$
$$\underline{S}_{6o} = \{\sigma \in \underline{S}_6 | \sigma \in P_3^o\} = \{\sigma_6\},$$
$$\underline{k}_{6e} = |\underline{S}_{6e}| = 0, and\ \underline{k}_{6o} = |\underline{S}_{6o}| = 1,$$

with **Definition 2**, get $\underline{k}_{6e} - \underline{k}_{6o} = 0 - 1 = -1 < 0$ so the value of $sign(\underline{A}) = -1$. Next, calculate the dominant matrix $\overline{A}$ using reference [24] that is written as

$$z^{\overline{A}} = \begin{pmatrix} z^3 & z^5 & z^2 \\ z^5 & z^{-1} & z^0 \\ z^2 & z^3 & z^3 \end{pmatrix}$$

because $det\left(z^{\overline{A}}\right) = z^5 - z^6 - z^{13} + z^7 + z^{10} - z^3 \neq 0$ results in $dom(\overline{A}) = 3$. Based on **Table 1**. $dom(\overline{A}) = 3 = \overline{t}_6$. Subsequently obtained $\overline{S}_6, \overline{S}_{6e}, \overline{S}_{6o}, \overline{k}_{6e}$ and $\overline{k}_{6o}$ which are written as

$$\begin{aligned} \overline{S}_6 &= \{\sigma \in P_3 | \overline{t}_3 = \otimes_{j=1}^{n} (\overline{a}_{j\sigma(j)})\} = \{\sigma_6\}, \\ \overline{S}_{6e} &= \{\sigma \in \overline{S}_6 | \sigma \in P_3^e\} = \{\}, \\ \overline{S}_{6o} &= \{\sigma \in \overline{S}_6 | \sigma \in P_3^o\} = \{\sigma_6\}, \\ \overline{k}_{6e} &= |\overline{S}_{6e}| = 0, and\ \overline{k}_{6o} = |\overline{S}_{6o}| = 1, \end{aligned}$$

with **Definition 2**, get $\overline{k}_{6e} - \overline{k}_{6o} = 0 - 1 = -1 < 0$ so the value of $sign(\overline{A}) = -1$.

The following **Theorem 1** describes the sufficient condition of the system of linear equations $A \overline{\otimes} x = b$ with Cramer's rule in interval min-plus algebra.

Theorem 1. *If* $sign(\underline{A}_i) = sign(\underline{A}), sign(\overline{A}_i) = sign(\overline{A})$ *for* $1 \leq i \leq n$, *and* $dom(A) \approx [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$ *then* $x \approx [\underline{x}, \overline{x}]$ *is a solution to the system of linear equations* $A \overline{\otimes} x = b$ *with* $x_i \overline{\otimes} dom(A) = dom(A_i)$.

Proof. Let $A \overline{\otimes} x = b$ with $A \approx [\underline{A}, \overline{A}], x \approx [\underline{x}, \overline{x}]$, and $b \approx [\underline{b}, \overline{b}]$ be a system of linear equations. Assume $sign(\underline{A}_i) = sign(\underline{A}), sign(\overline{A}_i) = sign(\overline{A})$ for $1 \leq i \leq n$, and $dom(A) \approx [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$. Then, express the system in the form of $z^A \approx \left[z^{\underline{A}}, z^{\overline{A}}\right], \xi \approx \left[\underline{\xi}, \overline{\xi}\right]$, and $z^b \approx \left[z^{\underline{b}}, z^{\overline{b}}\right]$, we get the following equation.

$$z^A \xi = z^b \Leftrightarrow z^{\underline{A}} \underline{\xi} = z^{\underline{b}}\ and\ z^{\overline{A}} \overline{\xi} = z^{\overline{b}} \tag{4}$$

Since $dom(A) \approx [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$ and $\det(z^A) \neq [0.0]$, **Equation (4)** can be solved using Cramer's rule. The solution of the interval matrix A with Cramer's rule is described as follows.

i. For $\underline{A}$

$$\underline{\xi}_i = \frac{\det(z^{\underline{A}_i})}{\det(z^{\underline{A}})} \tag{5}$$

If $z \longrightarrow \infty$ then the value of $\underline{\xi}_i$ is determined by the dominant of the right-hand segment in **Equation (5)**. The value of $\det(z^{\underline{A}_i})$ will lead to the value of $dom(\underline{A}_i)$ within the shape of $z^{dom(\underline{A}_i)}$, moreover for $\det(z^{\underline{A}})$. Suppose

$$\underline{d}_i = dom(\underline{A}_i), 1 \leq i \leq n \tag{6}$$

Based on the $sign$ assumption, $\underline{\xi}_i > 0$ is obtained so that $\underline{\xi}_i$ can be written as

$$\underline{\xi}_i \approx z^{\underline{d}_i - dom(\underline{A})} \tag{7}$$

Furthermore, by substituting **Equation (7)** into $z^{\underline{A}} \underline{\xi} = z^{\underline{b}}$, we obtain

$$\sum_{j=1}^{n} z^{\underline{a}_{ij} + \underline{d}_j - dom(\underline{A})} \approx z^{\underline{b}_i} \tag{8}$$

Equation (8) in min-plus algebra has the meaning as

$$\bigotimes_{j=1}^{n}{}' \left(\underline{a}_{ij} + \underline{d}_j - dom(\underline{A})\right) = \underline{b}_i, 1 \leq i \leq n \tag{9}$$

Thus, if $\underline{x}_i = \underline{d}_i - dom(\underline{A}), 1 \leq i \leq n$ then $\underline{x}_i$ is a solution to the system of linear equations $\underline{A} \otimes \underline{x} = \underline{b}$.

ii. For $\overline{A}$

$$\bar{\xi}_i = \frac{\det\left(z^{\bar{A}_i}\right)}{\det\left(z^{\bar{A}}\right)} \tag{10}$$

If $z \longrightarrow \infty$ then the value of $\bar{\xi}_i$ is determined by the dominant of the right-hand segment in equation (10). The value of $\det\left(z^{\bar{A}_i}\right)$ will lead to the value of $dom\left(\bar{A}_i\right)$ within the shape of $z^{dom(\bar{A}_i)}$, moreover for $\det\left(z^{\bar{A}}\right)$. Suppose

$$\bar{d}_i = dom\left(\bar{A}_i\right), 1 \le i \le n \tag{11}$$

Based on the $sign$ assumption, $\bar{\xi}_i > 0$ is obtained so that $\bar{\xi}_i$ can be written as

$$\bar{\xi}_i \approx z^{\bar{d}_i - dom(\bar{A})} \tag{12}$$

Furthermore, by substituting **Equation (12)** into $z^{\bar{A}}\bar{\xi} = z^{\bar{b}}$, we obtain

$$\sum_{j=1}^{n} z^{\bar{a}_{ij} + \bar{d}_j - dom(\bar{A})} \approx z^{\bar{b}_i} \tag{13}$$

Equation (13) in min-plus algebra has the meaning as

$$\bigotimes_{j=1}^{n}{}' \left(\bar{a}_{ij} + \bar{d}_j - dom\left(\bar{A}\right)\right) = \bar{b}_i, 1 \le i \le n \tag{14}$$

Thus, if $\bar{x}_i = \bar{d}_i - dom\left(\bar{A}\right), 1 \le i \le n$ then $\bar{x}_i$ is a solution to the system of linear equations $\bar{A} \otimes \bar{x} = \bar{b}$.

Thus, the obtained $x \approx [\underline{x}, \bar{x}]$ is the solution of the system of linear equations $A \overline{\otimes} x = b$ with $x_i \overline{\otimes} dom(A) = dom(A_i)$.

Here is the application of **Theorem 1** to solve the system of linear equations $A \overline{\otimes} x = b$ in interval min-plus algebra

Example 2. Given Given a Linear Equation System $A \overline{\otimes} x = b$ with $A = \begin{pmatrix} [-3,1] & [-2.3] \\ [-1,5] & [-4,2] \end{pmatrix}, x = \begin{pmatrix} [\underline{x}_1, \bar{x}_1] \\ [\underline{x}_2, \bar{x}_2] \end{pmatrix}$, dan $b = \begin{pmatrix} [-2.4] \\ [-2,4] \end{pmatrix}$.

From the system of linear equations of interval min-plus algebra, a system of linear equations in min-plus algebra is obtained, namely

$$\begin{pmatrix} -3 & -2 \\ -1 & -4 \end{pmatrix} \otimes \underline{x} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \tag{15}$$

And

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \otimes \bar{x} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{16}$$

a. For the system of linear **Equation (15)**

$$\underline{A} = \begin{pmatrix} -3 & -2 \\ -1 & -4 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

Checked whether the $sign(\underline{A}) = sign(\underline{A}_i)$ value for $i = 1,2$ with a_j is the entries of the j-column of the matrix A and $j = 1,2$.

$$z^{\underline{A}} = \begin{pmatrix} z^{-3} & z^{-2} \\ z^{-1} & z^{-4} \end{pmatrix},$$

obtained $det(z^{\underline{A}}) = z^{-7} - z^{-3}, dom\left(\underline{A}\right) = -7, sign\left(\underline{A}\right) = +1$.

$$\underline{A}_1 = \begin{pmatrix} -2 & -2 \\ -2 & -4 \end{pmatrix}, z^{\underline{A}_1} = \begin{pmatrix} z^{-2} & z^{-2} \\ z^{-2} & z^{-4} \end{pmatrix},$$

obtained $det(z^{\underline{A}_1}) = z^{-6} - z^{-4}$, $dom(\underline{A}_1) = -6, sign(\underline{A}_1) = +1$.

$$\underline{A}_2 = \begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}, z^{\underline{A}_2} = \begin{pmatrix} z^{-3} & z^{-2} \\ z^{-1} & z^{-2} \end{pmatrix},$$

obtained $det(z^{\underline{A}_2}) = z^{-5} - z^{-3}$, $dom(\underline{A}_2) = -5, sign(\underline{A}_2) = +1$. Based on this calculation, the value of $sign(\underline{A}) = sign(\underline{A}_1) = sign(\underline{A}_2)$ is shown. Referring to **Theorem 1**, the system of linear **Equation (15)** can be solved using Cramer's rule.

$$\underline{x}_1 \otimes dom(\underline{A}) = dom(\underline{A}_1)$$
$$\underline{x}_1 \otimes -7 = -6$$
$$\underline{x}_1 = 1$$
$$\underline{x}_2 \otimes dom(\underline{A}) = dom(\underline{A}_2)$$
$$\underline{x}_2 \otimes -7 = -5$$
$$\underline{x}_2 = 2$$

then obtained

$$\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{17}$$

with

$$\begin{pmatrix} -3 & -2 \\ -1 & -4 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \min(-2,0) \\ \min(0,-2) \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

Thus, the solution of the system of linear **Equation (15)** is **Equation (17)**.

b. for the system of linear **Equation (16)**

$$\overline{A} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Checked whether the $sign(\overline{A}) = sign(\overline{A}_i)$ value for $i = 1, 2$ with a_j is the entries of the j-column of the matrix A and $j = 1, 2$.

$$z^{\overline{A}} = \begin{pmatrix} z^1 & z^3 \\ z^5 & z^2 \end{pmatrix},$$

obtained $det(z^{\overline{A}}) = z^3 - z^8$, $dom(\overline{A}) = 3, sign(\underline{A}) = +1$.

$$\overline{A}_1 = \begin{pmatrix} 4 & 3 \\ 4 & 2 \end{pmatrix}, z^{\overline{A}_1} = \begin{pmatrix} z^4 & z^3 \\ z^4 & z^2 \end{pmatrix},$$

obtained $det(z^{\overline{A}_1}) = z^6 - z^7$, $dom(\underline{A}_1) = 6, sign(\underline{A}_1) = +1$.

$$\overline{A}_2 = \begin{pmatrix} 1 & 4 \\ 5 & 4 \end{pmatrix}, z^{\overline{A}_2} = \begin{pmatrix} z^1 & z^4 \\ z^5 & z^4 \end{pmatrix},$$

obtained $det(z^{\overline{A}_2}) = z^5 - z^9$, $dom(\overline{A}_2) = 5, sign(\underline{A}_2) = +1$. Based on this calculation, the value of $sign(\overline{A}) = sign(\overline{A}_1) = sign(\overline{A}_2)$ is shown. Referring to **Theorem 1**, the system of linear **Equation (16)** can be solved using Cramer's rule.

$$\overline{x}_1 \otimes dom(\overline{A}) = dom(\overline{A}_1)$$
$$\overline{x}_1 \otimes 3 = 6$$
$$\overline{x}_1 = 3$$
$$\overline{x}_2 \otimes dom(\overline{A}) = dom(\overline{A}_2)$$
$$\overline{x}_2 \otimes 3 = 5$$
$$\overline{x}_2 = 2$$

then obtained

$$\overline{x} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \tag{18}$$

with

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \min(4,5) \\ \min(8,4) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Thus, the solution of the system of linear **Equation (16)** is **Equation (18)**.

Therefore, the solution of the system of linear equations $\begin{pmatrix} [-3,1] & [-2.3] \\ [-1,5] & [-4,2] \end{pmatrix} \overline{\otimes}\, x = \begin{pmatrix} [-2.4] \\ [-2,4] \end{pmatrix}$ is $x = \begin{pmatrix} [1,3] \\ [2,2] \end{pmatrix}$.

4. CONCLUSIONS

Based on the results and discussion, the sufficient conditions for a system of linear equations $A \overline{\otimes}\, x = b$ in interval min-plus algebra to be solved using Cramer's rule are $sign(\underline{A}_i) = sign(\underline{A}), sign(\overline{A}_i) = sign(\overline{A})$ for $1 \leq i \leq n$, and $dom(A) \approx [dom(\underline{A}), dom(\overline{A})] < [\varepsilon', \varepsilon']$. The Cramer rule is $x_i \overline{\otimes}\, dom(A) = dom(A_i)$. Readers who are interested in this topic can continue the research of Cramer's rule for matrix $A \approx [\underline{A}, \overline{A}]$ with $\underline{A}$ has inverse and $\overline{A}$ has inverse.

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