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LEVEL SOFT GROUP AND ITS PROPERTIES

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ABSTRACT

In this paper, we present an application of fuzzy subset and fuzzy subgroup to a soft set and a soft group, thereby creating a soft set and a soft group within the same group. Furthermore, we refer to the soft and soft groups as level soft sets and level soft groups. We also found out the level of soft sets and the operations on soft sets, such as intersection, union, and subset. We also examine what conditions a fuzzy subgroup and a soft group must meet to form a level soft group. Moreover, we scrutinize the properties of operations on a soft set, specifically intersection, union, and AND, and apply them to the level soft group to ascertain if they consistently produce a level soft group over the same set. Furthermore, we investigate the formation of a level soft and level soft group resulting from the homomorphism of the group and soft group. The research findings can enrich studies on the relationships between structures in fuzzy subgroups and soft groups and the application of soft group levels in further research.

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1. INTRODUCTION

The concept of fuzzy sets was first introduced by Zadeh [1] in 1965 as a mathematical approach to represent uncertainty in data through degrees of membership, which provides flexibility in describing the ambiguity of the elements of a set. Many researchers have integrated the fuzzy concept into various fields of science. One of the crucial developments in fuzzy theory was when Rosenfeld [2] combined this concept with group theory, leading to the development of the fuzzy subgroup. In developing this theory, fuzzy subgroups have become a significant research topic and have been studied extensively by various researchers. Among them, Abdurrahman investigated aspects such as the Image (Pre-image) Homomorphism of Interior Fuzzy Subgroups [3], Interior ω-fuzzy Subgroups [4], and Homomorphisms and (λ, μ]-Fuzzy Subgroups [5]. Furthermore, Shuaib, Shaheryar, and Asghar focused their research on the characterization of o-fuzzy subgroups [6]; Bejines, Jesús Chasco, Elorza, and Montes addressed the preservation of equivalence relations within fuzzy subgroups, while Boixader, Mayor, and Recasens studied the aggregation of fuzzy subgroups and the concept of T-vague groups. Beyond group theory, fuzzy sets have also been extended and examined within various other algebraic structures, as carried out by Hemabala and Kumar in the context of Γ-near rings [16] and by Abdurrahman in semiring structures [7], [8], [9], [10], [11]. This development enriches group theory in the fuzzy context and introduces a new approach to applying fuzzy structures to other algebraic structures.

Furthermore, in 1999, Molodtsov [12] introduced the concept of the soft set as an alternative to a fuzzy set to handle uncertainty without requiring complex membership functions. The soft set provides a more straightforward and flexible framework, which was eventually extended into group theory to produce the concept of soft group. Researchers such as [13], [14], [15], [16], [17] developed the idea of the soft group, which allows applying the soft set concept to the algebraic structure of groups, providing an exciting alternative in modelling data structures with uncertainty.

Many researchers have also conducted studies by combining fuzzy sets and soft sets with groups, including [18], [19], [20], [21], [22], [23], [24]. However, no research has specifically discussed the formation of levels soft set (levels soft group) from fuzzy subsets (fuzzy subgroups). Additionally, previous studies have not examined the characteristics of level soft sets concerning subset operations, intersections, and unions in soft sets or the conditions required for a fuzzy subgroup to form a level soft group.

This research focuses on developing level soft sets and level soft groups, a domain not thoroughly investigated in the existing literature. The study substantially contributes to the field by demonstrating fundamental operations in soft group theory, including intersection, union, and AND operations, when applied to level soft groups. These yield-level soft groups remain within the same group. This finding suggests that the structure of soft groups exhibits a stable property that enables these operations to produce soft groups consistently.

In addition, this research introduces the concept of level soft groups as an extension of soft groups, incorporating group homomorphisms. This approach provides a new perspective on the relationships between these structures. By exploring the interplay between soft groups and group homomorphisms, this research paves the way for a deeper understanding of the underlying mechanisms that govern the behaviour of these structures and their role in the development of group theory.

As a result, this research expands the scope of fuzzy and soft group theory and provides valuable insights for researchers and academics interested in further exploration. It is expected to encourage more in-depth and applied studies within group theory and soft sets, opening new opportunities to apply these concepts across various fields and advancing knowledge in the area.

2. RESEARCH METHODS

This paper employs a theoretical framework to develop the concepts of level soft set and level soft group. The research methodology involves a systematic approach comprising several key stages, including a comprehensive literature review, concept development, property analysis, and constructed property validation. The literature review examined the fundamental concepts of group theory, including groups, subgroups, group homomorphisms, fuzzy sets, fuzzy subgroups, soft sets, and soft groups.

To support the discussion section, we cite the works [12], [15], [25], [26], [27], [28]. These concepts provide a crucial theoretical foundation for understanding and developing the ideas of levels soft set and soft group. By leveraging this foundation, we can construct and analyze the properties related to levels of soft set and soft group and subsequently validate these properties. Thus, this enables us to develop a more comprehensive and in-depth theory about soft sets and soft groups, providing a foundation for further research and applications in this field.

In this paper, we assume that G is a group. A non-empty subset K of a group G is called a subgroup, denoted by $K \preccurlyeq G$, if and only if $ac^{-1} \in G$ for every $a, c \in G$. The subgroup $\{e_G\}$ and G are trivial subgroups of G, where e_G is the identity element of G. If $K \preccurlyeq G$ and $N \preccurlyeq G$, then $K \cap N \preccurlyeq G$.

Let G and S be groups and φ is a function from G into S. Then φ is called a group homomorphism of G into S if φ maintains the group operation, that is

$$\varphi(ac) = \varphi(a)\varphi(c)$$

for every $a, c \in G$.

A group homomorphism $\varphi: G \longrightarrow S$ is called an epimorphism if it is onto and it is called a monomorphism if it is one-one. The set $Ker\ \varphi = \{c \mid \varphi(c) = e_S\}$, is called the kernel of φ. The set $Im\ \varphi = \{\varphi(c) \mid c \in G\}$, is called the Image of φ. If $N \preccurlyeq G$, then $\varphi(N) \preccurlyeq S$.

In addition to this, A fuzzy subset of G is a function from G into a close interval $[0,1]$. The operations listed below are the most frequently used ones on fuzzy subsets. Let $\mathfrak{f}$ and $\mathfrak{h}$ be fuzzy subsets of G and $z \in G$,

$$\mathfrak{f} \widetilde{\bigcup} \mathfrak{h}(z) \stackrel{\text{def}}{=} \mathfrak{f}(z) \vee \mathfrak{h}(z) \text{ and } \mathfrak{f} \widetilde{\bigcap} \mathfrak{h}(z) \stackrel{\text{def}}{=} \mathfrak{f}(z) \wedge \mathfrak{h}(z).$$

If $\mathfrak{f}(z) \leq \mathfrak{h}(z)$ for every $z \in G$, then we called $\mathfrak{f} \mathrel{\widetilde{\subseteq}} \mathfrak{h}$. For $K(\neq \emptyset) \subseteq G$, a fuzzy subset $\mathfrak{x}_K$ is called the characteristic function of K if

$$\mathfrak{x}_K(z) \stackrel{\text{def}}{=} \begin{cases} 1, & z \in K \\ 0, & z \notin K \end{cases}$$

for every $z \in G$.

Let $\mathfrak{f}$ be a fuzzy subset of G and $a \in [0,1]$. The set $\{z \mid \mathfrak{f}(z) \geq a\}$, is called the $a-$ level subset of $\mathfrak{f}$ and it is denoted by $\mathfrak{f}_a$.

Proposition 1. *Let $\mathfrak{f}$ and $\mathfrak{h}$ be a fuzzy subset of G.* Then the following assertions hold:

a. *If $\mathfrak{f} \mathrel{\widetilde{\subseteq}} \mathfrak{h}$ and $c \in [0, 1]$, then $\mathfrak{f}_c \subseteq \mathfrak{h}_c$,*
b. *If $a \leq c$ and $a, c \in [0, 1]$, then $\mathfrak{f}_c \subseteq \mathfrak{f}_a$,*
c. *$\mathfrak{f} = \mathfrak{h}$ if and only if $\mathfrak{f}_c = \mathfrak{h}_c$ for any $c \in [0, 1]$.*

Proposition 2. *A fuzzy subset $\mathfrak{f}$ of G is a fuzzy subgroup of G if and only if $\mathfrak{f}(ac^{-1}) \geq \mathfrak{f}(a) \wedge \mathfrak{f}(c)$ for all $a, c \in G$.*

Proposition 3. *A fuzzy subset $\mathfrak{f}$ of G is a fuzzy subgroup of G if and only if the level subset $\mathfrak{f}_a$ is a subgroup of G for every $a \in \mathfrak{f}(G) \cup \{c \in [0,1] \mid c \leq \mathfrak{f}(e)\}$.*

Although distinct from the previous section, the concept of soft sets and soft groups still supports the overall argument. Let σ be a function from the parameter set $\mathcal{A}$ to the power set $\mathcal{P}(G)$. A soft set over G is presented as

$$\sigma_{\mathcal{A}} \stackrel{\text{def}}{=} \{(a, \sigma(a)) \big| a \in \mathcal{A}, \sigma(a) \subseteq G\}.$$

In other words, the function σ maps elements of the set $\mathcal{A}$ to a subset of G, and the resulting soft set in a collection of ordered pairs. $(a, \sigma(a))$, where $a \in \mathcal{A}$ and $\sigma(a) \subseteq G$.

Definition 1. *Let $\sigma_{\mathcal{A}}$ and $\delta_{\mathfrak{D}}$ Be soft set over G. The soft set $\sigma_{\mathcal{A}}$ is said to be a soft subset of the soft set $\delta_{\mathfrak{D}}$, denoted by $\sigma_{\mathcal{A}} \sqsubseteq \delta_{\mathfrak{D}}$,* if

a. $\mathcal{A} \subseteq \mathcal{D}$, and

b. $\sigma(a) \subseteq \delta(a)$ for every $a \in \mathcal{A}$.

Definition 2. *Let $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{D}}$ Be soft set over G. The soft set $\sigma_{\mathcal{A}}$ is said to be equal to the soft set $\delta_{\mathcal{D}}$, denoted by $\sigma_{\mathcal{A}} = \delta_{\mathcal{D}}$, if $\sigma_{\mathcal{A}} \sqsubseteq \delta_{\mathcal{D}}$ and $\delta_{\mathcal{D}} \sqsubseteq \sigma_{\mathcal{A}}$.*

Definition 3. *The intersection of the soft set $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{D}}$ Over G is the soft set $\pi_{\mathcal{S}}$ over G, denoted by $\sigma_{\mathcal{A}} \sqcap \delta_{\mathcal{D}} = \pi_{\mathcal{S}}$, such that $\mathcal{S} = \mathcal{A} \cap \mathcal{D}$ and $\pi(s) = \sigma(s) \cap \delta(s)$ for each $s \in \mathcal{S}$.*

Definition 4. *The union of the soft sets $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{D}}$ Over G is the soft set $\pi_{\mathcal{S}}$ over G, denoted by $\sigma_{\mathcal{A}} \sqcup \delta_{\mathcal{D}} = \pi_{\mathcal{S}}$, such that $\mathcal{S} = \mathcal{A} \cup \mathcal{D}$ and for each $s \in \mathcal{S}$,*

$$\pi(s) = \begin{cases} \sigma(s), & s \in \mathcal{A} - \mathcal{D} \\ \delta(s), & s \in \mathcal{D} - \mathcal{A} \\ \sigma(s) \cup \delta(s), & s \in \mathcal{A} \cap \mathcal{D} \end{cases}$$

Definition 5. *Let $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{S}}$ Be soft set over G. The AND operation of $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{S}}$ It is a soft set $\pi_{\mathcal{A}\times\mathcal{S}}$ over G, denoted by $\sigma_{\mathcal{A}} \wedge \delta_{\mathcal{S}} = \pi_{\mathcal{A}\times\mathcal{S}}$, such that $\pi(a,s) = \sigma(a) \cap \delta(s)$ for any $(a,s) \in \mathcal{A} \times \mathcal{S}$.*

The following example illustrates **Definition 1** through 5 of soft set theory. We begin by defining two fundamental sets, G and $\mathcal{E}$. Let G be defined as $G = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}\right\}$, which contains five elements representing our soft sets discourse universe. Similarly, let $\mathcal{E}$ be defined as $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$, which also consists of five elements that will serve as parameters in our soft sets. Next, we define three subsets of $\mathcal{E}$, namely $\mathcal{A} = \{e_2, e_3\}$, $\mathcal{C} = \{e_4\}$, $\mathcal{D} = \{e_1, e_2, e_3, e_5\}$. With these sets established, we can define three soft sets over G.

The first soft set $\sigma_{\mathcal{A}}$, is defined as $\sigma_{\mathcal{A}} = \left\{\left(e_2, \left\{\begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_3, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}\right)\right\}$. Thus, this indicates that the element e_2 is associated with the single-element $\begin{bmatrix}4\\0\end{bmatrix}$, while element e_3 is associated with two elements, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$. The second soft set, $\theta_{\mathcal{C}}$, is defined as $\theta_{\mathcal{C}} = \left\{\left(e_4, \left\{\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}\right\}\right)\right\}$, showing that element e_4 is associated with three elements: $\begin{bmatrix}2\\0\end{bmatrix}$, $\begin{bmatrix}4\\0\end{bmatrix}$, and $\begin{bmatrix}5\\0\end{bmatrix}$. The third soft set, $\delta_{\mathcal{D}}$, is defined as $\delta_{\mathcal{D}} = \left\{\left(e_1, \left\{\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_2, \left\{\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_3, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_5, \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}\right\}\right)\right\}$. This soft set illustrates the associations of elements in $\mathcal{D}$ with various elements in G, indicating, for example, that e_1 is associated with $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}4\\0\end{bmatrix}$, while e_5 is associated with $\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}5\\0\end{bmatrix}$.

From these definitions, we can identify significant relationships. First, we observe that $\sigma_{\mathcal{A}} \sqsubseteq \delta_{\mathcal{D}}$, indicating that all pairs in $\sigma_{\mathcal{A}}$ are also present in $\delta_{\mathcal{D}}$. Thus, this means that the information contained in $\sigma_{\mathcal{A}}$ is fully represented within $\delta_{\mathcal{D}}$. Conversely, we find that $\theta_{\mathcal{C}} \not\sqsubseteq \sigma_{\mathcal{A}}$, which shows that not all pairs in $\theta_{\mathcal{C}}$ are found in $\sigma_{\mathcal{A}}$, highlighting that $\theta_{\mathcal{C}}$ has unique associations beyond $\sigma_{\mathcal{A}}$. Next, we analyze the intersection of sets $\mathcal{A}$ and $\mathcal{D}$, which yields $\mathcal{A} \cap \mathcal{D} = \{e_2, e_3\}$. This result indicates that elements e_2 and e_3, they are common to both sets. Consequently, the intersection of soft sets $\sigma_{\mathcal{A}}$ and $\delta_{\mathcal{D}}$ is

$$\sigma_{\mathcal{A}} \sqcap \delta_{\mathcal{D}} = \left\{\left(e_2, \left\{\begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_3, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}\right)\right\}.$$

Confirming that the pairs associated with e_2 and e_3 in $\sigma_{\mathcal{A}}$ are also present in $\delta_{\mathcal{D}}$.

We then proceed to calculate the union of sets $\mathcal{A}$ and $\mathcal{C}$, resulting in $\mathcal{A} \cup \mathcal{C} = \{e_2, e_3, e_4\}$. This union includes all unique elements from both sets. Therefore, the union of soft sets $\sigma_{\mathcal{A}}$ and $\theta_{\mathcal{C}}$ is

$$\sigma_{\mathcal{A}} \sqcup \theta_{\mathcal{C}} = \left\{\left(e_2, \left\{\begin{bmatrix}4\\0\end{bmatrix}\right\}\right), \left(e_3, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}\right), \left(e_4, \left\{\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}\right\}\right)\right\},$$

The combined information from both soft sets results in a new set encompassing all associations.

Additionally, we observe that $\mathcal{A} \cup \mathcal{D} = \mathcal{D}$, meaning that all elements in $\mathcal{A}$ are already included in $\mathcal{D}$. Thus, we conclude that $\sigma_{\mathcal{A}} \sqcup \delta_{\mathcal{D}} = \delta_{\mathcal{D}}$, indicating that the information in $\sigma_{\mathcal{A}}$ does not add any new elements to $\delta_{\mathcal{D}}$.

Finally, we analyze the intersection of soft sets $\theta_{\mathcal{C}}$ and $\delta_{\mathcal{D}}$, which results in

$$\theta_{\mathcal{C}} \wedge \delta_{\mathcal{D}} = \left\{ \left((e_4, e_1), \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\} \right), \left((e_4, e_2), \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\} \right), \left((e_4, e_3), \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right\} \right), \left((e_4, e_5), \left\{ \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\} \right) \right\}.$$

Thus, this shows the pairs formed by combining elements from $\theta_{\mathcal{C}}$ and $\delta_{\mathcal{D}}$, indicating the relationships that can be established between these two soft sets.

Definition 6. *Soft set $\sigma_{\mathcal{A}}$ over G is said to be a soft group over G if and only if $\sigma(a) \preccurlyeq G$, for any $a \in \mathcal{A}$.*

To illustrate **Definition 6**, we present the following example, where $G = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \middle| \, a, b, c, d, e, f \in \mathbb{Z} \right\}$ is a group under the operation of matrix addition over the integers, and the parameter set $\mathcal{A}$ is given as $\mathcal{A} = 3\mathbb{Z}$. We then define a function $\sigma: \mathcal{A} \longrightarrow \mathcal{P}(G)$ That maps each element of $\mathcal{A}$ to a subset of G, such that

$$\sigma(z) = \left\{ \begin{bmatrix} nz & 0 & 0 \\ 0 & nz & 0 \\ 0 & 0 & nz \end{bmatrix} \middle| \, n \in \mathbb{Z} \right\}$$

For each $z \in \mathcal{A}$. Then we can easily see that $\sigma(a) \preccurlyeq G$, for any $a \in \mathcal{A}$. Thus, by Definition 6, we obtain that $\sigma_{\mathcal{A}}$ is a soft group over G.

3. RESULTS AND DISCUSSION

In the initial part of the discussion, we will construct a soft set over the group G using the fuzzy subset $\mathfrak{f}$ of G. To facilitate this construction, we will define a parameter set $\mathcal{A}$, which consists of two components: the Image of the function $\mathfrak{f}$, i.e., $\mathfrak{f}(G)$, and the set of all values c within the interval $[0, 1]$ such that c is less than or equal to $\mathfrak{f}(e)$, where e represents the identity element of G. The Image $\mathfrak{f}(G)$ Includes all possible membership values assigned by $\mathfrak{f}$ to the elements of G. At the same time, the second component ensures that we also consider values related to the identity element, thereby enriching the parameter set. This construction will provide a foundational framework for analyzing the relationships between the components of G and their corresponding degrees of membership in the fuzzy subset. It will set the stage for further exploration of soft sets in this mathematical context.

Proposition 4. *Let $\mathfrak{f}$ be a fuzzy subset of G and a set of parameters*

$$\mathcal{A} = \mathfrak{f}(G) \cup \{d \in [0,1] \mid d \leq \mathfrak{f}(e)\}.$$

If $\sigma_{\mathfrak{f}}: \mathcal{A} \longrightarrow \mathcal{P}(G)$ is a relation defined by $\sigma_{\mathfrak{f}}(\boldsymbol{c}) = \mathfrak{f}_{\boldsymbol{c}}$, for each $c \in \mathcal{A}$, then $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ It is a soft set over G.

Proof. Let $c = d$ in the set of parameters $\mathcal{A}$. Since $\mathfrak{f}$ is a fuzzy subset of G, by **Proposition 1**, we have $\mathfrak{f}_{\boldsymbol{c}} = \mathfrak{f}_{\boldsymbol{d}}$. Therefore, by definition $\sigma_{\mathfrak{f}}$, we have $\sigma_{\mathfrak{f}}(c) = \sigma_{\mathfrak{f}}(d)$, and so the relation $\sigma_{\mathfrak{f}}$ well-defined. In other words, $\sigma_{\mathfrak{f}}: \mathcal{A} \longrightarrow \mathcal{P}(G)$ is a function. Thus, $\sigma_{\mathfrak{f}_{\mathcal{A}}}$is *soft set* over G. ■

The soft set $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ over G, as in **Proposition 4** is hereafter called the level soft set over G.

Proposition 5. *Let $\mathfrak{f}$ and $\mathfrak{h}$ be a fuzzy subset of G. Let $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{B}}}$ Be two-level soft sets over G. Then*

a. *If $\mathfrak{f} \tilde{\subseteq} \mathfrak{h}$, then $\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqsubseteq \sigma_{\mathfrak{h}_{\mathcal{B}}}$, where $\mathcal{A} \subseteq \mathcal{B}$.*

b. $\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqcup \sigma_{\mathfrak{h}_{\mathcal{B}}} \sqsubseteq \sigma_{(\mathfrak{f} \tilde{\cup} \mathfrak{h})_{\mathcal{A} \cup \mathcal{B}}}$.

c. $\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqcap \sigma_{\mathfrak{h}_{\mathcal{B}}} = \sigma_{(\mathfrak{f} \tilde{\cap} \mathfrak{h})_{\mathcal{A} \cap \mathcal{B}}}$.

Proof.

a. Since $\mathfrak{f} \mathbin{\widetilde{\subseteq}} \mathfrak{h}$, so based on **Proposition 1** we have that $\mathfrak{f}_a \subseteq \mathfrak{h}_a$ for any $a \in \mathcal{A}$. Thus, by definition $\sigma_{\mathfrak{f}}$ and $\sigma_{\mathfrak{h}}$, we have, $\sigma_{\mathfrak{f}}(a) \subseteq \sigma_{\mathfrak{h}}(a)$. In other words, $\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqsubseteq \sigma_{\mathfrak{h}_{\mathcal{B}}}$.

b. Let $x \in \mathfrak{f}_a \cup \mathfrak{h}_a$ for any $a \in \mathcal{A} \cup \mathcal{B}$. Then $x \in \mathfrak{f}_a$ or $x \in \mathfrak{h}_a$. Because of this,

$$\mathfrak{f}(x) \geq a \text{ or } \mathfrak{h}(x) \geq a.$$

Thus,

$$\mathfrak{f}(x) \vee \mathfrak{h}(x) \geq a \Leftrightarrow \mathfrak{f} \mathbin{\widetilde{\cup}} \mathfrak{h}(x) \geq a.$$

In other words, $x \in \left(\mathfrak{f} \mathbin{\widetilde{\cup}} \mathfrak{h}\right)_a$ and so

$$\mathfrak{f}_a \cup \mathfrak{h}_a \subseteq \left(\mathfrak{f} \mathbin{\widetilde{\cup}} \mathfrak{h}\right)_a \Leftrightarrow \sigma_{\mathfrak{f}}(a) \cup \sigma_{\mathfrak{h}}(a) \subseteq \sigma_{\mathfrak{f} \widetilde{\cup} \mathfrak{h}}(a), \text{ for each } a \in \mathcal{A} \cup \mathcal{B}.$$

Therefore,

$$\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqcup \sigma_{\mathfrak{h}_{\mathcal{B}}} \sqsubseteq \sigma_{(\mathfrak{f} \widetilde{\cup} \mathfrak{h})_{\mathcal{A} \cup \mathcal{B}}}.$$

c. Since $\mathfrak{f}$ and $\mathfrak{h}$ are fuzzy subsets of G, we have, for any $a \in \mathcal{A} \cap \mathcal{B}$

$$\begin{aligned} \mathfrak{f}_a \cap \mathfrak{h}_a &= \{x \in G \mid \mathfrak{f}(x) \geq a\} \cap \{x \in G \mid \mathfrak{h}(x) \geq a\} \\ &= \{x \in G \mid \mathfrak{f}(x) \geq a \text{ and } \mathfrak{h}(x) \geq a\} \\ &= \{x \in G \mid \mathfrak{f}(x) \wedge \mathfrak{h}(x) \geq a\} \\ &= \left\{x \in G \,\middle|\, \mathfrak{f} \mathbin{\widetilde{\cap}} \mathfrak{h}(x) \geq a\right\} \\ &= \left(\mathfrak{f} \mathbin{\widetilde{\cap}} \mathfrak{h}\right)_a. \end{aligned}$$

Thus,

$$\sigma_{\mathfrak{f}}(a) \cap \sigma_{\mathfrak{h}}(a) = \sigma_{\mathfrak{f} \widetilde{\cap} \mathfrak{h}}(a).$$

Therefore, $\sigma_{\mathfrak{f}_{\mathcal{A}}} \sqcap \sigma_{\mathfrak{h}_{\mathcal{B}}} = \sigma_{(\mathfrak{f} \widetilde{\cap} \mathfrak{h})_{\mathcal{A} \cap \mathcal{B}}}$. ■

Now we introduce a group $G = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \,\middle|\, z \in \mathbb{Z} \right\}$ under matrix addition over the integers. Then we can easily prove that $\mathcal{L} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \,\middle|\, z \in 2\mathbb{Z} \right\} \preccurlyeq G$. Next, define a fuzzy subset $\mathfrak{f}: G \longrightarrow [0,1]$ where for each $\mathscr{b} \in G$,

$$\mathfrak{f}(\mathscr{b}) \overset{\text{def}}{=} \begin{cases} 0.7, & \mathscr{b} \in \mathcal{L} \\ 0.4, & \mathscr{b} \in G - \mathcal{L} \end{cases}.$$

In the following section, we will demonstrate that $\mathfrak{f}$ is a fuzzy subgroup of G. To show that $\mathfrak{f}$, we review two conditions below for all $a, c \in G$.

a. If $a, c \in \mathcal{L}$, then $a - c \in \mathcal{L}$ and so

$$\begin{aligned} \mathfrak{f}(a - c) &= 0.7 \\ &= 0.7 \wedge 0.7 \\ &= \mathfrak{f}(a) \wedge \mathfrak{f}(c). \end{aligned}$$

b. If at least one of a and c does not belong to $\mathcal{L}$, then

$$\begin{aligned} \mathfrak{f}(a - c) &\geq 0.4 \\ &= \mathfrak{f}(a) \wedge \mathfrak{f}(c). \end{aligned}$$

Thus, this shows that $\mathfrak{f}$ is a fuzzy subgroup of G. Hence, by **Proposition 3**, we conclude that, $\mathfrak{f}_d \preccurlyeq G$ for each $d \in \mathfrak{f}(G) \cup \{c \in [0,1] \mid c \leq \mathfrak{f}(e)\}$. Thus, according to the definition of the level soft set $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ In **Proposition 4**, we obtain:

a. $\sigma_{\mathfrak{f}}(d) = \mathfrak{f}_d = G \preccurlyeq G$ for any $d \in [0, 0.4]$.

b. $\sigma_{\mathfrak{f}}(d) = \mathfrak{f}_d = \mathcal{L} \preccurlyeq G$ for any $d \in (0.4, 0.7]$.

Consequently, we have a soft group derived from a level subset of a fuzzy subgroup, as we will present in the following proposition.

Proposition 6. *Let $\mathfrak{f}$ be a fuzzy subgroup of G, then the level soft set $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is a soft group over G.*

Proof. Let $\mathfrak{f}$ be a fuzzy subgroup of G. Then, by **Proposition 3,** we have $\mathfrak{f}_d$ is a subgroup of G for every $d \in \mathcal{A}$. Therefore, by definition $\sigma_{\mathfrak{f}_{\mathcal{A}}}$, we have $\sigma_{\mathfrak{f}}(d) = \mathfrak{f}_d \preccurlyeq G$ for any $d \in \mathcal{A}$ and so $\sigma_{\mathfrak{f}_{\mathcal{A}}}$, it is a soft group over G. ■

The soft group over G, $\sigma_{\mathfrak{f}_{\mathcal{A}}}$, as in **Proposition 6** is hereafter called level soft group over G.

Proposition 7. *Let $\mathfrak{f}$ and $\mathfrak{h}$ be a fuzzy subgroup of G, then $\sigma_{(\mathfrak{f}\tilde{\cap}\mathfrak{h})_{\mathcal{A}}}$ is a level soft group over G.*

Proof. Since $\mathfrak{f}$ and $\mathfrak{h}$ are fuzzy subgroups of G. Then, by **Proposition 2,** for any $x, z \in G$, we have

$$
\begin{aligned}
\mathfrak{f}\tilde{\cap}\mathfrak{h}\,(cz^{-1}) &= \mathfrak{f}(cz^{-1}) \wedge \mathfrak{h}(cz^{-1})\\
&\geq [\mathfrak{f}(c) \wedge \mathfrak{f}(z)] \wedge [\mathfrak{h}(c) \wedge \mathfrak{h}(z)]\\
&= \mathfrak{f}(c) \wedge [\mathfrak{f}(z) \wedge \mathfrak{h}(c)] \wedge \mathfrak{h}(z)\\
&= \mathfrak{f}(c) \wedge [\mathfrak{h}(c) \wedge \mathfrak{f}(z)] \wedge \mathfrak{h}(z)\\
&= [\mathfrak{f}(c) \wedge \mathfrak{h}(c)] \wedge [\mathfrak{f}(z) \wedge \mathfrak{h}(z)]\\
&= \mathfrak{f}\tilde{\cap}\mathfrak{h}\,(c) \wedge \mathfrak{f}\tilde{\cap}\mathfrak{h}\,(z)
\end{aligned}
$$

Thus, $\mathfrak{f}\tilde{\cap}\mathfrak{h}$ is a fuzzy subgroup of G. Therefore, based on **Proposition 6**, we conclude that $\sigma_{(\mathfrak{f}\tilde{\cap}\mathfrak{h})_{\mathcal{A}}}$ It is a level soft group over G. ■

According to the analysis of the proof presented in **Proposition 7**, we conclude that the intersection of two fuzzy subgroups of group G is itself a fuzzy subgroup of group G. Therefore, we articulate this condition in the following remark.

Remark 1. *Let $\mathfrak{f}$ and $\mathfrak{h}$ be a fuzzy subgroup of G, then $\mathfrak{f}\tilde{\cap}\mathfrak{h}$ is a fuzzy subgroup of G.*

Let's explain this proposition with the following example.

Example 1. Let $G = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in \mathbb{Z} \right\}$ be a group under the operation of matrix addition over the integers such that $\mathcal{W} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in 3\mathbb{Z} \right\} \preccurlyeq G$ and $\mathcal{K} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in 5\mathbb{Z} \right\} \preccurlyeq G$. Suppose $\mathfrak{f}$ dan $\mathfrak{h}$ are fuzzy subsets of G such that for each $c \in G$,

$$\mathfrak{f}(c) \overset{\text{def}}{=} \begin{cases} 0.9, & c \in \mathcal{W} \\ 0, & c \notin \mathcal{W} \end{cases}$$

and

$$\mathfrak{h}(c) \overset{\text{def}}{=} \begin{cases} 0.6, & c \in \mathcal{K} \\ 0, & c \notin \mathcal{K} \end{cases}.$$

Since $\mathcal{W} \preccurlyeq G$ and $\mathcal{K} \preccurlyeq G$, it follows that

$$\mathcal{W} \cap \mathcal{K} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in 15\mathbb{Z} \right\} \preccurlyeq G$$

so, for any $c \in \mathrm{G}$

$$(\mathfrak{f}\widetilde{\cap}\mathfrak{h})(c) \overset{\text{def}}{=} \begin{cases} 0.6, & c \in \mathcal{W} \cap \mathcal{K} \\ 0, & c \notin \mathcal{W} \cap \mathcal{K} \end{cases}.$$

To show that $\mathfrak{f}\widetilde{\cap}\mathfrak{h}$ is a fuzzy subgroup of G. We need to verify two conditions for any $m, n \in G$. These conditions are as follows:

Firstly, if $m, n \in \mathcal{W} \cap \mathcal{K}$, then $m - n \in \mathcal{W} \cap \mathcal{K}$, which yields

$$\begin{aligned}(\mathfrak{f}\widetilde{\cap}\mathfrak{h})(m-n) &= 0.6 \\ &= 0.6 \wedge 0.6 \\ &= \mathfrak{f}(a) \wedge \mathfrak{f}(c).\end{aligned}$$

Secondly, if either m or n (or both) is not in $\mathcal{W} \cap \mathcal{K}$, then

$$\begin{aligned}(\mathfrak{f}\widetilde{\cap}\mathfrak{h})(m-n) &\geq 0 \\ &= \mathfrak{f}(a) \wedge \mathfrak{f}(c).\end{aligned}$$

Consequently, we have $\mathfrak{f}\widetilde{\cap}\mathfrak{h}$ is a fuzzy subgroup of G. Let $\mathcal{A} = (\mathfrak{f}\widetilde{\cap}\mathfrak{h})(G) \cup \{d \in [0,1] \mid d \leq (\mathfrak{f}\widetilde{\cap}\mathfrak{h})(e)\}$. Consider the set-valued function $\sigma_{\mathfrak{f}\widetilde{\cap}\mathfrak{h}}: \mathcal{A} \longrightarrow \mathcal{P}(G)$ defined by $\sigma_{\mathfrak{f}\widetilde{\cap}\mathfrak{h}}(d) \overset{\text{def}}{=} (\mathfrak{f}\widetilde{\cap}\mathfrak{h})_d$ for any $d \in \mathcal{A}$. Based on **Proposition 6**, we have a level soft set $\sigma_{(\mathfrak{f}\widetilde{\cap}\mathfrak{h})_{\mathcal{A}}}$, which forms a soft group over G.

Considering the conditions of **Proposition 7** and the steps in the solution of **Example 1**, we derive two conditions presented in the following analogies: **Corollary 1** and **Corollary 2**.

Corollary 1. *Let $\mathfrak{f}_1, \mathfrak{f}_2, \cdots, \mathfrak{f}_n$ Be a fuzzy subgroup of G, then. $\sigma_{(\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_n)_{\mathcal{A}}}$ It is a level soft group over G.*

Proof. To prove this property, we utilize the conditions in **Proposition 6**, which enables us to focus on showing that $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_n$ is a fuzzy subgroup of G. We subsequently prove this result for $n \geq 2$ by employing mathematical induction.

Firstly, for $n = 2$, based on Remark 1, we have that $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2$ is a fuzzy subgroup of G.

Secondly, assume that for n = k, $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_k$ is a fuzzy subgroup of G. We will prove that for $n = k+1$, $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_k\widetilde{\cap}\mathfrak{f}_{k+1}$ is also a fuzzy subgroup of G. Since $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_k$ and $\mathfrak{f}_{k+1}$ are both fuzzy subgroups of G, it follows from Remark 1 that, $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_k\widetilde{\cap}\mathfrak{f}_{k+1}$ is a fuzzy subgroup of G. From all cases, it is established that $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_k\widetilde{\cap}\mathfrak{f}_{k+1}$ is a fuzzy subgroup of G. Therefore, by the principle of mathematical induction, we conclude that $\mathfrak{f}_1\widetilde{\cap}\mathfrak{f}_2\widetilde{\cap}\cdots\widetilde{\cap}\mathfrak{f}_n$ is a fuzzy subgroup of G, for all positive integers n. ∎

Corollary 2. *Let $\mathcal{L}$ be a non-empty subset of a G and $\mathfrak{f}$ be a fuzzy set in G defined by*

$$\mathfrak{f}(c) \overset{\text{def}}{=} \begin{cases} a, & c \in \mathcal{L} \\ d, & c \notin \mathcal{L} \end{cases}$$

for all $c \in \mathcal{L}$ and $a, d \in [0,1]$ with $a > d$. Then, $\mathfrak{f}$ is a subgroup fuzzy of G if and only if $\mathcal{L} \preccurlyeq G$.

Proof. Let $\mathfrak{f}$ be a subgroup fuzzy of G and let $c, z \in \mathcal{L}$. Then

$$\mathfrak{f}(cz^{-1}) \geq \mathfrak{f}(c) \wedge \mathfrak{f}(z) = a,$$

and so $\mathfrak{f}(cz^{-1}) = a$. Thus, $cz^{-1} \in \mathcal{L}$. In other words, we have $\mathcal{L} \preccurlyeq G$. Conversely, assume that $\mathcal{L} \preccurlyeq G$. Let $c, z \in G$. Since $\mathcal{L} \preccurlyeq G$, we consider the following two conditions:
Firstly, if at least one of c and z does not belong to $\mathcal{L}$, then

$$\mathfrak{f}(cz^{-1}) \geq d = \mathfrak{f}(c) \wedge \mathfrak{f}(z).$$

Secondly, if $c, z \in \mathcal{L}$, then $cz^{-1} \in \mathcal{L}$ and so

$$\mathfrak{f}(cz^{-1}) \geq a = \mathfrak{f}(c) \wedge \mathfrak{f}(z).$$

Thus, $\mathfrak{f}$ is a fuzzy subgroup of G. ∎

Proposition 8. *Let $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{S}}}$ be level soft group over G, where $\mathcal{A}\cap\mathcal{S} \neq \emptyset$. Then $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}}$ is a level soft group over G.*

Proof. Consider $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{S}}}$ as level soft groups over G, where $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}} = \sigma_{\mathfrak{d}_{\mathcal{D}}}$ and $\mathcal{D} = \mathcal{A}\cap\mathcal{S}$. According to **Proposition 6**, we conclude that, $\sigma_{\mathfrak{f}}(c) \preccurlyeq G$ for any $c \in \mathcal{A}$ and $\sigma_{\mathfrak{h}}(s) \preccurlyeq G$ for any $s \in \mathcal{S}$. As a result, from the property of the intersection of two subgroups in G, we have $\mathfrak{d}_{\mathcal{D}}(d) = \sigma_{\mathfrak{f}}(d)\cap\sigma_{\mathfrak{h}}(d) \preccurlyeq G$, for any $d \in \mathcal{D}$. Therefore, $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}}$ is level soft groups over G. ■

As an illustration of **Proposition 8**, we provide a specific example which will help us to clarify the concepts where is discussed in the **Proposition 8**. By examining this case, we can better to understand the implications of **Proposition 8**.

Example 2. Based on the conditions in **Example 1**, we show that $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}}$ is a level soft group over G.

Proof. Since both $\mathcal{W} = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in 3\mathbb{Z}\right\}$ and $\mathcal{K} = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \middle| z \in 5\mathbb{Z}\right\}$ are subgroups of G, and $\mathfrak{f}$ and $\mathfrak{h}$ are subset fuzzy of G. Therefore, based on **Corollary 2**, we have that $\mathfrak{f}$ and $\mathfrak{h}$ are subgroup fuzzy of G. Let $\mathcal{A} = \mathfrak{f}(G) = \{0, 0.9\}$ and $\mathcal{S} = \mathfrak{h}(G) = \{0, 0.6\}$, and so $\mathcal{A}\cap\mathcal{S} = \{0\}$. Consider the set-valued function $\sigma_{\mathfrak{f}}: \mathcal{A} \longrightarrow \mathcal{P}(G)$ and $\sigma_{\mathfrak{h}}: \mathcal{S} \longrightarrow \mathcal{P}(G)$ defined by $\sigma_{\mathfrak{f}}(d) \stackrel{\text{def}}{=} \mathfrak{f}_d$ for any $d \in \mathcal{A}$, and $\sigma_{\mathfrak{h}}(s) \stackrel{\text{def}}{=} \mathfrak{f}_s$ for any $s \in \mathcal{S}$. Based on **Proposition 6**, we have a level soft set $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{S}}}$ which forms a soft group over G, where

$$\sigma_{\mathfrak{f}_{\mathcal{A}}} = \left\{\left(0, \sigma_{\mathfrak{f}}(0)\right), \left(0.9, \sigma_{\mathfrak{f}}(0.9)\right)\right\} = \{(0, \mathfrak{f}_0), (0.9, \mathfrak{f}_{0.9})\} = \{(0, G), (0.9, \mathcal{W})\}$$

and

$$\sigma_{\mathfrak{h}_{\mathcal{S}}} = \left\{\left(0, \sigma_{\mathfrak{h}}(0)\right), \left(0.6, \sigma_{\mathfrak{h}}(0.6)\right)\right\} = \{(0, \mathfrak{h}_0), (0.6, \mathfrak{h}_{0.6})\} = \{(0, G), (0.6, \mathcal{L})\}.$$

Suppose that $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}} = \sigma_{\mathfrak{d}_{\{0\}}}$. Thus, for any $e \in \{0\}$, we have

$$\sigma_{\mathfrak{d}}(0) = \sigma_{\mathfrak{f}}(0)\cap\sigma_{\mathfrak{h}}(0) = G\cap G \preccurlyeq G$$

Therefore, $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcap\sigma_{\mathfrak{h}_{\mathcal{S}}} = \{(0, G)\}$ is level soft groups over G. ■.

Corollary 3. *The intersection of the level soft groups $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{A}}}$ over G is a level soft group over G.*

Proposition 9. *Let $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{B}}}$ Be a level soft group over G, where $\mathcal{A}\cap\mathcal{B} = \emptyset$. Then $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcup\sigma_{\mathfrak{h}_{\mathcal{B}}}$ is a level soft group over G.*

Proof. Let $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{B}}}$ be level soft groups over G, where $\sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcup\sigma_{\mathfrak{h}_{\mathcal{B}}} = \sigma_{\mathfrak{d}_{\mathcal{D}}}$ and $\mathcal{D} = \mathcal{A}\cup\mathcal{B}$. Then, by **Proposition 6**, we have $\sigma_{\mathfrak{f}}(a) \preccurlyeq G$ for any $a \in \mathcal{A}$ and $\sigma_{\mathfrak{h}}(c) \preccurlyeq G$ for any $c \in \mathcal{B}$. Since $\mathcal{A}\cap\mathcal{B} = \emptyset$, it follows that $\mathcal{A} - \mathcal{B} = \mathcal{A}$ and $\mathcal{B} - \mathcal{A} = \mathcal{B}$. Therefore, based on **Definition 4**, we obtain the condition for any $s \in \mathcal{D}$,

$$\sigma_{\mathfrak{d}}(s) = \sigma_{\mathfrak{f}}(s) \preccurlyeq G, s \in \mathcal{A} - \mathcal{B} \text{ or } \sigma_{\mathfrak{d}}(s) = \sigma_{\mathfrak{h}}(s) \preccurlyeq G, s \in \mathcal{B} - \mathcal{A}.$$

Thus, we conclude that, $\sigma_{\mathfrak{d}_{\mathcal{D}}} = \sigma_{\mathfrak{f}_{\mathcal{A}}}\sqcup\sigma_{\mathfrak{h}_{\mathcal{B}}}$ is a level soft group over G. ■

Proposition 10. *Let $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{B}}}$ be level soft group over G. Then $\sigma_{\mathfrak{f}_{\mathcal{A}}}\wedge\sigma_{\mathfrak{h}_{\mathcal{B}}}$ is a level soft group over G.*

Proof. Since $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ and $\sigma_{\mathfrak{h}_{\mathcal{B}}}$ be level soft group over G, where $\sigma_{\mathfrak{f}_{\mathcal{A}}}\wedge\sigma_{\mathfrak{h}_{\mathcal{B}}} = \sigma_{\mathfrak{d}_{\mathcal{A}\times\mathcal{D}}}$. Then, by **Proposition 6**, we have $\sigma_{\mathfrak{f}}(a) = \mathfrak{f}_a \preccurlyeq G$ for any $a \in \mathcal{A}$ and $\sigma_{\mathfrak{h}}(c) = \mathfrak{h}_c \preccurlyeq G$ for any $c \in \mathcal{B}$. Therefore, based on the property of the intersection of two subgroups, we obtain $\mathfrak{f}_a\cap\mathfrak{h}_c \preccurlyeq G$, and so $\sigma_{\mathfrak{d}}(a, c) = \sigma_{\mathfrak{f}}(a)\cap\sigma_{\mathfrak{h}}(c) \preccurlyeq G$, for any $(a, c) \in \mathcal{A} \times \mathcal{B}$. Consequently, $\sigma_{\mathfrak{f}_{\mathcal{A}}}\wedge\sigma_{\mathfrak{h}_{\mathcal{B}}} = \sigma_{\mathfrak{d}_{\mathcal{A}\times\mathcal{D}}}$, It is a level soft group over G. ■

Definition 7. *Suppose $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is a level soft group over G, and e_G denotes the identity element in G.*

a. *$\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is called an identity level soft group on G if $\sigma_{\mathfrak{f}}(c) = \{e_G\}$ For any $c \in \mathcal{A}$.*

b. *$\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is called an absolute level soft group on G if $\sigma_{\mathfrak{f}}(c) = G$ for any $c \in \mathcal{A}$.*

Proposition 11. *Suppose φ is a homomorphism from group G into group L, and $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is a level soft group over G, where $\varphi\sigma_{\mathfrak{f}} \overset{\text{def}}{=} \varphi \circ \sigma_{\mathfrak{f}}$. If $\sigma_{\mathfrak{f}}(c) = Ker\ \varphi$ for any $c \in \mathcal{A}$, then $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is the identity level soft group over L.*

Proof. Since $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ It is a level soft group over G. Then, by **Definition 6**, $\sigma_{\mathfrak{f}}(c) = \mathfrak{f}_c \preccurlyeq G$, for any $c \in \mathcal{A}$. Since $\varphi: G \longrightarrow L$ a group homomorphism, it follows from the properties of group homomorphisms that the Image of a subgroup of G under φ is a subgroup of L. Thus, we obtain

$$\varphi\sigma_{\mathfrak{f}}(c) = \varphi\left(\sigma_{\mathfrak{f}}(c)\right) = \varphi(\mathfrak{f}_c) < K.$$

Consequently, by **Definition 6**, we have $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is a level soft group over L. Since, $\sigma_{\mathfrak{f}}(c) = Ker\ \varphi$ for any $c \in \mathcal{A}$. Then, based on the definition of $Ker\ \varphi$, we obtain

$$\varphi\sigma_{\mathfrak{f}}(c) = \varphi\left(\sigma_{\mathfrak{f}}(c)\right) = \varphi(Ker\ \varphi) = \{e_L\}.$$

Therefore, by **Definition 7**, $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is the identity level soft group over L. ■

Proposition 12. *Suppose φ is a homomorphism from group G onto group L, and $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is a level soft group over G, where $\varphi\sigma_{\mathfrak{f}} \overset{\text{def}}{=} \varphi \circ \sigma_{\mathfrak{f}}$. If $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ Is an absolute **level** soft group over G, then $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is an absolute **level** soft group over L.*

Proof. Considering that $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is a level soft group over G, based on **Definition 6**, $\sigma_{\mathfrak{f}}(c) = \mathfrak{f}_c \preccurlyeq G$, for any $c \in \mathcal{A}$. Since $\sigma_{\mathfrak{f}}(c) = \mathfrak{f}_c \preccurlyeq G$ and $\varphi: G \longrightarrow L$ a group homomorphism, we have

$$\varphi\left(\sigma_{\mathfrak{f}}(c)\right) = \varphi(\mathfrak{f}_c) < K.$$

Therefore, according to **Definition 6**, $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is a level soft group over L. Since $\sigma_{\mathfrak{f}_{\mathcal{A}}}$ is an absolute **level** soft group over G, and $\varphi: G \longrightarrow L$ is an epimorphism. It follows, based on **Definition 7**, that $\sigma_{\mathfrak{f}}(c) = G$ for each $c \in \mathcal{A}$. Consequently,

$$\varphi\left(\sigma_{\mathfrak{f}}(c)\right) = \varphi(G) = K.$$

Thus, $(\varphi\sigma_{\mathfrak{f}})_{\mathcal{A}}$ is an absolute **level** soft group over L. ■

4. CONCLUSIONS

Based on the results and discussions we have conducted throughout this research, we have successfully established the fundamental properties of levels soft set, which include several essential operations, including subset, intersection, and union. In this context, we found that the intersection, union, and AND operations applied to two levels of soft group over a group consistently yield the same level of soft group over the group. This condition demonstrates that this structure is stable and can be extended without losing its fundamental properties. These findings emphasize the consistency and stability inherent in the structure we investigated, which is an essential aspect of the development of this theory. Furthermore, we also identified the existence of levels soft set (levels soft group) formed through group homomorphism and soft group. This process highlights the profound relationship between soft set theory and algebraic structures, providing new insights into how these concepts interact and contribute to developing a broader theory in this field, opening up opportunities for further research and practical applications in various disciplines.

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