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A COMPARATIVE STUDY ON NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS USING THE THREE NUMERICAL METHODS

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ABSTRACT

Numerical methods are crucial for solving ordinary differential equations (ODEs) that frequently arise in various fields of science and engineering. This study compares three numerical methods: the fourth-order Runge-Kutta method (RK4), the fourth-order Runge-Kutta Contra-harmonic Mean method (CoM4), and the fourth-order Adam-Bashforth-Moulton method (ABM4) in solving initial value problems of ODEs. Three IVPs of ODEs have been solved with varying step sizes using the three methods that have been proposed, and the solutions for each step size are examined. Numerical comparisons between RK4, CoM4, and ABM4 methods have been presented to solve three initial problems of ODE. Simulation results show that each method has advantages and limitations depending on the type of ODE being solved. We find that for very small step sizes, the numerical solutions agree the best with the exact solution. As such, all three proposed approaches are sufficient to solve the IVP ODE accurately and efficiently. Among the three proposed methods, we observe that the mean absolute error for the RK4 method is the smallest, followed by the ABM4 method.

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1. INTRODUCTION

Differential equations play an essential role in solving complex mathematical problems in almost every area of science and engineering. Many real-world issues arise in ordinary differential equations (ODEs) or systems of ODEs. Differential equations are commonly used for mathematical modeling in science and engineering. In most real-world situations, the differential equations that model the problem are too complex to be solved accurately, so one of two approaches is taken to approximate a solution. The first approach is to simplify the differential equations to equations that can be solved exactly. The second approach uses numerical approximation methods to solve the initial value problems (IVP) in ODEs. There are a few methods for solving IVP, and all numerical methods are not equally accurate and effective [1].

Many studies have been conducted to find solutions to ordinary differential equations and systems of ordinary differential equations. Unfortunately, most of the differential equations obtained after modeling real-world problems are non-linear, and this causes significant problems to be solved by analytical methods. Therefore, numerical methods have been developed and have proven to be very helpful in solving such ordinary differential equations. In addition, many computer programs have been developed to assist users in solving such equations. From the literature review, many authors have determined the IVP solution of the ODE using various numerical methods, namely the Euler method, the modified Euler method, and the 4th-order Runge-Kutta method (RK-4), among others. For example, [2] presents the analysis of the accuracy of IVPs for ODE using the Euler Method, and in [3], the author tries to find an accurate solution using the RK-4 method for IVPs ODE. Various numerical methods are discussed and given their accuracy order in [4], solving the IVP. Then, [5] conducted a comparative study on the numerical solution of the initial value problem by using Euler's method, modified Euler's method, and the Runge-Kutta method. From the results of this study, we concluded that the Runge-Kutta method is more powerful and efficient in finding numerical solutions to the initial value problem of an ODE. In [6], the authors compare fourth-order and butcher's fifth-order RK methods. Next, the author in [7] uses the RK4 approach to solve a mathematical model constructed using the SEIR framework. This study models the spread of hepatitis B in Ambon City over 20 years using data from the Central Statistics Agency and the Maluku Provincial Health Office. A compression study of multistep iterative methods for solving ordinary differential equations is discussed in [8]. Implicit methods for numerical solutions of singular initial value problems are a topic discussed by [9]. Furthermore, to solve the IVPs ODEs can be solved numerically using the predictor-corrector approach. This method is essentially two steps: first, it predicts an initial estimate (predictor), and then it refines this estimate (corrector). The fourth-order Adams-Bashforth method gives a better approximation result among the others mentioned methods (Euler, RK-4, RK-6) for solving initial value problems in ordinary differential equation Furthermore, ODEs can be solved numerically using the predictor-corrector approach to solve the IVP. This method is essentially two steps: first, it predicts an initial estimate (predictor) and then refines this estimate (corrector). The fourth-order Adams-Bashforth method gives a better approximation than the others mentioned methods (Euler, RK-4, RK-6) for solving initial value problems in ordinary differential equations [10]. [11] presents an efficient implementation of the fourth-order Adams-Bashforth-Moulton method for solving initial value problems in engineering applications. The study illustrates the accuracy and stability of the method through various test cases, highlighting its potential for improving computational efficiency in complex simulations. In real-world applications, a numerical method that uses a predictor and corrector scheme is the MacCormack Method. The MacCormack Method performs numerical simulations of pool and tsunami models for one-dimensional (1D) shallow water wave equations with flat and non-flat topography [12]. People can read the excellent and comprehensive texts on this subject in [13], [14], [15], [16], [17], [18], [19].

New Runge-Kutta formulations using methods other than the conventional arithmetic mean have introduced a few new formulas for the numerical solution of IVP ODEs. In [20], the author presents a new modification of Heun's method based on a contra-harmonic mean for solving high-efficiency initial value problems. A new fourth-order Runge-Kutta formula based on the Contra-Harmonic Mean (CoM) is discussed in [21]. [22] discussed the derivation of the fourth-order Runge-Kutta method based on functional values' harmonic mean (HM). This paper shows that an alternative formulation of the Runge-Kutta formula can be constructed using the harmonic mean scheme of functional values to conform to the classical fourth-order formula. The harmonic mean formula performs slightly better than the standard Runge-Kutta (Arithmetic Mean or AM) formula in terms of accuracy for the problem in the paper. In [23], the author constructs new numerical methods for solving the initial value problem (IVP) in ordinary differential equations based on a symmetrical quadrature integration formula using hybrid functions. These methods will provide a new computational tool for solving IVPs in ordinary differential equations and can be applied in various fields of science and engineering.

Motivated by the above work, this study aims to analyze the calculation of numerical solutions for the initial value problems of ordinary differential equations. The novelty of this paper lies in the introduction of the 4th-order Contra-Harmonic Mean Runge-Kutta method. It compares it to established methods (4th-order Runge-Kutta and Adam-Bashforth-Moulton) for solving the initial of 1st-order differential equations. These contributions collectively advance the field of numerical methods for solving IVPs in ODEs, providing new tools and insights for researchers and practitioners to deal with complex differential equations.

The paper is organized as follows: Section 2 presents research methods. Furthermore, results and discussion are given in Section 3. Finally, concluding remarks are provided in Section 4.

2. RESEARCH METHODS

The study of the comparison of the 4th-order Runge-Kutta method, 4th-order Runge-Kutta, Adam-Bashforth-Moulton, and Contra-Harmonic Mean were used to solve the initial value problem of ordinary differential equations through four stages of research as shown in **Figure 1** below.

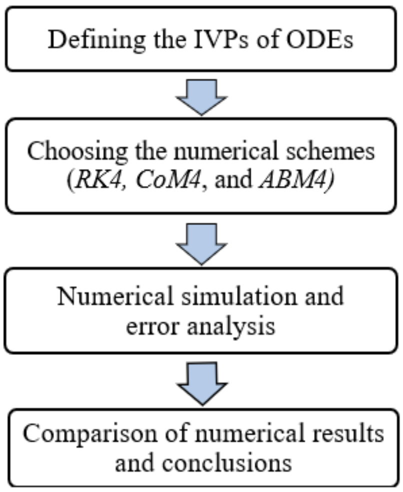


Figure 1. The Flowchart of Research Stages

Let us consider the first-order differential equation with initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y), \; y(t_0) = y_0. \quad (1)$$

where f is a given function on two real variables, y is an unknown function of the independent variable t, t_0 is the initial time, and y_0 is the initial value.

Although analytical methods are available to solve differential equations, their applicability is limited to certain special types, which rarely include those arising in practical problems. Continuous approximations of the solutions $y(t)$ and $y_i(t), i = 1, 2, 3, \dots, m$ may not be found. Instead, approximations to y and $y_i, i = 1, 2, 3, \dots, m$ will be produced at different values, in the interval $[a, b]$. The numerical method to obtain approximate solutions to the corresponding solution values for various chosen values of $t = t_n = a + nh, n = 1, 2, 3, \dots$ and h is the step size. The solutions to **Equation (1)** are given by the set of points $\{(t_n, y_n): n = 1, 2, 3, \dots\}$ and each solution (t_n, y_n) is an approximation to the corresponding point $\left(t_n, y(t_n)\right)$ on the solution curve.

Several ODEs with different initial conditions are used as case studies to evaluate the performance of the three numerical methods. The numerical techniques compared are as follows:

2.1 The Fourth-Order Runge-Kutta

The Fourth-Order Runge-Kutta method (RK4) is one of the most widely used techniques for solving ordinary differential equations (ODEs). The RK4 method approximates the solution of an initial value problem by considering the slope at several points within each step and then combining these slopes to produce a weighted average.

The RK4 method computes the solution at the next step y_{n+1} using the following formulas [16]:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2}$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n)\\
k_2 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1)\\
k_3 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2)\\
k_4 &= f(t_k + h, y_k + hk_3).
\end{aligned}$$

2.2 The Fourth-Order Runge-Kutta Contra-Harmonic Mean

The fourth-order Runge-Kutta contra-harmonic mean method (*CoM4*) is an advanced numerical technique for solving ordinary differential equations (ODEs). This method combines the traditional Runge-Kutta approach with the contra-harmonic mean to enhance accuracy and stability, particularly for problems with significant variations in the solution.

In 1995, [21] introduced the fourth-order Runge-Kutta method based on the contra-harmonic mean (*CoM4*). The *CoM4* computes the solution at the next step y_{n+1} using the following steps:

$$y_{n+1} = y_n + \frac{h}{3}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right) \tag{3}$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n)\\
k_2 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1)\\
k_3 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{8}hk_1 + \tfrac{3}{8}hk_2)\\
k_4 &= f(t_n + h, y_n + \tfrac{1}{4}hk_1 - \tfrac{3}{4}hk_2 + \tfrac{3}{2}hk_3).
\end{aligned}$$

2.3 The Fourth-Order Adam-Bashforth-Moulton

The Fourth-Order Adams-Bashforth-Moulton method (*ABM4*) is a predictor-corrector approach for solving ordinary differential equations (ODEs). It combines the explicit Adams-Bashforth method for prediction and the implicit Adams-Moulton method for correction. This combination enhances the numerical solution's accuracy and stability [19].

The fourth-order ABM method involves two main steps: the predictor (Adams-Bashforth**)** and the corrector (Adams-Moulton).

a. **Predictor** (Adams-Bashforth 4th order): Uses previous values to estimate the solution at the next step y_{n+1} explicitly using the following steps:

$$y_{n+1}^{(p)} = y_n + \frac{h}{24}(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}) \tag{4}$$

b. **Corrector** (Adams-Moulton 4th order**)**: Refines the predictor's estimate by using the newly computed solution value y_{n+1} implicitly using the following steps:

$$y_{n+1} = y_n + \frac{h}{24}(9 f_{n+1}^{(p)} + 19 f_n - 5 f_{n-1} + f_{n-2}) \tag{5}$$

The absolute error for this formula is defined by Absolute Error=$|y(t_n) - y_n|$. A numerical method is said to be convergent if

$$\lim_{\substack{h\to 0 \\ 1\le n\le N}} |y(t_n) - y_n| = 0$$

In numerical methods, we can use the mean absolute error (MAE) to compare the results of different approaches to determine the most accurate results. For example, when solving IVP of differential equations, MAE helps measure how close the numerical solutions are to the exact solution. The smaller the MAE, the better the numerical method. The formula is defined as follows, where N denotes the total number of iterations.

$$MAE = \frac{1}{N}\sum_{n=1}^{N} |y(t_n) - y_n|$$

Using the MATLAB software, numerical simulations are carried out to obtain approximate solutions for each method. Finally, the absolute error and mean absolute error (MAE) of the three proposed methods for different step sizes are calculated to test their superiority.

3. RESULTS AND DISCUSSION

In this section, the numerical solution of IVP is discussed, which is solved using the *RK4* method on **Equation (2)** using the *CoM4* method on **Equation (2)** and using the *ABM4* method on **Equation (4)** and **Equation (5)**. The convergence of IVP is calculated by $e_n = |y(t_n) - y_n| < \delta$ where $y(t_n)$ denotes the approximate solution and y_n denotes the exact solution. Parameter δ depends on the problem, which varies from 10^{-7}.

To perform numerical comparisons, *RK4*, *CoM4*, and *ABM4* are applied into two examples as follows:

Example 1: Solve the initial value problem (IVP)

$$y' = e^{-2t} - 50y,\ y(0) = 0$$

using *RK4*, *CoM4*, and *ABM4* methods with step sizes h =0.02, h =0.0025, and compare the errors on [0, 1].

The exact solution is $y(t) = \frac{1}{48} e^{-50t}(-1 + e^{48t})$.

Example 2: Solve the initial value problem (IVP)

$$y' = 4\ln(t) + \frac{1}{t}y;\ \ y(1) = 1$$

using *RK4*, *CoM4*, and *ABM4* methods with step sizes h =0.01, h =0.005, and compare the errors on [1, 2]..

The exact solution is $y(t) = 2t\ln^2(t) + t$.

Example 3: Solve the initial value problem (IVP)

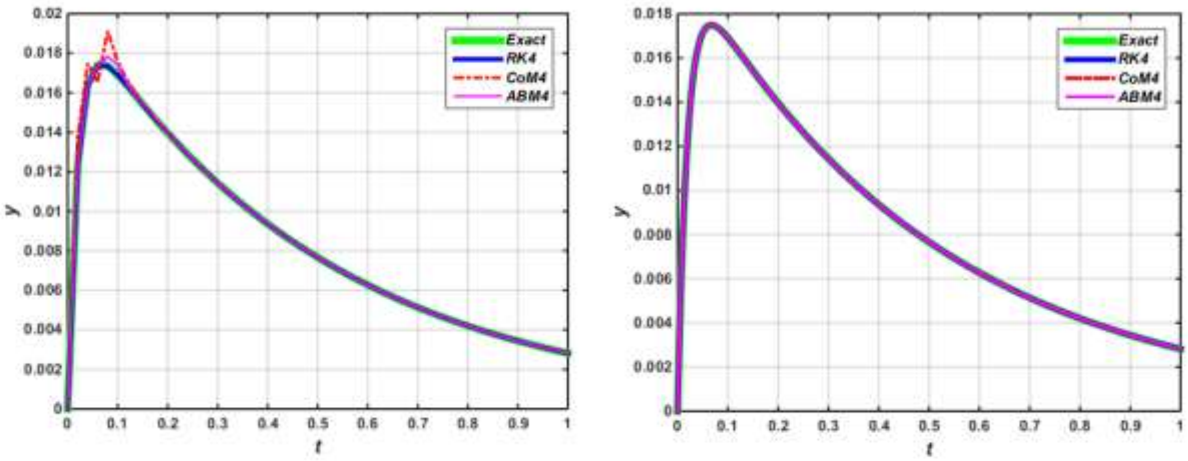
$$y' = (1-t)y^2 - y;\ \ y(0) = 1$$

using *RK4*, *CoM4*, and *ABM4* methods with step sizes h =0.05, h =0.01, and compare the errors on [0, 1].

The exact solution is $y(t) = \dfrac{1}{e^t - t}$.

3.1 Numerical Solution for Example 1

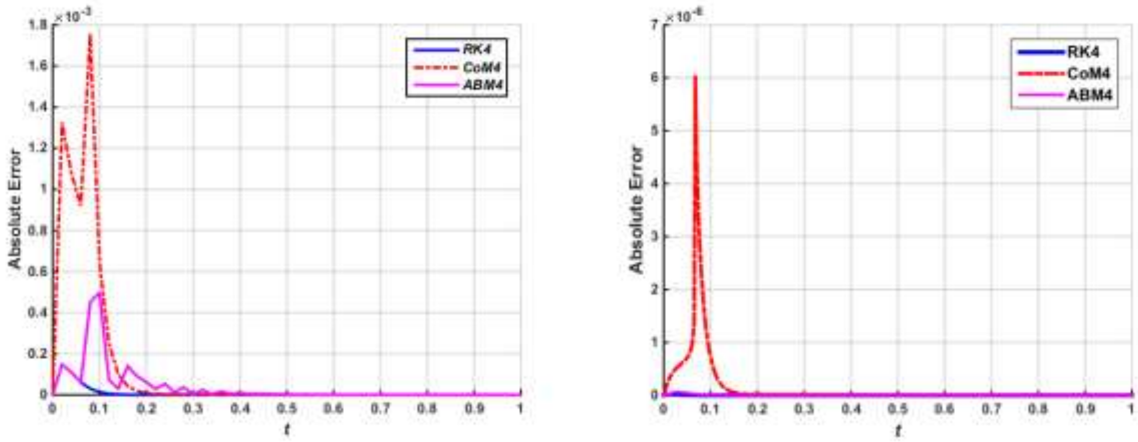
Figure 1 shows the numerical and exact solutions of Example 1 using the *RK4, CoM4*, and *ABM4* methods with step lengths of h = 0.02 and h = 0.0025. The results are computed with the help of MATLAB software and showed that all three methods at h = 0.0025 performed significantly better than at h = 0.02. **Table 1** and **Table 3** show the numerical solutions and the absolute errors of the numerical solutions represented by **Figure 1**.



(a) (b)

Figure 1. The *RK4, CoM4,* and *ABM4* Methods used to Solve Example 1 for (a) *h*=0.02 and (b) *h*=0.0025.

Figure 2 shows the absolute error obtained by using the *RK4, CoM4*, and *ABM4* methods obtained under **Table 2** and **Table 4**. Based on **Table 2** and **Table 4**, the mean absolute error of the *RK4*, *CoM4*, and *ABM4* methods for Example 1 is presented in **Table 5**. From **Table 5**, the MAE values of the *RK4, CoM4*, and *ABM4* methods for Example 1 with h = 0.02 and h = 0.0025 satisfy $RK4 < ABM4 < CoM4$. This shows that the *RK4* method gives more accurate or better results than the *ABM4* method, and the *ABM4* method gives more accurate or better results than the *CoM4* method.



(a) (b)

Figure 2. The Absolute Error of *RK4, CoM4,* and *ABM4* Methods used to Solve Example 1 for (a) *h*=0.02 and (b) *h*=0.0025

Table 1. The Numerical Solution, Exact Solution, and Absolute Error of the *RK4*, *CoM4* and *ABM4* Methods for Example 1 with *h*=0.02

n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
0	0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000
1	0.02	0.012352291625	0.012204287075	0.013677332507	0.012204287075
2	0.04	0.016412105482	0.016302357787	0.017488681935	0.016302357787
3	0.06	0.017440278507	0.017379361065	0.016518860781	0.017379361065
4	0.08	0.017371419793	0.017341492021	0.019127834405	0.017821218024
n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
4	0.08	0.017371419793	0.017341492021	0.019127834405	0.017821218024
5	0.10	0.016916516793	0.016902866938	0.017562531996	0.017414896624
⋮	⋮	⋮	⋮	⋮	⋮
45	0.90	0.003443726838	0.003443822900	0.003444368652	0.003443737367
46	0.92	0.003308696377	0.003308788672	0.003309313025	0.003308740381
47	0.94	0.003178960537	0.003179049213	0.003179553006	0.003178945463
48	0.96	0.003054311711	0.003054396910	0.003054880949	0.003054285314
49	0.98	0.002934550436	0.002934632294	0.002935097354	0.002934564980
50	1.00	0.002819485067	0.002819563716	0.002820010540	0.002819499655

Table 2. The Absolute Error of the *RK4, CoM4*, and *ABM4* Methods for Example with *h*=0.02

n	t_n	Absolute Error		
		RK4	*CoM4*	*ABM4*
0	0.00	0.000000000000	0.000000000000	0.000000000000
1	0.02	0.000148004550	0.001325040882	0.000148004550
2	0.04	0.000109747695	0.001076576452	0.000109747695
3	0.06	0.000060917443	0.000921417726	0.000060917443
4	0.08	0.000029927772	0.001756414612	0.000449798230
5	0.10	0.000013649856	0.000646015203	0.000498379831
4	0.08	0.000029927772	0.001756414612	0.000449798230
5	0.10	0.000013649856	0.000646015203	0.000498379831
⋮	⋮	⋮	⋮	⋮
45	0.90	0.000000096062	0.000000641814	0.000000010529
46	0.92	0.000000092295	0.000000616648	0.000000044004
47	0.94	0.000000088676	0.000000592469	0.000000015073
48	0.96	0.000000085199	0.000000569238	0.000000026397
49	0.98	0.000000081858	0.000000546918	0.000000014544
50	1.00	0.000000078649	0.000000525473	0.000000014587
Total		0.000378784143	0.006199912376	0.001882695540

Table 3. The Numerical Solution, Exact Solution, and Absolute Error of the *RK4*, *CoM4* and *ABM4* Methods for Example 1 with *h*=0.0025

n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
0	0.0000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
1	0.0025	0.002344074513	0.002344069333	0.002344159899	0.002344069333
2	0.0050	0.004401021889	0.004401012749	0.004401181308	0.004401012749
3	0.0075	0.006204638767	0.006204626669	0.006204862412	0.006204626669
4	0.0100	0.007784750283	0.007784736050	0.007785029705	0.007784759801
5	0.0125	0.009167676740	0.009167661042	0.009168004677	0.009167702606
⋮	⋮	⋮	⋮	⋮	⋮
395	0.9875	0.002890860672	0.002890860685	0.002890864041	0.002890860672
396	0.9900	0.002876442444	0.002876442457	0.002876445797	0.002876442444
397	0.9925	0.002862096127	0.002862096140	0.002862099463	0.002862096127
398	0.9950	0.002847821363	0.002847821376	0.002847824683	0.002847821363
399	0.9975	0.002833617795	0.002833617808	0.002833621098	0.002833617795
400	1.0000	0.002819485067	0.002819485080	0.002819488354	0.002819485067

Table 4. The Absolute Error of the *RK4*, *CoM4*, and *ABM4* Methods for Example 1 with *h*=0.0025

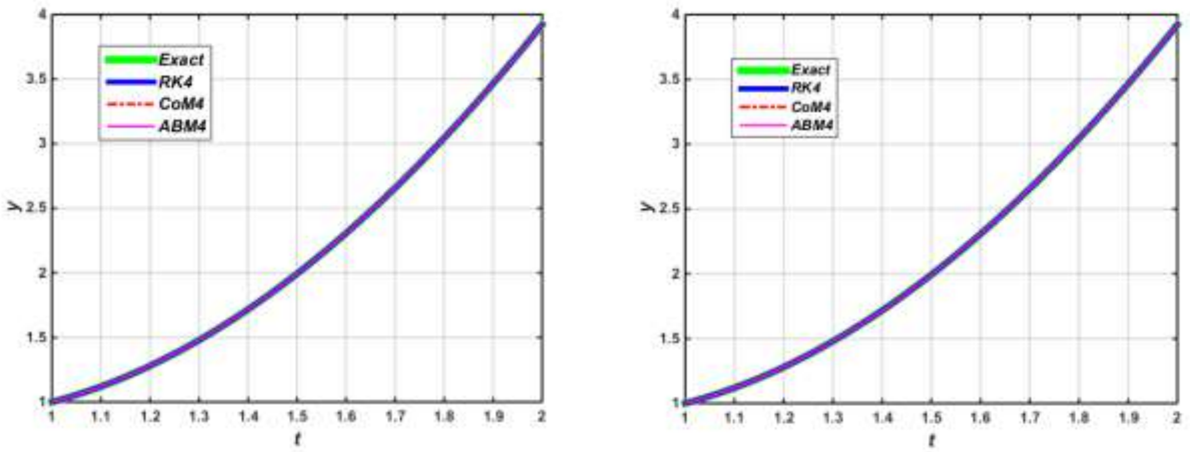
n	t_n	Absolute Error		
		RK4	*CoM4*	*ABM4*
0	0.0000	0.000000000000	0.000000000000	0.000000000000
1	0.0025	0.000000005179	0.000000085386	0.000000005179
2	0.0050	0.000000009140	0.000000159418	0.000000009140
3	0.0075	0.000000012098	0.000000223646	0.000000012098
4	0.0100	0.000000014233	0.000000279422	0.000000009518
5	0.0125	0.000000015698	0.000000327937	0.000000025867
⋮	⋮	⋮	⋮	⋮
395	0.9875	0.000000000013	0.000000003369	0.000000000000
396	0.9900	0.000000000013	0.000000003353	0.000000000000
397	0.9925	0.000000000013	0.000000003336	0.000000000000
398	0.9950	0.000000000013	0.000000003319	0.000000000000
399	0.9975	0.000000000013	0.000000003303	0.000000000000
400	1.0000	0.000000000013	0.000000003286	0.000000000000
Total		0.000000380543	0.000055973348	0.0000013745

Table 5. Mean Absolute Error (MAE) of the *RK4*, *CoM4*, and *ABM4* methods for Example 1

Method	MAE	
	***h*=0.02**	***h*=0.0025**
RK4	7.5757e-06	9.5136e-10
CoM4	1.2399e-04	1.3993e-07
ABM4	3.7654e-05	34363e-9

3.2 Numerical Solution for Example 2

Table 6 shows the numerical and exact solutions for Example 2 utilizing the *RK4*, *CoM4*, and *ABM4* methods when h = 0.01 while Table 8 when h = 0.005. According to the data, all three numerical methods perform noticeably better at h = 0.005 than they do at h = 0.01. Figure 3 demonstrates the numerical solutions of the *RK4*, *CoM4*, and *ABM4* methods based on the calculation results using h = 0.01 and $h = 0.01$. From Figure 3, it can be seen that the numerical solutions of the three methods are very close to the exact solution.



(a) (b)

Figure 3. The *RK4*, *CoM4*, and *ABM4* methods used to solve Example 2 for (a) h=0.01 and (b) h=0.005.

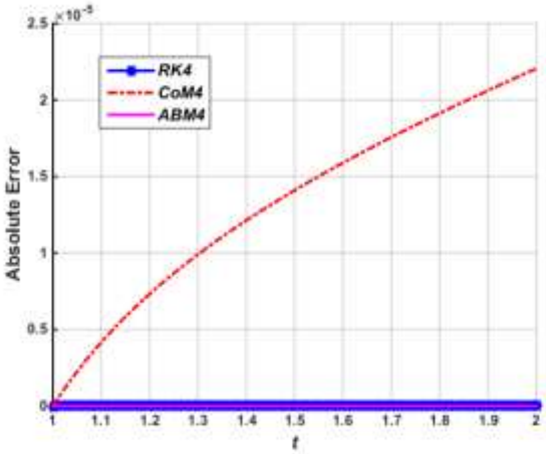
Table 6. The Numerical Solution, Exact Solution, and Absolute Error of the *RK4*, *CoM4* and *ABM4* Methods for Example 2 with h=0.01

n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
0	1.00	1.000000000000	1.000000000000	1.000000000000	1.000000000000
1	1.01	1.010199998350	1.010199998343	1.010200483173	1.010199998343
2	1.02	1.020799973858	1.020799973844	1.020800925327	1.020799973844
3	1.03	1.031799868947	1.031799868927	1.031801270279	1.031799868927
4	1.04	1.043199589828	1.043199589801	1.043201425479	1.043199589851
5	1.05	1.054999008251	1.054999008218	1.055001263795	1.054999008316
⋮	⋮	⋮	⋮	⋮	⋮
95	1.95	3.689384676412	3.689384675979	3.689406028886	3.689384677895
96	1.96	3.735188715145	3.735188714709	3.735210210390	3.735188716637
97	1.97	3.781334171270	3.781334170831	3.781355808761	3.781334172771
98	1.98	3.827820345022	3.827820344579	3.827842124242	3.827820346530
99	1.99	3.874646538495	3.874646538049	3.874668458942	3.874646540012
100	2.00	3.921812055673	3.921812055224	3.921834116853	3.921812057198

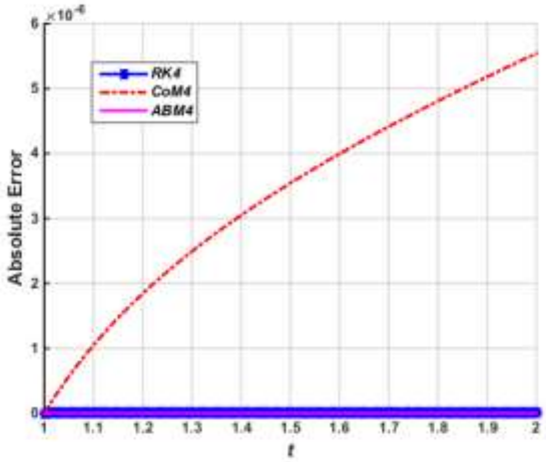
The absolute error achieved under Table 7 and Table 9 utilizing the *RK4*, *CoM4*, and *ABM4* methods is displayed in Figure 4. Table 10 displays the mean absolute error of the *RK4*, *CoM4*, and *ABM4* methods for Example 2 based on Table 7 and Table 9. From Table 10, it can be seen that the MAE values of the *RK4*, *CoM4*, and *ABM4* methods for Example 2 with h = 0.01 and h = 0.005 satisfy $RK4 < ABM4 < CoM4$. This means that the RK4 method provides more accurate results than the ABM4 method and the ABM4 method provides more accurate results than the CoM4 method.

Table 7. The Absolute Error of the *RK4, CoM4,* and *ABM4* Methods for Example 2 with *h*=0.01

n	t_n	Absolute Error		
		RK4	*CoM4*	*ABM4*
0	1.00	1.000000000000	1.000000000000	1.000000000000
1	1.01	0.000000000007	0.000000484823	0.000000000007
2	1.02	0.000000000014	0.000000951470	0.000000000014
3	1.03	0.000000000020	0.000001401332	0.000000000020
4	1.04	0.000000000027	0.000001835652	0.000000000023
5	1.05	0.000000000033	0.000002255544	0.000000000065
⋮	⋮	⋮	⋮	⋮
95	1.95	0.000000000433	0.000021352475	0.000000001484
96	1.96	0.000000000436	0.000021495246	0.000000001492
97	1.97	0.000000000439	0.000021637490	0.000000001500
98	1.98	0.000000000443	0.000021779220	0.000000001509
99	1.99	0.000000000446	0.000021920447	0.000000001517
100	2.00	0.000000000449	0.000022061180	0.000000001525
Total		0.000000025090	0.001311199394	0.000000093755



(a)



(b)

Figure 4. The Absolute Error of *RK4, CoM4,* and *ABM4* Methods Used to Solve Example 1 for (a) *h*=0.01 and (b) *h*=0.005.

Table 8. The Numerical Solution and Exact Solution of the *RK4, CoM4* and *ABM4* Methods for Example 2 with *h*=0.005

n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
0	1.000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
1	1.005	1.005049999896	1.005049999896	1.005050061434	1.005049999896
2	1.010	1.010199998350	1.010199998349	1.010200120234	1.010199998349
3	1.015	1.015449991687	1.015449991687	1.015450172775	1.015449991687
4	1.020	1.020799973858	1.020799973857	1.020800213051	1.020799973858
5	1.025	1.026249936489	1.026249936488	1.026250232732	1.026249936491
⋮	⋮	⋮	⋮	⋮	⋮

n	t_n	Numerical Solution and Exact Solution			
		Exact	***RK4***	***CoM4***	***ABM4***
195	1.975	3.804534712106	3.804534712078	3.804540165149	3.804534712210
196	1.980	3.827820345022	3.827820344994	3.827825815845	3.827820345126
197	1.985	3.851190982839	3.851190982811	3.851196471410	3.851190982943
198	1.990	3.874646538495	3.874646538467	3.874652044783	3.874646538600
199	1.995	3.898186925048	3.898186925019	3.898192449021	3.898186925153
200	2.000	3.921812055673	3.921812055645	3.921817597301	3.921812055778

Table 9. The Absolute Error of the *RK4, CoM4,* and *ABM4* methods for Example 2 with *h*=0.005

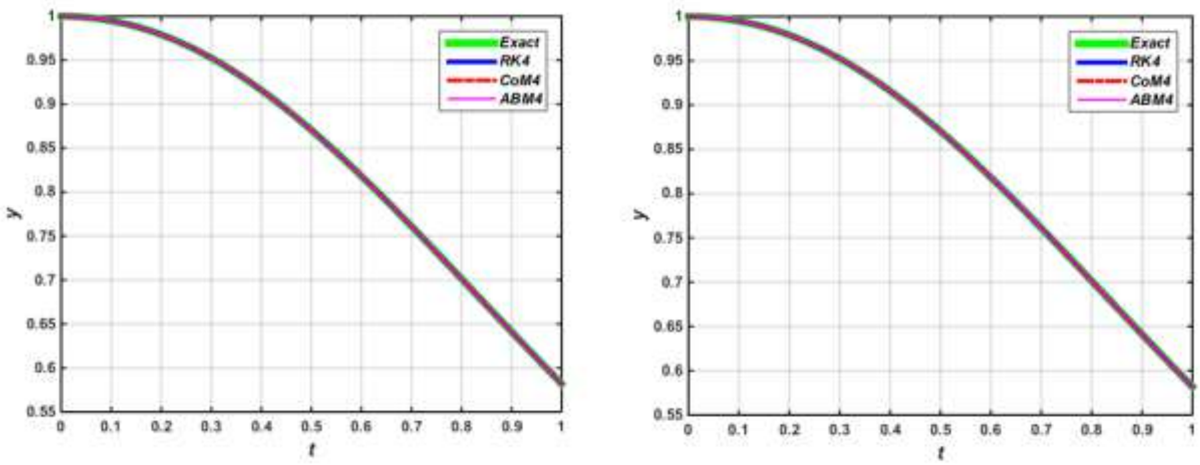
n	t_n	Absolute Error		
		RK4	***CoM4***	***ABM4***
0	1.000	0.000000000000	0.000000000000	0.000000000000
1	1.005	0.000000000000	0.000000061538	0.000000000000
2	1.010	0.000000000000	0.000000121884	0.000000000000
3	1.015	0.000000000001	0.000000181088	0.000000000001
4	1.020	0.000000000001	0.000000239193	0.000000000001
5	1.025	0.000000000001	0.000000296243	0.000000000002
⋮	⋮	⋮	⋮	⋮
195	1.975	0.000000000028	0.000005453043	0.000000000104
196	1.980	0.000000000028	0.000005470823	0.000000000104
197	1.985	0.000000000028	0.000005488571	0.000000000105
198	1.990	0.000000000028	0.000005506288	0.000000000105
199	1.995	0.000000000028	0.000005523973	0.000000000105
200	2.000	0.000000000028	0.000005541629	0.000000000106
Total		0.000000003164	0.000661612872	0.000000013268

Table 10. Mean Absolute Error (MAE) of the *RK4, CoM4,* and *ABM4* methods for Example 2

Method	MAE	
	***h*=0.01**	***h*=0.005**
RK4	2.5090e-10	1.5820e-11
CoM4	1.3112e-05	3.3081e-06
ABM4	9.3755e-10	6.6340e-11

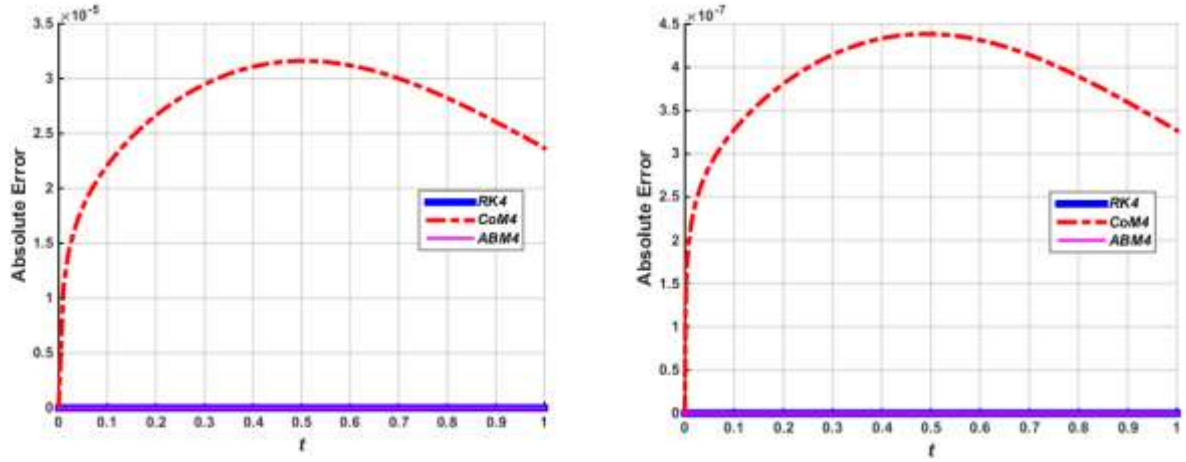
3.3 Numerical Solution for Example 3

The numerical and exact solutions for Example 3 utilizing the *RK4, CoM4,* and *ABM4* methods with step lengths of h = 0.01 and h = 0.001 are displayed in **Figure 5**. From **Figure 5**, it can be seen that the numerical solutions of the three methods are very close to the exact solution.According to the data, all three numerical methods perform noticeably better at h = 0.001 than they do at h = 0.01. **Table 11** shows the exact solutions and numerical solutions of the *RK4, CoM4, ABM4* methods, and the exact solution based on calculations with a step size of h=0.01, while Table 13 is for h=0.001.



(a) (b)

Figure 5. **The *RK4*, *CoM4*, and *ABM4* Methods Used to solve Example 3 for (a) *h*=0.05 and (b) *h*=0.01**



(a) (b)

Figure 6. **The Absolute Error of *RK4*, *CoM4*, and *ABM4* Methods Used to solve Example 3 for (a) *h*=0.05 and (b) *h*=0.01**

Table 11. **The Numerical Solution, Exact Solution, and Absolute Error of the *RK4*, *CoM4* and *ABM4* methods for Example 3 with *h*=0.01**

n	t_n	Numerical Solution and Exact Solution			
		Exact	*RK4*	*CoM4*	*ABM4*
0	0.00	1.00000000000	1.00000000000	1.00000000000	1.00000000000
1	0.01	0.999949835432	0.999949835435	0.999938755038	0.999949835435
2	0.02	0.999798700503	0.999798700509	0.999784654265	0.999798700509
3	0.03	0.999545672554	0.999545672562	0.999529801489	0.999545672562
4	0.04	0.999189882630	0.999189882641	0.999172657542	0.999189882683
5	0.05	0.998730517259	0.998730517273	0.998712196393	0.998730517356
⋮	⋮	⋮	⋮	⋮	⋮
95	0.95	0.611355440927	0.611355441031	0.611330559899	0.611355438914
96	0.96	0.605438115350	0.605438115454	0.605413473237	0.605438113355
97	0.97	0.599540347037	0.599540347141	0.599515944914	0.599540345061
98	0.98	0.593663388284	0.593663388387	0.593639227077	0.593663386328
99	0.99	0.587808451012	0.587808451115	0.587784531506	0.587808449079
100	1.00	0.581976706869	0.581976706972	0.581953029711	0.581976704959

Table 12. The Absolute Error of the *RK4*, *CoM4*, and *ABM4* methods for Example 3 with *h*=0.01

n	t_n	Absolute Error		
		RK4	***CoM4***	***ABM4***
0	0.00	0.000000000000	0.000000000000	0.000000000000
1	0.01	0.000000000003	0.000011080394	0.000000000003
2	0.02	0.000000000006	0.000014046238	0.000000000006
3	0.03	0.000000000009	0.000015871064	0.000000000009
4	0.04	0.000000000012	0.000017225088	0.000000000053
5	0.05	0.000000000015	0.000018320866	0.000000000097
⋮	⋮	⋮	⋮	⋮
95	0.95	0.000000000104	0.000024881028	0.000000002013
96	0.96	0.000000000104	0.000024642113	0.000000001995
97	0.97	0.000000000104	0.000024402124	0.000000001976
98	0.98	0.000000000103	0.000024161207	0.000000001956
99	0.99	0.000000000103	0.000023919506	0.000000001933
100	1.00	0.000000000103	0.000023677159	0.000000001910
Total		0.000000008594	0.002759437232	0.000000100031

The absolute error achieved under **Table 12** and **Table 14** utilizing the *RK4*, *CoM4*, and *ABM4* methods is displayed in **Figure 6**. **Table 15** displays the mean absolute error of the *RK4*, *CoM4*, and *ABM4* methods for Example 2 based on the absolute errors of the numerical solutions in **Table 12** and **Table 14**. As in the cases in Example 1 and 2, it can be seen that the MAE values of the *RK4*, *CoM4*, and *ABM4* methods for Example 3 with h = 0.01 and h = 0.005 also satisfy $RK4 < ABM4 < CoM4$. This means that the *RK4* method provides more accurate results than the *ABM4* method and the ABM4 method provides more accurate results than the *CoM4* method.

Table 13. The Numerical Solution, Exact Solution, and Absolute Error of the *RK4*, *CoM4* and *ABM4* methods for Example 3 with *h*=0.001

n	t_n	Numerical Solution and Exact Solution			
		Exact	***RK4***	***CoM4***	***ABM4***
0	0.000	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000
1	0.001	0.99999949983354	0.99999949983354	0.99999938875363	0.99999949983354
2	0.002	0.99999799867001	0.99999799867001	0.99999785890621	0.99999799867001
3	0.003	0.99999549551691	0.99999549551691	0.99999533876951	0.99999549551691
4	0.004	0.99999198938683	0.99999198938683	0.99999182050526	0.99999198938683
5	0.005	0.99998747929737	0.99998747929737	0.99998730094614	0.99998747929737
⋮	⋮	⋮	⋮	⋮	⋮
995	0.995	0.58488960814096	0.58488960814097	0.58488927955184	0.58488960814078
996	0.996	0.58430654803666	0.58430654803668	0.58430621978005	0.58430654803649
997	0.997	0.58372372673386	0.58372372673387	0.58372339880983	0.58372372673368
998	0.998	0.58314114535803	0.58314114535804	0.58314081776665	0.58314114535785
999	0.999	0.58255880503069	0.58255880503070	0.58255847777203	0.58255880503051
1000	1.000	0.58197670686933	0.58197670686934	0.58197637994346	0.58197670686915

Table 14. The Absolute Error of the *RK4*, *CoM4*, and *ABM4* methods for Example 3 with *h*=0.001

n	t_n	Absolute Error		
		RK4	***CoM4***	***ABM4***
0	0.000	0.00000000000000	0.00000000000000	0.00000000000000
1	0.001	0.00000000000000	0.00000011107992	0.00000000000000
2	0.002	0.00000000000000	0.00000013976380	0.00000000000000
3	0.003	0.00000000000000	0.00000015674740	0.00000000000000
4	0.004	0.00000000000000	0.00000016888157	0.00000000000000
5	0.005	0.00000000000000	0.00000017835123	0.00000000000000
⋮	⋮	⋮	⋮	⋮
995	0.995	0.00000000000001	0.00000032892155	0.00000000000018
996	0.996	0.00000000000001	0.00000032858912	0.00000000000018
997	0.997	0.00000000000001	0.00000032825661	0.00000000000018
998	0.998	0.00000000000001	0.00000032792403	0.00000000000018
999	0.999	0.00000000000001	0.00000032759138	0.00000000000018
1000	1.000	0.00000000000001	0.00000032725866	0.00000000000018
Total		0.000000000009	0.000387274937	0.00000000009764

Table 15. Mean Absolute Error (MAE) of the *RK4*, *CoM4*, and *ABM4* methods for Example 3

Method	MAE	
	***h*=0.01**	***h*=0.001**
RK4	8.5940e-11	9.0000e-15
CoM4	2.7594e-05	3.8728e-07
ABM4	1.0003e-09	9.7640e-14

4. CONCLUSIONS

This research paper compares three numerical methods: 4th order Runge-Kutta, 4th order Runge-Kutta based contra-harmonic mean, and Adam-Bashforth-Moulton for solving ordinary differential equations (ODEs) with initial value problems (IVPs). The numerical results obtained by the *RK4*, *CoM4*, and *ABM4* methods agree with exact solutions. The results of each initial value problem from the three examples guarantee the convergence to an exact solution if the step size is reduced. Based on the MAE values, the accuracy for all three numerical methods in all three examples is valid; the *RK4* method is better than the *ABM4* method, and the *ABM4* method is better than the *CoM4* method for the two-step sizes used.

To ensure higher accuracy, computational efficiency, solution reliability, and stability of numerical methods to solve the ODE's initial value problems, we recommend using a higher-order adaptive numerical method that can adjust the step size based on the calculated error and consider error control.

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