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PAIR MEAN CORDIAL LABELING OF HURDLE, KEY, LOTUS, AND NECKLACE GRAPHS

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ABSTRACT

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Keywords:

F-Tree;
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Uniform Bow Graph;
Y-Tree.

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be a graph with $\boldsymbol{p}$ vertices and $\boldsymbol{q}$ edges. Define $\boldsymbol{\rho} = \begin{cases} \frac{p}{2} & \boldsymbol{p\ is\ even} \\ \frac{p-1}{2} & \boldsymbol{p\ is\ odd} \end{cases}$ *and* $\boldsymbol{M} = \{\pm\mathbf{1}, \pm\mathbf{2}, \dots, \pm\boldsymbol{\rho}\}$. *Consider a mapping* $\boldsymbol{\lambda}: \boldsymbol{V} \to \boldsymbol{M}$ *by assigning different labels in* $\boldsymbol{M}$ *to the different elements of* $\boldsymbol{V}$ *when* $\boldsymbol{p}$ *is even and different labels in* $\boldsymbol{M}$ *to* $\boldsymbol{p} - \mathbf{1}$*elements of V and repeating a label for the remaining one vertex when* $\boldsymbol{p}$ *is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge* $\boldsymbol{uv}$ *of G, there exists a labeling* $(\boldsymbol{\lambda}(\boldsymbol{u}) + \boldsymbol{\lambda}(\boldsymbol{v}))/\mathbf{2}$ *if* $\boldsymbol{\lambda}(\boldsymbol{u}) + \boldsymbol{\lambda}(\boldsymbol{v})$ *is even and* $(\boldsymbol{\lambda}(\boldsymbol{u}) + \boldsymbol{\lambda}(\boldsymbol{v}) + \mathbf{1})/\mathbf{2}$ *if* $\boldsymbol{\lambda}(\boldsymbol{u}) + \boldsymbol{\lambda}(\boldsymbol{v})$ *is odd such that* $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ *where* $\bar{\mathbb{S}}_{\lambda_1}$ *and* $\bar{\mathbb{S}}_{\lambda_1^c}$ *respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling is called pair mean cordial graph(PMC-graph). In this paper, we investigate the pair mean cordial labeling of some graphs like hurdle graph, lotus graph, necklace graph, F-tree, Y-tree, subdivided shell graph, uniform bow graph and key graph.*

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1. INTRODUCTION

This paper contains only simple, finite, and undirected graphs. The vertex set and edge set of a graph G will be represented by the symbols $V(G)$ and $E(G)$. The size of the graph G is defined as the cardinality of the edge set. A (p, q) graph is a graph with p vertices and $q$edges. A graph is the most effective kind of discrete structure due to its real-world uses and scientific applications. Terms and notations not defined here are used in the sense of F. Harary [1]. Graph labeling is a mathematical area that aims to understand the properties and relationships of graphs. It is also a basic part of graph theory. Graph labeling is the process of assigning values to the vertices under particular conditions. Rosa established the concept of graceful graphs and graceful labeling techniques [2].

Labeled graphs have applications across a wide range of fields. This procedure has great applicability in various areas, such as computer science, network analysis, biology, chemistry, social sciences, etc. Graph labeling can also be applied in social media, mobile telecommunications, network routing, graph visualization, scheduling problems, code optimization, and medical fields. Gallian [3] reports on a thorough survey of several graph labeling techniques.

Somasundaram and R. Ponraj [4] have introduced the notion of mean labeling of graphs. K. Ezhilarasi et al. [5] proved that subdivided shell flower graphs are ρ- labeling. Jesintha and Stanley [6] examined all uniform bow graphs to ensure they were graceful. Lucky edge labeling of car graph, lotus graph, and prism graph are discussed in [7]. Jia-Bao Liu et al. [8] have studied a new graph called a necklace graph.V. Sharon Philomena. and K. Thirusangu [9] have introduced a new type of graph called a key graph. Radio labeling of hurdle graphs and biregular rooted trees is examined in [10]. L.Vennila and P.Vidhyarani [11] have investigated the Skolem difference odd geometric mean labeling of path and cycle-related graphs.

The notion of cordial labeling was due to I. Cahit [12]. Total product and total edge product cordial labeling of dragonfly graphhas investigated in [13]. A. Petrano and R. Rulete [14] [15] shows that cartesian product of $P_m \times C_n$ and $C_m \times C_n$ and generalized peterson graph $P(m, n)$and some special graphs are total product cordial. U. M. Prajapati et al. [16] that cordiality in the context of duplication in crown related graphs and also investigated cordial labeling for complements of some graphs [17]. Pair difference cordial graph notion was examined by Ponraj et al. [18]. Prime cordial labeling of generalized prism graph $Y_{m,n}$ is discussed in [19] and M. A. Seoud et. al. studied the cordial and balanced labeling of graphs in [20].

We have introduced a new type of graph labeling called pair mean cordial labeling [21] and investigated pair mean cordiality of some graphs in [22], [23], [24], [25]. In this paper, we examine the pair mean cordial labeling of some graphs like hurdle graph, lotus graph, necklace graph, F-tree, Y-tree, subdivided shell graph, uniform bow graph, and critical graph.

2. RESEARCH METHODS

The literature reviews related to pair mean cordiality of different graphs is used in the research presented in this article. Books, journals, papers, and articles were the sources of information employed in this study. The pair mean cordiality of the hurdle graph, lotus graph, necklace graph, F-tree, Y-tree, subdivided shell graph, uniform bow graph, and key graph have been investigated using the study methodologies.

3. RESULTS AND DISCUSSION

The pair man cordial labeling behavior of some special graphs like hurdle graph, lotus graph, necklace graph, F-tree, Y-tree, subdivided shell graph, uniform bow graph, and key graph has been investigated.

3.1 Preliminaries

Definition 1. [7] A graph is obtained from a shell graph by adding a vertex between each pair of adjacent vertices on the cycle and adding an edge in the apex, and two or more chords are known as a Lotus graph Ls_n.

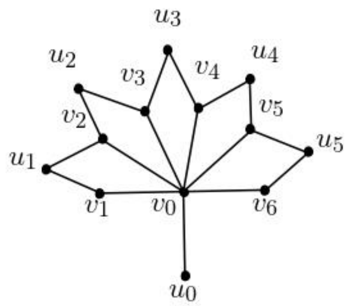

Figure 1. Lotus graph Ls_5.

Definition 2. [8] The necklace graph denoted by Ne_n is a cubic halin graph obtained by joining a cycle with all vertices of degree 1 of a caterpillar (also called a comb) having n vertices of degree 3 and n + 2 vertices of degree 1.

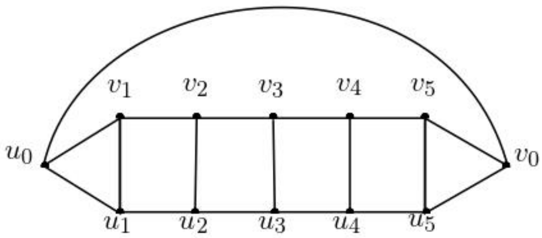

Figure 2. Necklace graph Ne_5

Definition 3. A F-tree $F(P_n)$ is a graph obtained from path on $n \geq 3$ vertices by appending two pendant edges one to an end vertex and the other to a vertex adjacent to an end vertex.

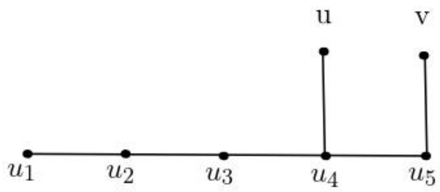

Figure 3. F-tree $F(P_5)$

Definition 4. [11] A Y-tree Y_{n+1}, $n \geq 2$ is a graph obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end vertex.

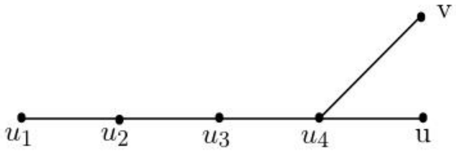

Figure 4. Y-tree Y_5

Definition 5. [5] A subdivided shell graph is a graph obtained by subdividing only edges in the path of a shell graph.

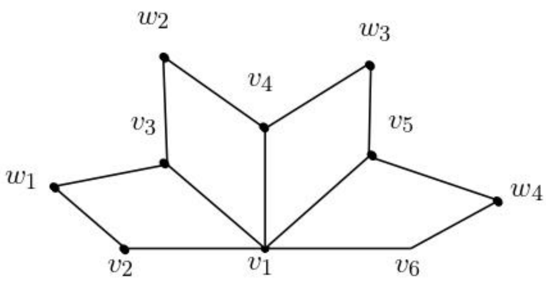

Figure 5. **Subdivided shell graph with $n = 6$**

Definition 6. [10] A graph obtained from a path P_n by attaching pendent edges to every internal vertices of the path. It is called Hurdle graph with $n - 2$ hurdles and is denoted by Hd_n.

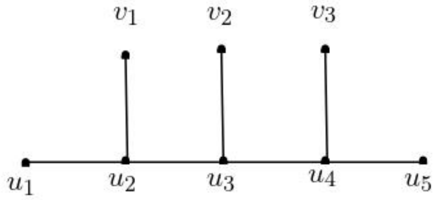

Figure 6. **Hurdle Graph Hd_5**

Definition 7. [6] A multiple shell is defined as a collection of edge-disjoint shells with their apex in common. Hence, a double shell consists of two disjointed shells with a common apex. A uniform bow graph is defined as a double shell in which each shell has the same order.

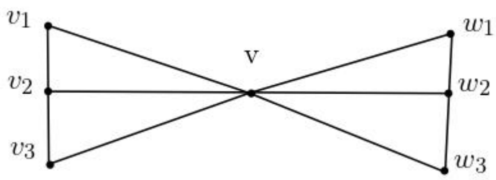

Figure 7. **Uniform bow graph with $n = 4$**

Definition 8. [9] A key graph is a graph obtained from $K_2$by appending one vertex of $C_5$to one end point and Hoffman tree $P_n \odot K_1$ to the other end point of K_2.

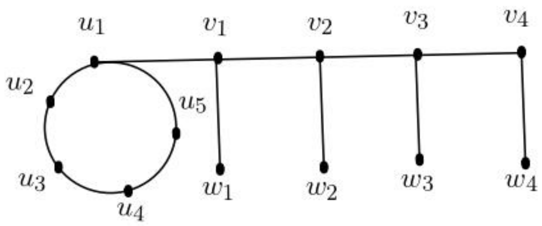

Figure 8. **Key graph ky_4**

3.2 Main Results

Theorem 1. *The lotus graph Ls_n is pair mean cordial for all $n \geq 2$.*

Proof. Consider the lotus graphLs_n. Let $V(Ls_n) = \{u_0, v_0, u_i, v_j : 1 \leq i \leq n \; and \; 1 \leq j \leq n + 1\}$ and $E(Ls_n) = \{v_0 v_i, u_0 v_0 : 1 \leq i \leq n + 1\} \cup \{v_i u_i, u_i v_{i+1} : 1 \leq i \leq n\}$ respectively be the vertex and edge set of the Lotus graph Ls_n. Then it has $2n + 3$ vertices and $3n + 2$ edges. This proof is divided into two cases:

Case (i): n is odd

Let us assign the labels $2, 3, \ldots, \frac{n+5}{2}$ respectively to the vertices $v_1, v_2, \ldots, v_{\frac{n+3}{2}}$ and $\frac{-n-5}{2}, \frac{-n-7}{2}, \ldots, -n - 1$ to the vertices $v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \ldots, v_{n+1}$ respectively. Next we give the labels $-1, -2, \ldots, \frac{-n-3}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{n+3}{2}}$ and $\frac{n+7}{2}, \frac{n+9}{2}, \ldots, n+1$ to the vertices $u_{\frac{n+5}{2}}, u_{\frac{n+7}{2}}, \ldots, u_n$ respectively. Finally assign the labels $-n-1, 1$ to the vertices u_0, v_0 respectively.

Case (ii): n is even

In this case, we assign the labels $2, 3, \ldots, \frac{n+4}{2}$ respectively to the vertices $v_1, v_2, \ldots, v_{\frac{n+2}{2}}$ and $\frac{-n-4}{2}, \frac{-n-6}{2}, \ldots, -n-1$ to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \ldots, v_{n+1}$ respectively. Then we give the labels $-1, -2, \ldots, \frac{-n-2}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{n+2}{2}}$ and $\frac{n+6}{2}, \frac{n+8}{2}, \ldots, n+1$ to the vertices $u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \ldots, u_n$ respectively. Finally assign the labels $-n-1, 1$ to the vertices u_0, v_0 respectively.

Table 1. This Vertex Labeling λ is a Pair Mean Cordial of the Lotus Graph Ls_n for all $n \geq 2$.

Nature of n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
n is odd	$\frac{3n+1}{2}$	$\frac{3n+3}{2}$
n is even	$\frac{3n+2}{2}$	$\frac{3n+2}{2}$

Example 1. The example of the pair mean cordial labeling of the Lotus graph Ls_n is shown in **Figure 9.**

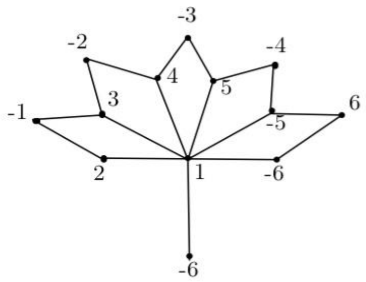

Figure 9. Pair mean cordial labeling of the Lotus graph Ls_5.

Theorem 2. *The necklace graph Ne_n is pair mean cordial for all $n \geq 4$.*

Proof. Let $V(Ne_n) = \{u_0, v_0, u_i, v_i : 1 \leq i \leq n\}$ and $E(Ne_n) = \{u_0u_1, u_0v_1, u_iv_i, v_0u_n, v_0v_n : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}$. Then necklace graph Ne_n has $2n+2$ vertices and $3n+3$ edges. Let $\lambda(u_0) = 1$ and $\lambda(v_0) = -n-1$. First let us assign the labels $-1, 3, -3, 5, -5$ respectively to the vertices u_1, u_2, u_3, u_4, u_5 and $2, -2, 4, -4, 6$ to the vertices v_1, v_2, v_3, v_4, v_5 respectively. We consider the following cases.

Case (i): $n \equiv 0 (mod\ 4)$

In this case, we assign the labels $-6, -10, \ldots, -n+2$ respectively to the vertices $u_6, u_{10}, \ldots, u_{n-2}$ and $8, 12, \ldots, n$ to the vertices $u_7, u_{11}, \ldots, u_{n-1}$ respectively. Then we assign the labels $9, 13, \ldots, n+1$ respectively to the vertices $u_8, u_{12}, \ldots, u_n$ and $-9, -13, \ldots, -n+3$ to the vertices $u_9, u_{13}, \ldots, u_{n-3}$ respectively. Also, we give the labels $7, 11, \ldots, n-1$ respectively to the vertices $v_6, v_{10}, \ldots, v_{n-2}$ and $-7, -11, \ldots, -n+1$ to the vertices $v_7, v_{11}, \ldots, v_{n-1}$ respectively. Next, we give the labels $-8, -12, \ldots, -n$ respectively to the vertices $v_8, v_{12}, \ldots, v_n$ and $10, 14, \ldots, n-2$ to the vertices $v_9, v_{13}, \ldots, v_{n-3}$ respectively.

Case (ii): $n \equiv 1 (mod\ 4)$

Now, we assign the labels $-6, -10, \ldots, -n+3$ respectively to the vertices $u_6, u_{10}, \ldots, u_{n-3}$ and $8, 12, \ldots, n-1$ to the vertices $u_7, u_{11}, \ldots, u_{n-2}$ respectively. Then we assign the labels $9, 13, \ldots, n$ respectively to the vertices $u_8, u_{12}, \ldots, u_{n-1}$ and $-9, -13, \ldots, -n$ to the vertices $u_9, u_{13}, \ldots, u_n$ respectively. Furthermore, we give the labels $7, 11, \ldots, n-2$ respectively to the vertices $v_6, v_{10}, \ldots, v_{n-3}$ and $-7, -11, \ldots, -n+2$ to the vertices $v_7, v_{11}, \ldots, v_{n-2}$ respectively. Next, we give the labels $-8, -12, \ldots, -n+1$ respectively to the vertices $v_8, v_{12}, \ldots, v_{n-1}$ and $10, 14, \ldots, n+1$ to the vertices $v_9, v_{13}, \ldots, v_n$ respectively.

Case (iii): $n \equiv 2(mod\ 4)$

Assign the labels $-6, -10, \dots, -n$ respectively to the vertices $u_6, u_{10}, \dots, u_n$ and $8,12, \dots, n-2$ to the vertices $u_7, u_{11}, \dots, u_{n-3}$ respectively. Then we assign the labels $9, 13, \dots, n-1$ respectively to the vertices $u_8, u_{12}, \dots, u_{n-2}$ and $-9, -13, \dots, -n+1$ to the vertices $u_9, u_{13}, \dots, u_{n-1}$ respectively. Also, we give the labels $7, 11, \dots, n+1$ respectively to the vertices $v_6, v_{10}, \dots, v_n$ and $-7, -11, \dots, -n+3$ to the vertices $v_7, v_{11}, \dots, v_{n-3}$ respectively. Next, we give the labels $-8, -12, \dots, -n+2$ respectively to the vertices $v_8, v_{12}, \dots, v_{n-2}$ and $10,14, \dots, n$ to the vertices $v_9, v_{13}, \dots, v_{n-1}$ respectively.

Case (iv): $n \equiv 3(mod\ 4)$

Furthermore, we assign the labels $-6, -10, \dots, -n+1$ respectively to the vertices $u_6, u_{10}, \dots, u_{n-1}$ and $8,12, \dots, n+1$ to the vertices $u_7, u_{11}, \dots, u_n$ respectively. Then we assign the labels $9, 13, \dots, n-2$ respectively to the vertices $u_8, u_{12}, \dots, u_{n-3}$ and $-9, -13, \dots, -n+2$ to the vertices $u_9, u_{13}, \dots, u_{n-2}$ respectively. Also, we give the labels $7, 11, \dots, n$ respectively to the vertices $v_6, v_{10}, \dots, v_{n-1}$ and $-7, -11, \dots, -n$ to the vertices $v_7, v_{11}, \dots, v_n$ respectively. Next, we give the labels $-8, -12, \dots, -n+3$ respectively to the vertices $v_8, v_{12}, \dots, v_{n-3}$ and $10,14, \dots, n-1$ to the vertices $v_9, v_{13}, \dots, v_{n-2}$ respectively.

Table 2. **This Vertex Labeling λ is a Pair Mean Cordial of the Necklace GraphNe_n for all $n \geq 4$.**

Nature of n	$\overline{\mathbb{S}}_{\lambda_1}$	$\overline{\mathbb{S}}_{\lambda_1^c}$
$n \equiv 0(mod\ 4)$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$
$n \equiv 1(mod\ 4)$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$
$n \equiv 2(mod\ 4)$	$\frac{3n+2}{2}$	$\frac{3n+4}{2}$
$n \equiv 3(mod\ 4)$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$

Example 2. The example of the pair mean cordial labeling of the necklace graph is shown in **Figure 10.**

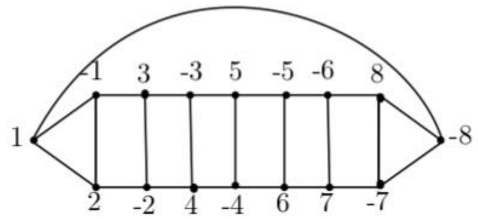

Figure 10. **Pair mean cordial labeling of the necklace graph Ne_7.**

Theorem 3. F-tree $F(P_n)$ is pair mean cordial for all $n \geq 3$.

Proof. Let $V(F(P_n)) = \{u, v, u_i: 1 \leq i \leq n\}$ and $E(F(P_n)) = \{uu_{n-1}, vu_n, u_iu_{i+1}: 1 \leq i \leq n-1\}$. Clearly the F-tree $F(P_n)$ has $n+2$ vertices and $n+1$ edges. We consider the following cases:

Case (i): n is odd

Let us assume that $\lambda(u) = \frac{n+1}{2}$ and $\lambda(v) = \frac{-n-1}{2}$. Then we assign the labels $1,2, \dots, \frac{n+1}{2}$ respectively to the vertices $u_1, u_3, \dots, u_n$ and $-1, -2, \dots, \frac{-n+1}{2}$ to the vertices $u_2, u_4, \dots, u_{n-1}$ respectively.

Case (ii): n is even

Now $\lambda(u) = \frac{-n-2}{2}$ and $\lambda(v) = \frac{n+2}{2}$. Then we assign the labels $1,2, \dots, \frac{n}{2}$ respectively to the vertices $u_1, u_3, \dots, u_{n-1}$ and $-1, -2, \dots, \frac{-n}{2}$ to the vertices $u_2, u_4, \dots, u_n$ respectively.

Table 3. **Vertex Labeling λ is a Pair Mean Cordial of $F(P_n)$ for all $n \geq 3$.**

Nature of n	$\overline{\mathbb{S}}_{\lambda_1}$	$\overline{\mathbb{S}}_{\lambda_1^c}$
n is odd	$\frac{n+1}{2}$	$\frac{n+1}{2}$
n is even	$\frac{n}{2}$	$\frac{n+2}{2}$

Example 3. The example of the pair mean cordial labeling of the F-tree $F(P_n)$ is shown in **Figure 11.**

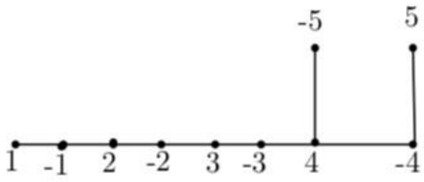

Figure 11. Pair mean cordial labeling of the F-tree $F(P_8)$.

Theorem 4. *The Y-tree $Y(n+1)$ is pair mean cordial for all $n \geq 2$.*
Proof. Let $V(Y(n+1)) = \{u, v, u_i: 1 \leq i \leq n\}$ and $E(Y(n+1)) = \{uu_n, vu_n, u_iu_{i+1}: 1 \leq i \leq n-1\}$. Clearly the F-tree $F(P_n)$ has $n+2$ vertices and $n+1$ edges. We have the following two cases arise:

Case (i): n is odd
Let us assume that $\lambda(u) = \frac{-n-1}{2}$ and $\lambda(v) = \frac{-n+1}{2}$. Then assign the labels $u_i, 1 \leq i \leq n$ as in Case (1) of theorem (2.1). $1, 2, \dots, \frac{n+1}{2}$ respectively to the vertices $u_1, u_3, \dots, u_n$ and $-1, -2, \dots, \frac{-n+1}{2}$ to the vertices $u_2, u_4, \dots, u_{n-1}$ respectively.

Case (ii): n is even
Now $\lambda(u) = \frac{n+1}{2}$ and $\lambda(v) = \frac{-n-1}{2}$. Next assign the labels $u_i, 1 \leq i \leq n$ as in **Case (ii)** of **Theorem (3)**.

Table 4. This Vertex Labeling λ is a Pair Mean Cordial of $Y(n+1)$ for all $n \geq 2$.

Nature of n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
n is odd	$\frac{n}{2}$	$\frac{n+2}{2}$
n is even	$\frac{n+1}{2}$	$\frac{n+1}{2}$

Example 4. The example of the pair mean cordial labeling of the Y-tree Y_{n+1} is shown in **Figure 12.**

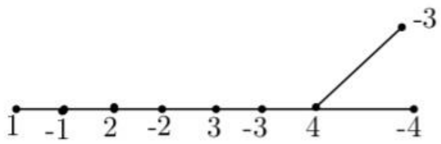

Figure 12. Pair mean cordial labeling of the Y-tree Y_9.

Theorem 5. *The subdivided shell graph is pair mean cordial for all $n \geq 4$.*
Proof. Let G_0 be subdivided shell graph. Then the vertex and edge set of subdivided shell graph are defined by $V(G_0) = \{v_i, w_j: 1 \leq i \leq n \text{ and } 1 \leq j \leq n-2\}$ and $E(G_0) = \{v_1v_i, v_{j+1}w_j, w_jv_{j+2}: 1 \leq i \leq n \text{and } 1 \leq j \leq n-2\}$. Clearly the subdivided shell graph has $2n-2$ vertices and $3n-5$ edges.

Case (i): n is odd
Now, we assign the labels $1, 2, \dots, \frac{n+3}{2}$ to the vertices $v_1, v_2, \dots, v_{\frac{n+3}{2}}$ respectively and $\frac{-n-3}{2}, \frac{-n-5}{2}, \dots, -n+1$ respectively to the vertices $v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \dots, v_n$. Then we give the labels $-1, -2, \dots, \frac{-n-1}{2}$ to the vertices $w_1, w_2, \dots, w_{\frac{n+1}{2}}$ respectively and $\frac{n+5}{2}, \frac{n+7}{2}, \dots, n-1$ respectively to the vertices $w_{\frac{n+3}{2}}, w_{\frac{n+5}{2}}, \dots, w_{n-2}$.

Case (ii): n is even

Let us assign the labels $1, 2, \dots, \frac{n+3}{2}$ to the vertices $v_1, v_2, \dots, v_{\frac{n+2}{2}}$ respectively and $\frac{-n-2}{2}, \frac{-n-4}{2}, \dots, -n+1$ respectively to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \dots, v_n$. Next we give the labels $-1, -2, \dots, \frac{-n}{2}$ to the vertices $w_1, w_2, \dots, w_{\frac{n}{2}}$ respectively and $\frac{n+4}{2}, \frac{n+6}{2}, \dots, n-1$ respectively to the vertices $w_{\frac{n+2}{2}}, w_{\frac{n+4}{2}}, \dots, w_{n-2}$.

Table 5. Vertex Labeling λ is a Pair Mean Cordial of Subdivided Shell Graph for all $n \geq 4$.

Nature of n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
n is odd	$\frac{3n-5}{2}$	$\frac{3n-5}{2}$
n is even	$\frac{3n-6}{2}$	$\frac{3n-4}{2}$

Example 5. The example of the pair mean cordial labeling of the subdivided shell graph is shown in **Figure 13.**

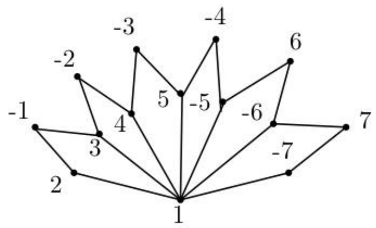

Figure 13. Pair mean cordial labeling of the subdivided shell graph with $n = 8$.

Theorem 6. *The hurdle graph Hd_n is pair mean cordial for all $n \geq 3$.*

Proof. Let $V(Hd_n) = \{u_i, v_j : 1 \leq i \leq n \, and \, 1 \leq j \leq n-2\}$ and $E(Hd_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{i+1} v_i : 1 \leq i \leq n-2\}$. Then there are $2n-2$ vertices and $2n-3$ edges. Let us assign the labels $1, 2, \dots, n-1$ to the vertices $u_1, u_2, \dots, u_{n-1}$ respectively and $-1, -2, \dots, -n+2$ respectively to the vertices $v_1, v_2, \dots, v_{n-2}$. Finally assign the label $-n+1$ to the vertex u_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = n-2$ *and* $\bar{\mathbb{S}}_{\lambda_1^c} = n-1$.

Example 6. The example of the pair mean cordial labeling of the hurdle graph Hd_n is shown in **Figure 14.**

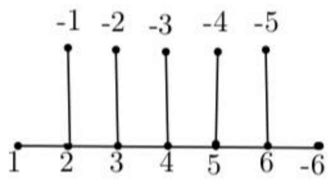

Figure 14. Pair mean cordial labeling of the Hurdle graph Hd_7.

Theorem 7. *The uniform bow graph is not a pair mean cordial for all $n \geq 7$.*

Proof. Let H_0 be the uniform bow graph. Then the vertex and edge set of the uniform bow graph are defined by $V(H_0) = \{v, v_i, w_i : 1 \leq i \leq n-1\}$ and $E(H_0) = \{vv_i, vw_i, v_j v_{j+1}, w_j w_{j+1} : 1 \leq i \leq n-1 \, and \, 1 \leq j \leq n-2\}$. Hence the uniform bow graph has $2n-1$ vertices and $4n-6$ edges.

Case(i): $n = 3$ and $n \geq 7$

Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(v) + \lambda(w) = 1$ or $\lambda(v) + \lambda(w) = 2$. Hence the maximum number of edges label with 1 is $2n-4$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n-4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n-2-(2n-4) = 2 > 1$, a contradiction.

Case(ii): $4 \leq n \leq 6$

If $n = 4$, define $\lambda(v) = -1$. Then we assign the labels $1, 2, -3, 3, -2, 3$ respectively to the vertices $v_1, v_2, v_3, w_1, w_2, w_3$. If $n = 5$, define $\lambda(v) = -2$. Next, we assign the labels $1, 2, -1, 3, 4, -3, 4-4$ respectively to the vertices $v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4$. If $n = 6$, take$\lambda(v) = -3$. Then we assign the labels $2, -1, 3, -2, 4$ respectively to the vertices v_1, v_2, v_3, v_4, v_5 and $5, -4, 5, -5, 1$ to the vertices w_1, w_2, w_3, w_4, w_5 respectively. In this case,$\bar{\mathbb{S}}_{\lambda_1} = 2n-3$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2n-3$.

Example 7. The example of the pair mean cordial labeling of the uniform bow graph is shown in **Figure 15.**

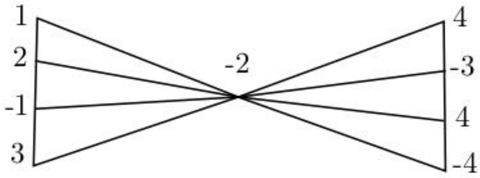

Figure 15. Pair mean cordial labeling of the uniform bow graph with $n = 5$.

Theorem 8. The key graph Ky_n, is pair mean cordial for all $n \geq 1$.

Proof. Let $C_5: u_1u_2 \dots u_5u_1$ be a cycle. Let $V(Ky_n) = V(C_5) \cup \{v_i, w_i: 1 \leq i \leq n\}$ and $E(Ky_n) = E(C_5) \cup \{u_1v_1, v_iw_i: 1 \leq i \leq n\} \cup \{v_iv_{i+1}: 1 \leq i \leq n-1\}$. Clearly the key graph Ky_nhas $2n+5$ vertices and $2n+5$ edges. Assign the labels $1, -1, 2, -2, 3$ to the vertices $u_1, u_2, \dots, u_5$ respectively. Now, we assign the labels $4, 5, \dots, n+2$ to the vertices $v_1, v_2, \dots, v_{n-1}$ respectively and assign the label $-n-1$ to the vertex v_n. Then we give the labels $-3, -4, \dots, -n-2$ to the vertices $w_1, w_2, \dots, w_n$ respectively. Hence $\overline{\mathbb{S}}_{\lambda_1} = n+2$ and $\overline{\mathbb{S}}_{\lambda_1^c} = n+3$.

Example 8. The example of the pair mean cordial labeling of the key graph $Ky_n$is shown in **Figure 16.**

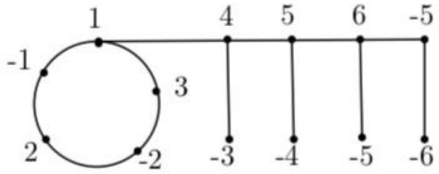

Figure 16. Pair mean cordial labeling of the key graph Ky_4.

4. CONCLUSIONS

In this article, we examined the pair man cordial labeling behavior of special graphs such as the hurdle graph, lotus graph, necklace graph, F-tree, Y-tree, subdivided shell graph, uniform bow graph, and key graph. We conclude that future open problems will include the pair mean cordial labeling of theta graph, shadowgraph, windmill graph, olive tree, coconut tree, generalized web graph, tensor product graph, and pappus graph.

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