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LOCAL IRREGULARITY VERTEX COLORING OF BICYCLIC GRAPH FAMILIES

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ABSTRACT

The graph in this research is a simple and connected graph with $V(G)$ as vertex set and $E(G)$ as an edge set. We used deductive axiomatic and pattern recognition method. Local irregularity vertex coloring is defined as mapping $l: V(G) \to \{1,2,\dots,k\}$ as vertex irregular k-labeling and $w: V(G) \to N$ where $w(u) = \sum_{v\in N(u)} l(v)$. The conditions for w to be a local irregularity vertex coloring, if $opt(l) = min\{max\{l_i\}\}$ with l_i as irregularity vertex labeling and for every $uv \in E(G), w(u) \neq w(v)$. The minimum number of colors produced from local irregularity vertex coloring of graph G is called chromatic number local irregularity, denoted by $\chi_{lis}(G)$. In this research, we analyze about the local irregularity vertex coloring and determine the chromatic number of local irregularity of bicyclic graphs.

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1. INTRODUCTION

A graph G is a set consisting of a set of vertex denoted by $V(G)$ and a set of edges denoted by $E(G)$ [1]. The set of vertex is said not to be empty, while the set of edges may be empty [2]. Based on the above definition, it can be said that a graph may have no edges and only one vertex [3]. In graph terminology, we recognize cardinality, neighborhood, and vertex degree. Cardinality is the number of members of a set [4]. Two neighboring vertices (connected directly) are two vertices connected by an edge [5]. The degree of a vertex is the number of neighboring vertices [6].

In graph theory, we are familiar with graph labeling. Graph labeling is a way of assigning labels in the form of positive integers to vertex, edges, or both with certain conditions [7]. Graph labeling is a mapping of a set of numbers called labels to graph elements, usually vertex or edges. In general, the labels given are positive integers [8]. Suppose there is a function $f: V(G) \to S$ where S is a set of non-negative numbers and $V(G)$ is the set of vertex in the graph, then f is called the vertex labeling of a graph G and the member of $f(v)$ where $v \in V(G)$ is called the edges label [9]. The irregularity labeling is defined as the labeling $f: E \to \{1,2,\dots,k\}$ with k is a positive integer such that $w_{f(x)} = \sum_{(y \epsilon N(x))} f(xy)$ is different for all vertices, where $N(x)$ is the neighborhood of the vertex x [8]. In another definition, irregularity labeling is a mapping that maps the set of edges of a graph G to the set of positive integers, such that any two distinct vertex in G have different weights [10].

A distinctive problem in graph labeling involves graph coloring. There are three types of graph coloring concepts, namely vertex coloring, edge coloring, and region coloring. Vertex coloring is the assignment of color to each vertex of $V(G)$ where every two neighboring vertices have different colors [11]. This graph coloring method is an application of graph theory that is used in solving map coloring and schedule management problems with methods contained in the graph coloring [12].

There is a special case in vertex coloring of a graph, namely local irregularity vertex coloring. Local irregularity vertex coloring is a combination of irregularity labeling and vertex coloring, which is done by minimizing the vertex label and minimizing the number of vertex colors in a graph G [13]. The vertex weight of local irregularity vertex coloring is the sum of the neighboring vertex labels [14]. The number of colors used in vertex coloring is called the chromatic number. The chromatic number is denoted $\chi(G)$ is the smallest integer k such that the graph G has a vertex coloring with k-color [15]. A vertex coloring on a graph is said to be efficient if the number of colors used to color a graph is the chromatic number of the graph [16]. Previously, it has been studied about local irregularity vertex coloring of comb product on star graphs [17], local irregularity vertex coloring on unicyclic graph families [18], local irregularity vertex coloring of the corona product of a tree graph [19], local irregularity vertex coloring of vertex amalgamation of path graph [20], local irregularity vertex coloring on sun graph, fan graph, star graph, and double star graph [21].

The definition of local irregularity vertex coloring can be seen in the following definition [14]: Suppose $l: V(G) \to \{1,2,\dots,k\}$ is a labeling k-vertex of local irregularity and $w: V(G) \to N$ where $w(u) = \sum_{v \in N(u)} l(v)$. The condition for w to be a local irregular vertex coloring if $opt(l) = min\{max\{l_i\};$ $l_i\ is\ irregularity\ vertex\ labeling\}$ and for every $uv \in E(G)$, $w(u) \neq w(v)$. The chromatic number of local irregularity denoted by $\chi_{lis}(G)$ is defined as $\chi_{lis}(G) = \min\{|w(V(G))|\}$ where w is the local irregularity vertex coloring. There are several studies about local irregularity vertex coloring, but the research about local irregularity vertex coloring of bicyclic graphs has not been conducted yet. Therefore, the author is interested in researching local irregularity vertex coloring of bicyclic graphs. A simple connected graph G with $|V(G)| = n$ is said to be bicyclic, if $|E(G)| = n + 1$ [22]. A simple connected graph G is called a bicyclic graph, if it contains exactly two circles with $q = p + 1$, where $p = |V(G)|$ and $q = |E(G)|$ [23]. A bicyclic graph is a specific type of directed graph that contains two cycles. These cycles can be traversed in a specific direction, with each cycle forming a closed loop. The presence of these two cycles distinguishes a bicyclic graph from other types of directed graphs [24]. In this research, there are 3 types of graphs from the bicyclic graph family, namely bicyclic graf of type I $B_1(n,m)$, bicyclic graph of type II $B_2(n,m,r)$, and bicyclic graph of type III $B_3(n,m,t)$.

2. RESEARCH METHODS

The research methods used in this research are axiomatic deductive method and pattern detection. The axiomatic deductive method is a research method that uses the principles of deductive proof that apply in mathematical logic by using existing axioms or theorems and then applied in coloring the local irregularity vertex on the bicyclic graph family. Then to obtain the pattern and chromatic number of the local irregularity on the family of bicyclic graphs, the pattern detection method is used by finding the pattern and chromatic number on the family of bicyclic graphs of type I $B_1(n,m)$, type II $B_2(n,m,r)$, type III $B_3(n,m,t)$. The illustration of a bicyclic graph family is shown in the **Figure 1**.

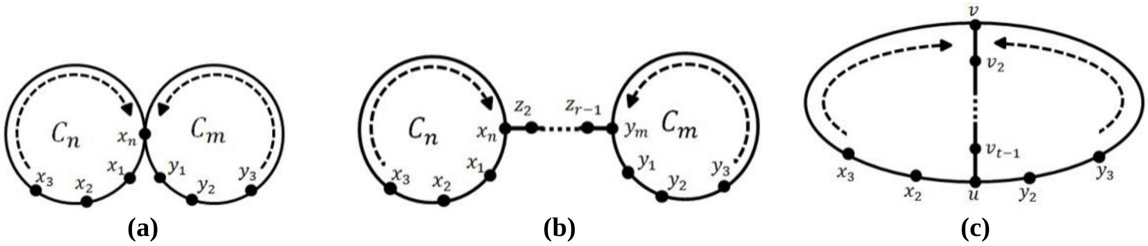

Figure 1. **Illustration of a Bicyclic Graph Family, (a) Bicyclic Graph of Type I $B_1(n,m)$, (b) Bicyclic Graph of Type II $B_2(n,m,r)$, (c) Bicyclic Graph of Type III $B_3(n,m,t)$**

The research procedure carried out in this research can be seen in **Figure 2**.

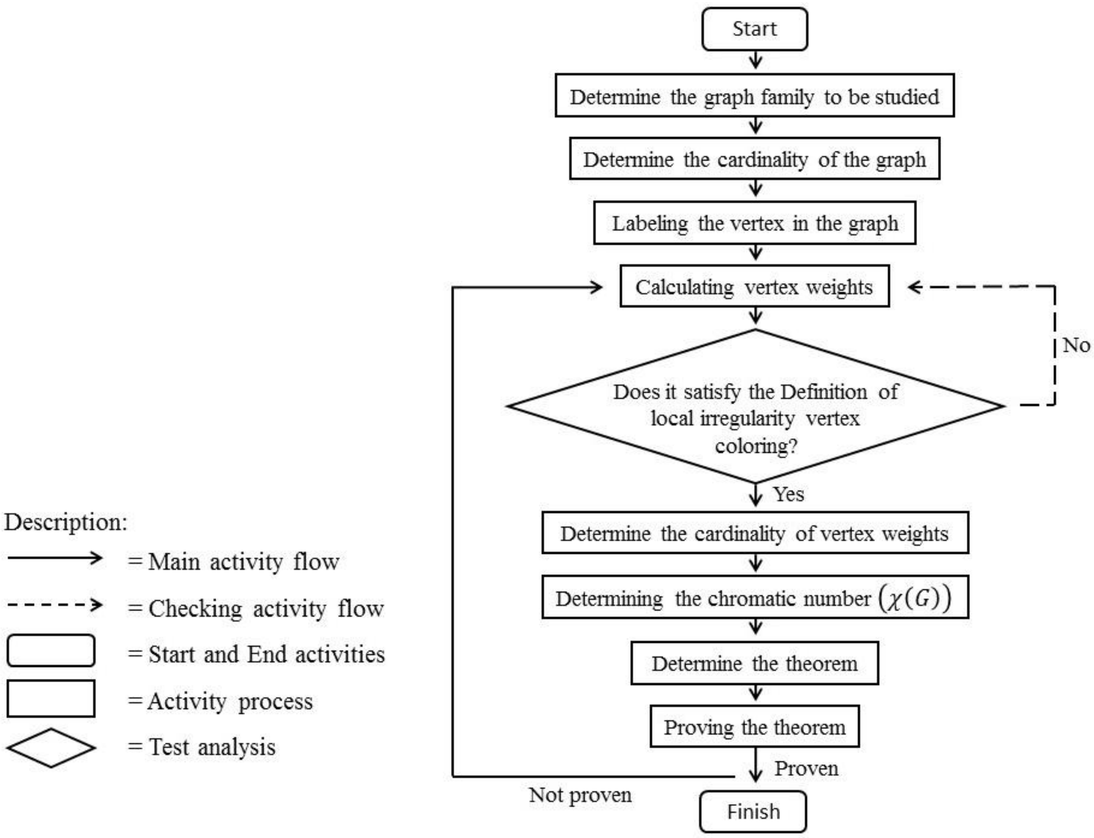

Figure 2. **Research Procedure**

In order to prove some theorems in this research, there are lemma and some observations used in this research. Here are the following lemma and observations:

Lemma 1. [14] *Suppose graph G is simple and connected, then $\chi_{lis}(G) \geq \chi(G)$.*

Observation 1. [14] *For a connected graph G, if every two neighboring vertices have different degrees, then $opt(l) = 1$.*

Observation 2. [14] *For a connected graph G, if every two neighboring vertices have the same degrees, then $opt(l) \geq 2$.*

3. RESULTS AND DISCUSSION

There are three new theorems obtained from this research regarding the chromatic number of local irregularities $\chi_{lis}(G)$ on the family of bicyclic graphs, which are as follows:

Observation 3. *Suppose the graph $B_1(n,m)$ is a bicyclic graph with $n,m \geq 3$, then*

$$\chi\big(B_1(n,m)\big) = \begin{cases} 2 & n,m \text{ is even} \\ 3 & n \text{ is odd}; \cup \\ & m \text{ is odd} \end{cases}$$

Theorem 1. *Suppose the graph $B_1(n,m)$ is a bicyclic graph with $n,m \geq 3$, then $\chi_{lis}\big(B_1(n,m)\big) = 3$.*

Proof. A bicyclic graph of type I $B_1(n,m)$ has vertex set $V\big(B_1(n,m)\big) = \{x_i; 1 \leq i \leq n\} \cup \{y_j; 1 \leq j \leq m-1\}$ and edges set $E\big(B_1(n,m)\big) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup (x_n x_1) \cup \{y_j y_{j+1}; 1 \leq j \leq m-2\} \cup (y_{m-1} x_n) \cup (x_n y_1)$, therefore the number of vertex is $\big|V\big(B_1(n,m)\big)\big| = n+m-1$ and the number of edges is $\big|E\big(B_1(n,m)\big)\big| = n+m$.

Case 1. n,m *is even*; $n,m \geq 3$

In the graph $B_1(n,m)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E\big(B_1(n,m)\big)$. Based on **Observation 2**, the value of the labeling is $opt(l) \geq 2$. Next, proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 3**, we get $\chi_{lis}\big(B_1(n,m)\big) \geq \chi\big(B_1(n,m)\big) = 2$. Assume $\chi_{lis}\big(B_1(n,m)\big) = 2$, meaning there are 2 colors. If $x_i, y_j \in V\big(B_1(n,m)\big)$ is labeled, then the following possible vertex weights are obtained:

a) If $l(x_i) = 1; l(y_j) = 1$, then $w(x_1) = w(x_2),\ x_1x_2 \in E\big(B_1(n,m)\big)$.
b) If $l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even; $l(y_j) = 1$ for j is odd; $l(y_j) = 2$ for j is even, then $w(x_1) = w(x_n), x_n x_1 \in E\big(B_1(n,m)\big)$.
c) If $l(x_n) = 1; l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even; $l(y_j) = 1$ for j is odd; $l(y_j) = 2$ for j is even, then the vertex color is more than 2.

Based on the above statement, there are at least 3 colors at the vertices $x_i, y_j \in V\big(B_1(n,m)\big)$ with the value of $opt(l) = 2$. Next, select the vertices x_1, x_2, and x_n,

$$w(x_1) = l(x_2) + l(x_n) = 2 + 1 = 3 \tag{1}$$

$$w(x_2) = l(x_1) + l(x_3) = 1 + 1 = 2 \tag{2}$$

$$w(x_n) = l(x_1) + l(x_{n-1}) + l(y_1) + l(y_{n-1}) = 1 + 1 + 1 + 1 = 4 \tag{3}$$

Based on **Equation (1)**, **Equation (2)**, and **Equation (3)**, we get $w(x_1) \neq w(x_2),\ x_1x_2 \in E\big(B_1(n,m)\big)$ and $w(x_1) \neq w(x_n),\ x_n x_1 \in E\big(B_1(n,m)\big)$ satisfies the definition of local irregularity vertex coloring. So the lower bound of the graph $B_1(n,m)$ when n,m *is even*; $n,m \geq 3$ is $\chi_{lis}\big(B_1(n,m)\big) \geq 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_1(n,m)$, we define the labeling $l: V\big(B_1(n,m)\big) \to \{1,2\}$ as follows:

$$l(x_i) = \begin{cases} 1 & i \text{ is odd}; i = n \\ 2 & i \text{ is even}; i < n \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \text{ is odd} \\ 2 & j \text{ is even}; j < m \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i \text{ is even}; i < n \\ 3 & i = 1, n-1 \\ 4 & i \text{ is odd}; 1 < i < n-1; i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \text{ is even}; j < m \\ 3 & j = 1, m-1 \\ 4 & j \text{ is odd}; 1 < j < m-1 \end{cases}$$

Based on the weight function above, each neighboring vertices has a different vertex weight. The set of vertex weights on the graph $B_1(n,m)$ when n,m *is even*; $n,m \geq 3$ which is $\{2,3,4\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$. Thus, we get $3 \leq \chi_{lis}(B_1(n,m)) \leq 3$, then $\chi_{lis}(B_1(n,m)) = 3$; when n,m *is even*; $n,m \geq 3$.

Case 2. n,m *is odd*; $n,m \geq 3$

In the graph $B_1(n,m)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E(B_1(n,m))$. Based on **Observation 2**, the value of the labeling is $opt(l) \geq 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 3**, we get $\chi_{lis}(B_1(n,m)) \geq \chi(B_1(n,m)) = 3$. So the lower bound when n,m *is odd*; $n,m \geq 3$ is $\chi_{lis}(B_1(n,m)) \geq 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_1(n,m)$, we divide into 5 subcases, namely when $n = m = 3$, when $n = m \neq 3$, when $n \neq m; n = 3; m \equiv 1(mod\ 4)$ *or* $m = 3; n \equiv 1(mod\ 4)$, when $n \neq m$; $n = 3$; $m \equiv 3(mod\ 4)$ *or* $m = 3; n \equiv 3(mod\ 4)$, and when $n \neq m; n,m > 3$.

Subcase 2.1 is when $n = m = 3$. The following is the labeling defined $l: V(B_1(n,m)) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = 1,3 \\ 2 & i = 2 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j = 1 \\ 2 & j = 2 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2 \\ 3 & i = 1 \\ 6 & i = 3 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j = 2 \\ 3 & j = 1 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when n,m *is odd*; $n = m = 3$ which is $\{2,3,6\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$.

Subcase 2.2 is when $n = m \neq 3$. The following is the labeling defined $l: V(B_1(n,m)) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i \text{ is odd; or} \\ & i = n-1; \text{ or} \\ & i = n \\ 2 & i \text{ is even}; i < n-1 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \text{ is odd; or} \\ & j = m-1 \\ 2 & j \text{ is even}; j < m-1 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i \text{ is even} \\ 3 & i = 1, n-2 \\ 4 & i = n; \text{ or} \\ & i \text{ is odd}; 1 < i < n-2 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \text{ is even} \\ 3 & j = 1, n-2 \\ 4 & j \text{ is odd}; 1 < j < m-2 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when n, m is $odd; n = m \neq 3$ which is $\{2,3,4\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$.

Subcase 2.3 is when $n \neq m; n = 3; m \equiv 1(mod\ 4)$ or $m = 3; n \equiv 1(mod\ 4)$. The following is the labeling defined $l: V(B_1(n,m)) \rightarrow \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = 1,3; n = 3; or \\ & i \equiv 0,1,2(mod\ 4); m = 3 \\ 2 & i = 2; n = 3; or \\ & i \equiv 3(mod\ 4); m = 3 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \equiv 0,1,2(mod\ 4); n = 3; or \\ & j = 1,3; m = 3 \\ 2 & j \equiv 3(mod\ 4); n = 3; or \\ & j = 2; m = 3 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2; n = 3; or \\ & i \text{ is odd}; i < n; m = 3 \\ 3 & i = 1; n = 3; or \\ & i \text{ is even}; m = 3 \\ 5 & i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \text{ is odd}; j < m; n = 3; or \\ & j = 2; m = 3 \\ 3 & j \text{ is even}; n = 3; or \\ & j = 1; m = 3 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when n, m is $odd; n \neq m; n = 3; m \equiv 1(mod\ 4)$ or $m = 3; n \equiv 1(mod\ 4)$ which is $\{2,3,5\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$.

Subcase 2.4 is when $n \neq m; n = 3; m \equiv 3(mod\ 4)$ or $m = 3; n \equiv 3(mod\ 4)$. The following is the labeling defined $l: V(B_1(n,m)) \rightarrow \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = 1,3; n = 3; or \\ & i \equiv 0,1,3(mod\ 4); m = 3 \\ 2 & i = 2; n = 3; or \\ & i \equiv 2(mod\ 4); m = 3 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \equiv 0,1,3(mod\ 4); n = 3; or \\ & j = 1,3; m = 3 \\ 2 & j \equiv 2(mod\ 4); n = 3; or \\ & j = 2; m = 3 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2; n = 3; or \\ & i \text{ is even}; m = 3 \\ 3 & i = 1; n = 3; or \\ & i \text{ is odd}; i < n; m = 3 \\ 6 & i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \text{ is even}; n = 3; or \\ & j = 2; m = 3 \\ 3 & j \text{ is odd}; j < m; n = 3; or \\ & j = 1; m = 3 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n,m\ is\ odd; n \neq m; n = 3; m \equiv 3(mod\ 4)\ or$ $m = 3; n \equiv 3(mod\ 4)$ which is {2,3,6}, then $\left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$.

Subcase 2.5 is when $n \neq m; n,m > 3$. The following is the labeling defined $l: V\big(B_1(n,m)\big) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i\ is\ odd; 1 \le i < n-1; or \\ & i = n-1;\ or \\ & i = n \\ 2 & i\ is\ even; 2 \le i < n-1 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j\ is\ odd; 1 \le j < m-1; or \\ & j = m-1 \\ 2 & j\ is\ even; 2 \le j < m-1 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i\ is\ even; i < n \\ 3 & i = 1, n-2 \\ 4 & i = n; or \\ & i\ is\ odd; 1 < i < n-2 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j\ is\ even; j < m \\ 3 & j = 1, m-2 \\ 4 & j\ is\ odd; 1 < j < m-2 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n,m\ is\ odd; n \neq m; n = 3; m > 3$. which is {2,3,4}, then $\left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$.

Based on the five subcases above, the upper bound of the chromatic number of local irregularity is obtained $\chi_{lis}\big(B_1(n,m)\big) \le \left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$. Thus, we get $3 \le \chi_{lis}\big(B_1(n,m)\big) \le 3$, then $\chi_{lis}\big(B_1(n,m)\big) = 3$; when $n,m\ is\ odd; n,m \ge 3$.

Case 3. $n\ is\ odd;\ m\ is\ even; n,m \ge 3$

In the graph $B_1(n,m)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E\big(B_1(n,m)\big)$. Based on **Observation 2**, the value of the labeling is $opt(l) \ge 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 3**, we get $\chi_{lis}\big(B_1(n,m)\big) \ \ge \chi\big(B_1(n,m)\big) = 3$. So the lower bound when $n\ is\ odd; m\ is\ even;$ $n,m \ge 3$ is $\chi_{lis}\big(B_1(n,m)\big) \ge 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_1(n,m)$, we divide into 2 subcases, namely when $n = 3$ and when $n > 3$.

Subcase 3.1 is when $n = 3$. The following is the labeling defined $l: V\big(B_1(n,m)\big) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = 1,3 \\ 2 & i = 2 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \equiv 0,1,3(mod\ 4); m \equiv 0(mod\ 4); or \\ & j \equiv 0,1,2(mod\ 4); m \equiv 2(mod\ 4) \\ 2 & j \equiv 2(mod\ 4); m \equiv 0(mod\ 4); or \\ & j \equiv 3(mod\ 4); m \equiv 2(mod\ 4) \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2 \\ 3 & i = 1 \\ 5 & i = 3 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \ is \ even; j < m; m \equiv 0(mod\ 4); or \\ & j \ is \ odd; m \equiv 2(mod\ 4) \\ 3 & j \ is \ odd; m \equiv 0(mod\ 4); or \\ & j \ is \ even; j < m; m \equiv 2(mod\ 4) \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n\ is\ odd,; m\ is\ even; n = 3$ which is {2,3,5}, then $\left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$.

Subcase 3.2 is when $n > 3$. The following is the labeling defined $l: V\big(B_1(n,m)\big) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i \ is \ odd; 1 \le i < n-1; or \\ & i = n-1;\ or \\ & i = n \\ 2 & i \ is \ even; 2 \le i < n-1 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \equiv 0,1,3(mod\ 4); m \equiv 0(mod\ 4); or \\ & j \equiv 0,1,2(mod\ 4); m \equiv 2(mod\ 4) \\ 2 & j \equiv 2(mod\ 4); m \equiv 0(mod\ 4); or \\ & j \equiv 3(mod\ 4); m \equiv 2(mod\ 4) \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i \ is \ even; i < n \\ 3 & i = 1, n-2 \\ 4 & i = n; or \\ & i \ is \ odd; 1 < i < n-2 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \ is \ even; j < m; m \equiv 0(mod\ 4); or \\ & j \ is \ odd; m \equiv 2(mod\ 4) \\ 3 & j \ is \ odd; m \equiv 0(mod\ 4); or \\ & j \ is \ even; j < m; m \equiv 2(mod\ 4) \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n\ is\ odd;\ m\ is\ even; n > 3$ which is {2,3,4}, then $\left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$.

Based on the above two subcases, the upper bound of the chromatic number of local irregularity is obtained $\chi_{lis}\big(B_1(n,m)\big) \le \left|w\left(V\big(B_1(n,m)\big)\right)\right| = 3$. Thus, we get $3 \le \chi_{lis}\big(B_1(n,m)\big) \le 3$, then $\chi_{lis}\big(B_1(n,m)\big) = 3$; when $n\ is\ odd;\ m\ is\ even; n,m \ge 3$.

Case 4. $n\ is\ even; m\ is\ odd; n,m \ge 3$

In the graph $B_1(n,m)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E\big(B_1(n,m)\big)$. Based on **Observation 2**, the value of the labeling is $opt(l) \ge 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 3**, we get $\chi_{lis}\big(B_1(n,m)\big) \ge \chi\big(B_1(n,m)\big) = 3$. So the lower bound when $n\ is\ even; m\ is\ odd; n,m \ge 3$ is $\chi_{lis}\big(B_1(n,m)\big) \ge 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_1(n,m)$, we divide into 2 subcases, namely when $m = 3$ and when $m > 3$.

Subcase 4.1 is when $m = 3$. The following is the labeling defined $l: V\big(B_1(n,m)\big) \to \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = n; or \\ & i \equiv 0,1,3(mod\ 4); n \equiv 0(mod\ 4); or \\ & i \equiv 0,1,2(mod\ 4); n \equiv 2(mod\ 4) \\ 2 & i \equiv 2(mod\ 4); n \equiv 0(mod\ 4); or \\ & i \equiv 3(mod\ 4); n \equiv 2(mod\ 4) \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j = 1 \\ 2 & j = 2 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i\ is\ even; i < n; n \equiv 0(mod\ 4); or \\ & i\ is\ odd; n \equiv 2(mod\ 4) \\ 3 & i\ is\ odd; n \equiv 0(mod\ 4); or \\ & i\ is\ even; i < n; n \equiv 2(mod\ 4) \\ 5 & i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j = 2 \\ 3 & j = 1 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n\ is\ even;\ m\ is\ odd; m = 3$ which is $\{2,3,5\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$.

Subcase 4.2 is when $m > 3$. The following is the labeling defined $l: V(B_1(n,m)) \rightarrow \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i \equiv 0,1,3(mod\ 4); n \equiv 0(mod\ 4); or \\ & i \equiv 0,1,2(mod\ 4); n \equiv 2(mod\ 4) \\ & i = n \\ 2 & i \equiv 2(mod\ 4); n \equiv 0(mod\ 4); or \\ & i \equiv 3(mod\ 4); n \equiv 2(mod\ 4) \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j\ is\ odd; or \\ & j = m - 1; \\ 2 & j\ is\ even; 2 \le j < m - 1 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i\ is\ even; i < n; n \equiv 0(mod\ 4); or \\ & i\ is\ odd; n \equiv 2(mod\ 4) \\ 3 & i\ is\ odd; n \equiv 0(mod\ 4); or \\ & i\ is\ even; i < n; n \equiv 2(mod\ 4) \\ 4 & i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j\ is\ even \\ 3 & j = 1, n - 2 \\ 4 & j\ is\ odd; 1 < j < m - 2 \end{cases}$$

The set of vertex weights on the graph $B_1(n,m)$ when $n\ is\ even;\ m\ is\ odd; m > 3$ which is $\{2,3,4\}$, then $\left|w\left(V(B_1(n,m))\right)\right| = 3$.

Based on the above two subcases, the upper bound of the chromatic number of local irregularity is obtained $\chi_{lis}(B_1(n,m)) \le \left|w\left(V(B_1(n,m))\right)\right| = 3$. Thus, we get $3 \le \chi_{lis}(B_1(n,m)) \le 3$, then $\chi_{lis}(B_1(n,m)) = 3$; when $n\ is\ even; m\ is\ odd; n,m \ge 3$. ■

An illustration of local irregularity vertex coloring on the bicyclic graph of type I $B_1(n,m); n = m = 3$ is shown in **Figure 3**.

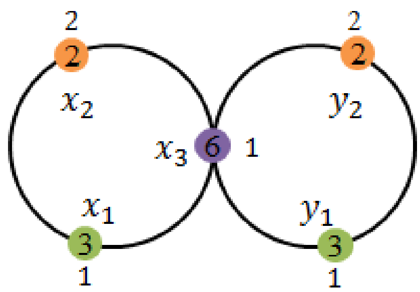

Figure 3. Illustration of Local Irregularity Vertex Coloring on the Bicyclic Graph of Type I $B_1(n,m)$

Observation 4. *Suppose the graph $B_2(n,m,r)$ is a bicyclic graph with $n,m \ge 3$ and $r \ge 2$, then*

$$\chi(B_2(n,m,r)) = \begin{cases} 2 & n,m\ is\ even \\ 3 & n\ is\ odd; \cup \\ & m\ is\ odd \end{cases}$$

Theorem 2. *Suppose the graph $B_2(n,m,r)$ is a bicyclic graph with $n,m \geq 3$ and $r \geq 2$, then*

$$\chi_{lis}(B_2(n,m,r)) = \begin{cases} 3 & n,m \text{ is even}; n,m \geq 3; r \geq 2; \cup \\ & n,m \text{ is odd}; r \text{ is even}; n=m=3; r>2; \cup \\ & n,m \text{ is odd}; r \text{ is even}; n=m>3; \cup \\ & n,m \text{ is odd}; r \text{ is even}; n \neq m; \cup \\ & n,m \text{ is odd}; r \text{ is odd}; \cup \\ & n \text{ is odd}; m \text{ is even}; \cup \\ & n \text{ is even}; m \text{ is odd} \\ 4 & n=m=3; r=2 \end{cases}$$

Proof. A bicyclic graph of type II $B_2(n,m,r)$ has vertex set $V(B_2(n,m,r)) = \{x_i; 1 \leq i \leq n\} \cup \{z_h; 2 \leq h \leq r-1\} \cup \{y_j; 1 \leq j \leq m\}$ and edges set $E(B_2(n,m,r)) = \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup (x_n x_1) \cup (x_n z_2) \cup \{z_h z_{h+1}; 2 \leq h \leq r-2\} \cup (z_{r-1} y_m) \cup (y_m y_1) \cup \{y_j y_{j+1}; 1 \leq j \leq m-1\}$, therefore the number of vertex is $|V(B_2(n,m,r))| = n+m+r-2$ and the number of edges is $|E(B_2(n,m,r))| = n+m+r-1$.

Case 1. n,m *even*$; n,m \geq 3; r \geq 2$

In the graph $B_2(n,m,r)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E(B_2(n,m,r))$. Based on **Observation 2**, the value of the labeling is $opt(l) \geq 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 4**, we get $\chi_{lis}(B_2(n,m,r)) \geq \chi(B_2(n,m,r)) = 2$. Assume $\chi_{lis}(B_2(n,m,r)) = 2$, meaning there are 2 colors. If $x_i, y_j, z_h \in V(B_2(n,m,r))$ is labeled, then the following possible vertex weights are obtained:

a) If $l(x_i) = 1; l(y_j) = 1; l(z_h) = 1$, then $w(x_1) = w(x_2), x_1 x_2 \in E(B_2(n,m,r))$.
b) If $l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even; $l(y_j) = 1$ for j is odd; $l(y_j) = 2$ for j is even; $l(z_h) = 1$ for h is odd, $h > 1$; $l(z_h) = 2$ for h is even, then $w(x_1) = w(x_n), x_n x_1 \in E(B_2(n,m,r))$.
c) If $l(x_n) = 1$; $l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even, $i < n$; $l(y_m) = 1$ for r is odd; $l(y_j) = 1$ for j is even; $l(y_j) = 2$ for j is even, r is even or $j < m$, r is odd; $l(z_h) = 1$ for h is odd; $l(z_h) = 2$ for h is even, then the vertex color is more than 2.

Based on the above statement, there are at least 3 colors at the vertices $x_i, y_j, z_h \in V(B_2(n,m,r))$. Next, select the vertices x_1, x_2, and x_n,

$$w(x_1) = l(x_2) + l(x_n) = 2 + 1 = 3 \tag{4}$$

$$w(x_2) = l(x_1) + l(x_3) = 1 + 1 = 2 \tag{5}$$

$$w(x_n) = l(x_1) + l(x_{n-1}) + l(z_2) = 1 + 1 + 2 = 4 \tag{6}$$

Based on **Equation (4)**, **Equation (5)**, and **Equation (6)**, we get $w(x_1) \neq w(x_2), x_1 x_2 \in E(B_2(n,m,r))$ and $w(x_n) \neq w(x_1), x_n x_1 \in E(B_2(n,m,r))$ satisfies the definition of local irregularity vertex coloring. So the lower bound of the graph $B_2(n,m,r)$ when n,m *is even*$; n,m \geq 3; r \geq 2$ is $\chi_{lis}(B_2(n,m,r)) \geq 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_2(n,m,r)$ we define the labeling $l: V(B_2(n,m,r)) \to \{1,2\}$ as follows:

$$l(x_i) = \begin{cases} 1 & i \text{ is odd}; or \\ & i = n \\ 2 & i \text{ is even}; i < n \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j \text{ is odd}; or \\ & j = m; r \text{ is odd} \\ 2 & j, r \text{ is even}; or \\ & j \text{ is even}; j < m; r \text{ is odd} \end{cases}$$

$$l(z_h) = \begin{cases} 1 & h \text{ is odd} \\ 2 & h \text{ is even} \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i \text{ is even}; i < n \\ 3 & i = 1, n-1 \\ 4 & i \text{ is odd}; 1 < i < n-1; \text{or} \\ & i = n \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j \text{ is even}; j < m \\ 3 & j = m; r \text{ is even}; \text{or} \\ & j = 1, m-1; r \text{ is odd} \\ 4 & j \text{ is odd}; r \text{ is even}; \text{or} \\ & j, r \text{ is odd}; 1 < j < m-1 \end{cases}$$

$$w(z_h) = \begin{cases} 2 & h \text{ is even}; h < r \\ 4 & h \text{ is odd} \end{cases}$$

Based on the weight function above, each neighboring vertices has a different vertex weight. The set of vertex weights on the graph $B_2(n,m,r)$ when $n, m \text{ is even}; n, m \geq 3; r \geq 2$ which is $\{2,3,4\}$, then $\left|w\left(V\left(B_2(n,m,r)\right)\right)\right| = 3$. Thus, we get $3 \leq \chi_{lis}\left(B_2(n,m,r)\right) \leq 3$, then $\chi_{lis}\left(B_2(n,m,r)\right) = 3$; when $n, m \text{ is even}; n, m \geq 3; r \geq 2$.

Case 2. $n = m = 3; r = 2$

In the graph $B_2(n,m,r)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E\left(B_2(n,m,r)\right)$. Based on **Observation 2**, the value of the labeling is $opt(l) \geq 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 4**, we get $\chi_{lis}\left(B_2(n,m,r)\right) \geq \chi\left(B_2(n,m,r)\right) = 3$. Assume $\chi_{lis}\left(B_2(n,m,r)\right) = 3$, meaning there are 3 colors. If $x_i, y_j, z_h \in V\left(B_2(n,m,r)\right)$ is labeled, then the following possible vertex weights are obtained:

a) If $l(x_i) = 1; l(y_j) = 1$, then $w(x_1) = w(x_2), x_1x_2 \in E\left(B_2(n,m,r)\right)$.
b) If $l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even; $l(y_j) = 1$ for j is odd; $l(y_j) = 2$ for j is even, then $w(x_3) = w(y_3)$, $x_3y_3 \in E\left(B_2(n,m,r)\right)$.
c) If $l(x_i) = 1$ for i is odd; $l(x_i) = 2$ for i is even; $l(y_j) = 1$ for j is odd; $l(y_j) = 3$ for j is even, then the vertex color is more than 3.

Based on the above statement, there are at least 3 colors at the vertices $x_i, y_j, z_h \in V\left(B_2(n,m,r)\right)$. Next, select the vertices x_1, x_2, x_3, and y_3,

$$w(x_1) = l(x_2) + l(x_3) = 2 + 1 = 3 \tag{7}$$

$$w(x_2) = l(x_1) + l(x_3) = 1 + 1 = 2 \tag{8}$$

$$w(x_3) = l(x_1) + l(x_2) + l(y_3) = 1 + 2 + 1 = 4 \tag{9}$$

$$w(y_3) = l(y_1) + l(y_2) + l(x_3) = 1 + 3 + 1 = 5 \tag{10}$$

Based on **Equation (7)**, **Equation (8)**, **Equation (9)**, and **Equation (10)**, we get $w(x_1) \neq w(x_2), x_1x_2 \in E\left(B_2(n,m,r)\right)$, $w(x_2) \neq w(x_3), x_2x_3 \in E\left(B_2(n,m,r)\right), w(x_3) \neq w(x_1)$, $x_3x_1 \in E\left(B_2(n,m,r)\right)$, and $w(x_3) \neq w(y_3)$, $x_3y_3 \in E\left(B_2(n,m,r)\right)$ satisfies the definition of local irregularity vertex coloring. So the lower bound of the graph $B_2(n,m,r)$ when $n = m = 3; r = 2$ is $\chi_{lis}\left(B_2(n,m,r)\right) \geq 4$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_2(n,m,r)$, we define the labeling $l: V\left(B_2(n,m,r)\right) \to \{1,2,3\}$ as follows:

$$l(x_i) = \begin{cases} 1 & i = 1,3 \\ 2 & i = 2 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j = 1,3 \\ 3 & j = 2 \end{cases}$$

From the labeling with $opt(l) = 3$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2 \\ 3 & i = 1 \\ 4 & i = 3 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j = 2 \\ 4 & j = 1 \\ 5 & j = 3 \end{cases}$$

Based on the weight function above, each neighboring vertices has a different vertex weight. The set of vertex weights on the graph $B_2(n,m,r)$ when $n = m = 3; r = 2$ which is $\{2,3,4,5\}$, then $\left|w\left(V(B_2(n,m,r))\right)\right| = 4$. Thus, we get $4 \leq \chi_{lis}(B_2(n,m,r)) \leq 4$, then $\chi_{lis}(B_2(n,m,r)) = 4$; when $n = m = 3; r = 2$.

Case 3. $n = m = 3; r > 2; r$ is even

In the graph $B_2(n,m,r)$ there are neighboring vertex that have the same degree, for example x_1 and x_n with $x_n x_1 \in E(B_2(n,m,r))$. Based on **Observation 2**, the value of the labeling is $opt(l) \geq 2$. Next proves the lower bound of the chromatic number of the local irregularity vertex coloring. Based on **Lemma 1** and **Observation 4**, we get $\chi_{lis}(B_2(n,m,r)) \geq \chi(B_2(n,m,r)) = 3$. So the lower bound when $n = m = 3; r > 2; r$ is even is $\chi_{lis}(B_2(n,m,r)) \geq 3$.

Next, to prove the upper bound of the chromatic number of the local irregularity of the graph $B_2(n,m,r)$, we divide into 2 subcases, namely when $r = 4$ and when $r > 4$.

Subcase 3.1 is when $r = 4$. The following is the labeling defined $l: V(B_2(n,m,r)) \rightarrow \{1,2,3\}$:

$$l(x_i) = \begin{cases} 1 & i = 1 \\ 2 & i = 2 \\ 3 & i = 3 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j = 1,3 \\ 2 & j = 2 \end{cases}$$

$$l(z_h) = \begin{cases} 1 & h = 3 \\ 3 & h = 2 \end{cases}$$

From the labeling with $opt(l) = 3$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 4 & i = 2 \\ 5 & i = 1 \\ 6 & i = 3 \end{cases}$$

$$w(y_j) = \begin{cases} 4 & j = 2 \\ 5 & j = 1 \\ 6 & j = 3 \end{cases}$$

$$w(z_h) = \begin{cases} 4 & h = 2 \\ 5 & h = 3 \end{cases}$$

The set of vertex weights on the graph $B_2(n,m,r)$ when $n = m = 3; r = 4$ which is $\{4,5,6\}$, then $\left|w\left(V(B_2(n,m,r))\right)\right| = 3$.

Subcase 3.2 is when $r > 4$. The following is the labeling defined $l: V(B_2(n,m,r)) \rightarrow \{1,2\}$:

$$l(x_i) = \begin{cases} 1 & i = 1,3 \\ 2 & i = 2 \end{cases}$$

$$l(y_j) = \begin{cases} 1 & j = 1,3 \\ 2 & j = 2 \end{cases}$$

$$l(z_h) = \begin{cases} 1 & h = r-1; \text{or} \\ & h\ is\ even \\ 2 & h\ is\ odd; 1 < h < r-1 \end{cases}$$

From the labeling with $opt(l) = 2$, the following weight function is obtained:

$$w(x_i) = \begin{cases} 2 & i = 2 \\ 3 & i = 1 \\ 4 & i = 3 \end{cases}$$

$$w(y_j) = \begin{cases} 2 & j = 2 \\ 3 & j = 1 \\ 4 & j = 3 \end{cases}$$

$$w(z_h) = \begin{cases} 2 & h\ is\ odd; h > 1 \\ 3 & h\ is\ even; h < r \end{cases}$$

The set of vertex weights on the graph $B_2(n, m, r)$ when $n = m = 3; r\ is\ even; r > 4$ which is {2,3,4}, then $\left|w\left(V(B_2(n, m, r))\right)\right| = 3$.

Based on the above two subcases, the upper bound of the chromatic number of local irregularity is obtained $\chi_{lis}(B_2(n, m, r)) \leq \left|w\left(V(B_2(n, m, r))\right)\right| = 3$. Thus, we get $3 \leq \chi_{lis}(B_2(n, m, r)) \leq 3$, then $\chi_{lis}(B_2(n, m, r)) = 3$; when $n = m = 3; r > 2; r\ is\ even.$ ■

An illustration of local irregularity vertex coloring on the bicyclic graph of type II $B_2(n, m, r)$; $n = m = 3; r = 4$ is shown in **Figure 4.**

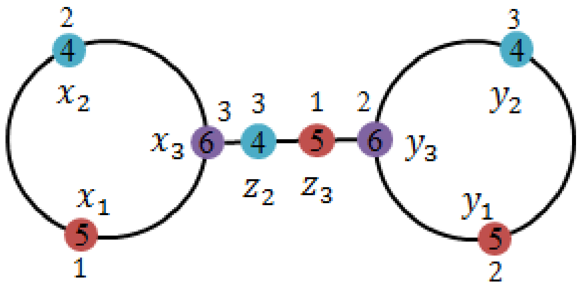

Figure 4. Illustration of Local Irregularity Vertex Coloring on the Bicyclic Graph of Type II $B_2(n, m, r)$

Observation 5. *Suppose the graph $B_3(n, m, t)$ is a bicyclic graph with $n, m \geq 3$ and $1 < t \leq min\{n, m\}$, then*

$$X(B_3(n, m, t)) = \begin{cases} 2 & n, m\ is\ even \\ 3 & n\ is\ odd; \cup \\ & m\ is\ odd \end{cases}$$

Theorem 3. *Chromatic number of local irregularity of bicyclic graph of type III $B_3(n, m, t)$ with $n, m \geq 3$ and $1 < t \leq min\{n, m\}$ are:*

I. $X_{lis}(B_3(n, m, t)) = 2$

$$for \begin{cases} n, m\ is\ even; t\ is\ odd; n = m = t + 1; \cup \\ n, m, t\ is\ even; n = m = t; \cup \\ n, m\ is\ even; t\ is\ odd; n \neq m; t = 3; n = 4; m > 4\ or\ m = 4; n > 4 \end{cases}$$

II. $X_{lis}(B_3(n, m, t)) = 3$

$$for \begin{cases} n, m, t\ is\ odd; n = m; \cup \\ n, m\ is\ odd; n \neq m; t = 3; n, m \neq t; \cup \\ n, m, t\ is\ odd; n \neq m; t > 3; n = t; m > t + 2\ or\ m = t; n > t + 2; \cup \\ n, m, t\ is\ odd; n \neq m; t > 3; n, m \neq t; \cup \\ n, m\ is\ odd; n \neq m; t\ is\ even; \cup \\ n, m\ is\ even; t\ is\ odd; n = m; n > t + 1; \cup \\ n, m, t\ is\ even; n = m; n > t + 2; \cup \\ n, m\ is\ even; n \neq m; t = 3; n, m > 4; \cup \\ n, m\ is\ even; t\ is\ odd; n \neq m; t > 3; \cup \\ n, m, t\ is\ even; n \neq m; n, m > t + 2; \cup \\ n, t\ is\ odd; m\ is\ even; \cup \\ n\ is\ odd; m, t\ is\ even; t > 2; m > t; \cup \\ n\ is\ even; m, t\ is\ odd; \cup \\ n, t\ is\ even; m\ is\ odd; t > 2; n > t \end{cases}$$

III. $X_{lis}(B_3(n, m, t)) = 4$

$$for \begin{cases} n, m\ is\ odd; n \neq m; t = 3; n = t\ or\ m = t; \cup \\ n, m, t\ is\ odd; n \neq m; t > 3; n = t; m = t + 2\ or\ m = t; n = t + 2; \cup \\ n, m, t\ is\ even; n = m; n = t + 2; \cup \\ n, m, t\ is\ even; n \neq m; n \leq t + 2\ or\ m \leq t + 2; \cup \\ n\ is\ odd; m, t\ is\ even; t = 2; \cup \\ n\ is\ odd; m, t\ is\ even; t > 2; m = t; \cup \\ n, t\ is\ even; m\ is\ odd; t = 2; \cup \\ n, t\ is\ even; m\ is\ odd; t > 2; n = t \end{cases}$$

An illustration of local irregularity vertex coloring on the bicyclic graph of type III $B_3(n, m, t); n, m\ is\ odd; n \neq m \neq t; t = 3$ is shown in **Figure 5**.

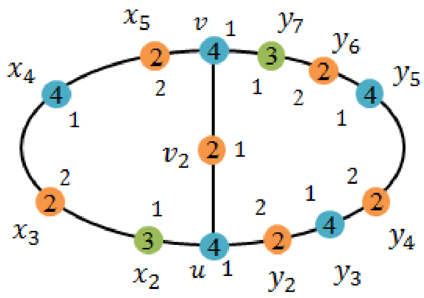

Figure 5. Illustration of Local Irregularity Vertex Coloring on the Bicyclic Graph of Type III $B_3(n, m, t)$

4. CONCLUSIONS

In this research, we have discussed the local irregularity vertex coloring of bicyclic graphs of type I $B_1(n, m)$, type II $B_2(n, m, r)$, type III $B_3(n, m, t)$. Thus, we obtained the chromatic number of local irregularity in the bicyclic graph of type I $B_1(n, m)$ is 3. The chromatic number of local irregularity in the bicyclic graph of type II $B_2(n, m, r)$ is 3 and 4 with certain conditions. The chromatic number of local irregularity in the bicyclic graph of type III $B_3(n, m, t)$ is 2, 3, and 4 with certain conditions. Based on the results of research on local irregularity vertex coloring of the bicyclic graph family, we provide an open problem to the readers.

Open Problem: Analyze the local irregularity vertex coloring of the bicyclic graph on n-vertices and its operation.

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REFERENCES

[1] G. Saputra *et al.*, "ON THE LOCAL IRREGULARITY VERTEX COLORING OF PENCIL GRAPHS," *AXIOMA*, vol. 9, no. 1, pp. 97–105, 2024.

[2] Slamin, *TEORI GRAF DAN APLIKASINYA EDISI KEDUA*. R12 Group, 2023. [Online]. Available: https://www.academia.edu/95939687/Teori_Graf_dan_Aplikasinya_Edisi_Kedua_Copyright_Slamin_216_halaman_16_x_23_cm_Hak_cipta_ada_pada_penulis

[3] I. H. Agustin, Dafik, Slamin, R. Adawiyah, and E. Y. Kurniawati, "ON THE STUDY OF LOCAL ANTIMAGIC VERTEX COLORING OF GRAPHS AND THEIR OPERATIONS," *ICCGANT 2020*, 2021, doi: 10.1088/1742-6596/1836/1/012018.

[4] Slamin, *TEORI GRAF DAN APLIKASINYA*. CV. Dream Litera Buana, 2019. [Online]. Available: https://repository.unej.ac.id/jspui/bitstream/123456789/91043/1/F. IK_Buku_Slamin_Teori Graf dan Aplikasinya.pdf?shem=ssusba

[5] A. Maulani and A. K. Lestari, “IMPLEMENTASI PEWARNAAN TITIK PADA PENGELOMPOKKAN PENJUALAN BAHAN GORDEN DI WS INTERIOR,” *J. Siger Mat.*, vol. 4, no. 2, pp. 43–51, 2023.

[6] A. B. Agatha, R. M. Prihandini, and A. P. Kusuma, “Graph Theory: Application of Graphs In Map Coloring,” *ResearchGate*, 2024.

[7] I. C. Nisa, N. Hinding, and H. Basir, “MODULAR IRREGULAR LABELING ON COMPLETE GRAPHS,” vol. 10 (3), no. Desember, pp. 171–184, 2022, doi: https://doi.org/10.26858/jdm.v10i3.37426.

[8] K. A. Sugeng, Z. Z. Barack, N. Hinding, and R. Simanjuntak, “MODULAR IRREGULAR LABELING ON DOUBLE-STAR AND FRIENDSHIP GRAPHS,” *J. Math.*, vol. 2021, pp. 1–6, 2021, doi: 10.1155/2021/4746609.

[9] G. Chartrand, C. Egan, and P. Zhang, *HOW TO LABEL A GRAPH*. SPRINGER BRIEFS IN MATHEMATICS, 2019. doi: http://doi.org/10.1007/978-3-030-16863-6.

[10] M. I. Pratama, “NILAI TOTAL KETIDAKTERATURAN SISI DAN TITIK PADA GRAF BERLIAN,” *J. Math. Comput. Stat.*, vol. 4, no. 2, p. 124, 2021, doi: 10.35580/jmathcos.v4i2.34509.

[11] N. R. O. Putri, E. Saputra, and A. A. Lisnasari, “IMPLEMENTATION OF GRAPH COLORING IN UMMUL MUKMININ HIGH SCHOOL STUDENT’S DORMITORY USING WELCH-POWELL ALGORITHM,” *BAREKENG J. Math. Its Appl.*, vol. 17, no. 1, pp. 593–600, 2023.

[12] A. N. Andriani, M. Tonis, and A. Zaky, “OPTIMIZATION OF EMERGENCY INSTALLATION DOCTOR PRACTICE SCHEDULING USING GRAPH COLORING METHOD IN PETALA BUMI HOSPITAL RIAU PROVINCE OPTIMASI PENJADWALAN PRAKTEK DOKTER INSTALASI GAWAT,” vol. 3, no. 2, pp. 1–8, 2022, [Online]. Available: https://journal.univawalbros.ac.id/index.php/jham/article/download/269/142?shem=ssusba

[13] I. Nuroeni, A. I. Kristiana, S. Hussen, S. Setiawani, and R. Adawiyah, “PEWARNAAN TITIK KETAKTERATURAN LOKAL PADA HASIL OPERASI SUBDIVISI GRAF HELM,” *J. Difer.*, vol. 5, no. 2, pp. 117–125, 2023, doi: 10.20473/conmatha.v5i2.47904.

[14] A. I. Kristiana, R. Alfarisi, I. Hesti Agustin, and M. Venkatachalam, “LOCAL IRREGULARITY VERTEX COLORING OF GRAPHS,” *Int. J. Civ. Eng. Technol.*, vol. 10, no. 03, pp. 1606–1616, 2019, [Online]. Available: http://www.iaeme.com/IJCIET/index.asp1606http://www.iaeme.com/ijmet/issues.asp?JType=IJCIET&VType=10&IType=3http://www.iaeme.com/IJCIET/issues.asp?JType=IJCIET&VType=10&IType=3

[15] M. Abdy, R. Syam, and T. Tina, “BILANGAN KROMATIK PEWARNAAN TITIK PADA GRAF DUAL DARI GRAF RODA,” *J. Math. Comput. Stat.*, vol. 4, no. 2, p. 95, 2021, doi: 10.35580/jmathcos.v4i2.24443.

[16] R. Adawiyah, M. Aima, and A. I. Kristiana, “ON INCLUSIVE LOCAL IRREGULARITY VERTEX COLORING OF BOOK GRAPH FAMILY,” *BAREKENG J. Math. Its Appl.*, vol. 17, no. 2, pp. 601–608, 2023.

[17] I. L. Mursyidah, Dafik, R. Adawiyah, A. I. Kristiana, and I. H. Agustin, “ON LOCAL IRREGULARITY VERTEX COLORING OF COMB PRODUCT ON STAR GRAPHS,” *J. Phys. Conf. Ser.*, 2021, doi: 10.1088/1742-6596/1836/1/012023.

[18] K. Munawaroh, A. I. Kristiana, E. R. Albirri, Dafik, and R. Adawiyah, “PEWARNAAN TITIK KETAKTERATURAN LOKAL PADA KELUARGA GRAF UNICYCLIC,” *CGANT*, vol. 2, no. 2, pp. 6–21, 2021, doi: 10.25037/cgantjma.v2i2.59.

[19] M. Hidayat, A. I. Kristiana, R. Adawiyah, Dafik, S. Setiawani, and R. Alfarisi, “ON LOCAL IRREGULARITY OF THE VERTEX COLORING OF THE CORONA PRODUCT OF A TREE GRAPH,” *Ural Math. J.*, vol. 8, no. 2, pp. 94–114, 2022, doi: 10.15826/umj.2022.2.008.

[20] R. F. Sandi, A. I. Kristiana, L. A. Monalisa, Slamin, and R. Adawiyah, “PEWARNAAN TITIK KETAKTERATURAN LOKAL PADA HASIL OPERASI AMALGAMASI TITIK GRAF LINTASAN,” *Contemp. Math. Appl.*, vol. 5, no. 2, pp. 88–101, 2023.

[21] F. Madina and R. Rahadjeng, “PEWARNAAN TITIK KETAKTERATURAN LOKAL PADA BEBERAPA KELAS GRAF,” *MATHunesa J. Ilm. Mat.*, vol. 12, no. 2, pp. 406–417, 2024.

[22] A. Khan, G. Haidar, N. Abbas, M. U. I. Khan, A. U. K. Niazi, and A. U. I. Khan, “METRIC DIMENSIONS OF BICYCLIC GRAPHS,” *Mathematics*, vol. 11, no. 4, pp. 1–17, 2023, doi: 10.3390/math11040869.

[23] V. Narayanan, “DECOMPOSITION OF BICYCLIC GRAPHS INTO PATHS AND CYCLES,” pp. 1–26, 2022, doi: https://doi.org/10.21203/rs.3.rs-2178974/v1.

[24] A. Ambarwati and V. H. Krisnawati, “CONSTRUCTION OF BICYCLIC GRAPH AND ITS APPLICATION IN TRANS JOGJA ROUTES,” *BAREKENG J. Math. Its Appl.*, vol. 17, no. 4, pp. 2095–2106, 2023.