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THE TRIPLE IDEMPOTENT GRAPH OF THE RING $\mathbb{Z}_n$

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ABSTRACT

Let R be a commutative ring, and $I(R)$ denote the set of all idempotent elements of R. The triple idempotent graph of R, denoted by $TI(R)$, is defined as an undirected simple graph whose vertex set $R-\{0,1\}$. Two distinct vertices u and v in $TI(R)$ are adjacent if and only if there exists $w \in R-\{0,1\}$ where $w \neq u$ and $w \neq v$ such that $uv \notin I(R), uw \notin I(R), vw \notin I(R)$, and $uvw \in I(R)$. This definition generalizes the notion of an idempotent divisor graph by involving a triple product, which allows deeper exploration of the combinatorial behavior of idempotents in rings. In this research, we investigate the properties of the triple idempotent graph of the ring of integers modulo n, denoted by $TI(\mathbb{Z}_n)$. As a results, we establish that $\mathrm{diam}(TI(\mathbb{Z}_n)) = 2$ and $gr(TI(\mathbb{Z}_n)) = 3$, provided that the graph is connected. Furthermore, $TI(\mathbb{Z}_n)$ is Hamiltonian if n is a prime and $n \geq 13$, and Eulerian if n is a prime and $n \geq 7$.

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1. INTRODUCTION

Research that links graph theory and algebraic structures has a fairly long history. In 1988, Beck [1] introduced the concept of a zero-divisor graph, which connects ring theory with graph theory. Anderson and Livingston [2] subsequently modified this concept by defining the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, with the vertex set $Z(R)^* = Z(R) - \{0\}$, where $Z(R)$ is the set of zero-divisors in R. In this graph, two distinct vertices x and y in $Z(R)$ are adjacent if and only if $xy = 0$. They showed that $\Gamma(R)$ is a connected graph with girth $\Gamma(R)$ in $\{3,4,\infty\}$. The graph $\Gamma(R)$ has become a valuable tool for analyzing the algebraic structure of commutative rings, particularly in understanding the behavior of zero-divisors. Akhtar and Lee [3] studied the connectivity of the zero-divisor graph associated with a finite commutative ring R, investigating conditions under which the ring R ensures that the graph $\Gamma(R)$ remains connected. This study demonstrated that, despite its simple construction, the $\Gamma(R)$ can exhibit varied and intricate connectivity properties, depending on the algebraic nature of the underlying ring. For example, the presence of idempotent elements, the number of minimal prime ideals, or the ring's decomposition into direct products of local rings can significantly influence the connectivity structure. This connection of the graph with the algebraic properties of zero-divisors in commutative rings remains a primary focus in understanding its fundamental mathematical structure. In addition to investigating the connectivity, diameter, and girth of graphs, some researchers also examine the conditions under which a graph associated with a ring becomes a Hamiltonian graph, a planar graph, or a Eulerian graph. For example, Nongsiang and Saikia [4] studied the conditions for the non-nilpotent graph of a finite group to become a Hamiltonian graph. Similarly, Mahmudi et al. [5] examined the maximal graph's diameter, girth, and Eulerian and Hamiltonian conditions from a commutative ring. Subsequently, many papers have investigated various kinds of graphs associated with rings, as discussed in [6], [7], [8], [9], and [10].

The significance of examining graphs related to algebraic structures resides not only in their theoretical relevance but also in their extensive applications. At the fundamental level, these investigations enhance our comprehension of commutative rings by offering a visual and combinatorial framework for analyzing zero-divisors, idempotents, and associated algebraic characteristics. For instance, the structural characteristics have proven essential in revealing new relationships among ring elements and their interrelations [11]. This framework has also been broadened to analyze other algebraic structures, including groups, monoids, and semirings, demonstrating the adaptability of these graph-theoretic methods [12]. Beyond pure mathematics, the applications of zero-divisor and idempotent graphs are diverse and impactful. One significant application can be found in coding theory and cryptography, where the structural characteristics of these graphs have been used to design secure and efficient cryptographic schemes ([13], [14]). Their adjacency and connectivity properties have been leveraged to formulate robust algorithms for error detection and correction in coding theory. Likewise, the interaction between algebraic structures and graph theory has been used to study complex systems, including social, biological, and communication networks ([15], [16]). By modeling these systems as graphs formulated from algebraic structures, researchers are able to examine their connectivity, robustness, and various other essential properties, thereby providing significant insights into their dynamics and behavior.

The study of graphs derived from algebraic structures also holds promise for computational mathematics and computer science. For instance, the algebraic properties of rings, such as idempotence, have inspired the development of efficient algorithms for parallel computing and optimization ([17], [18]). These algorithms use the structural regularities found in graphs associated with rings, like the unit graph in [14], to achieve improved computational performance. Moreover, the connections between graph theory and algebra have informed advancements in machine learning and artificial intelligence, where graph-theoretic techniques are used to design more effective algorithms for data analysis and decision-making ([19], [20]). Other significant areas of application are mathematical physics and quantum mechanics. The modeling of physical systems often involves a combination of algebraic and graph-theoretic techniques. For example, graph-theoretic principles have been utilized to analyze the structure and behavior of quantum systems, providing new perspectives on their fundamental mathematical foundations ([21], [22]). Likewise, investigating zero-divisor and idempotent graphs has facilitated progress in combinatorics and graph theory by establishing connections between these domains and abstract algebraic structures, resulting in identifying innovative problems and solutions [23].

As the interaction between graph theory and algebra progresses, the opportunity for discoveries persists significantly. The adaptability and applicability of these graphs render them an essential instrument for examining theoretical and practical issues. This research intends to enhance the current body of

knowledge by further examining the properties of graphs obtained from rings, emphasizing their algebraic and graph-theoretic traits. Recently, Mohammad and Shuker [24] introduced a graph called the idempotent divisor graph, denoted by $JI(R)$, with the vertex set $R^* = R - \{0\}$, where two vertices v_1 and v_2 are connected if and only if $v_1 v_2 = e$, for some non-unit idempotent element $e \in R$ (i.e., $e^2 = e \neq 1$). This research not only expands the scope of graph theory related to algebraic structures but also provides new insights into the relationship between idempotents and the graphical properties of commutative rings. The idempotent divisor graph focuses on adjacency through multiplication resulting in an idempotent element. In contrast, the triple idempotent graph, introduced by Kurniawan et al. [25], examines adjacency via a more restrictive triple condition, offering a different perspective on the structural role of idempotents in a ring. Let $I(R)$ be the set of idempotent elements of ring R. Kurniawan et al. [25] introduced the definition of the triple idempotent graph of a commutative ring R, denoted by $TI(R)$, as an undirected simple graph with the vertex set $R - \{0,1\}$. Two different vertices u and v are adjacent if there is an element $w \in R - \{0,1\}$ where $w \neq u$ and $w \neq v$ such that $uv \notin I(R), uw \notin I(R), vw \notin I(R)$, and $uvw \in I(R)$. In their paper, Kurniawan et al. [25] obtained several properties related to the connectivity of the triple idempotent graph of the ring $\mathbb{Z}_n$. Notably, the triple idempotent graph of the ring $\mathbb{Z}_n$ is connected if n is prime and $n \geq 7$. However, beyond connectivity, other structural properties of $TI(\mathbb{Z}_n)$ remain unexplored, such as its degree distribution, minimum degree, diameter, girth, and conditions under which it is Eulerian or Hamiltonian. This paper aims to fill this gap by systematically analyzing these additional graph-theoretic properties of the triple idempotent graph of the ring $\mathbb{Z}_n$, denoted by $TI(\mathbb{Z}_n)$. Through this investigation, we seek to deepen the understanding of the structural characteristics of these graphs.

2. RESEARCH METHODS

This study employs a theoretical approach grounded in an extensive literature review and analytical reasoning. The focus of the research is the triple idempotent graph of the ring of integers modulo n, denoted by $TI(\mathbb{Z}_n)$. The investigation aims to explore the structural properties of this graph that have not been thoroughly studied in previous works. These include vertex degree, minimum degree, diameter, girth, and conditions for Eulerian or Hamiltonian structures.

To support the analysis, several fundamental concepts are first introduced, including the definition and properties of idempotent elements in the ring $\mathbb{Z}_n$ and the specific criteria used to define adjacency in the triple idempotent graph. The conceptual framework guiding this study is based on the interplay between algebraic elements in $\mathbb{Z}_n$ and their graphical representation through the triple idempotent relation. This framework allows the investigation to bridge ring-theoretic properties with graph-theoretic structures in a meaningful and rigorous way. The methodology involves a detailed review of prior research related to algebraic graph theory, particularly studies that examine graphs associated with rings and the role of idempotents within such structures. This theoretical foundation is essential for identifying research gaps and situating the present work within the broader academic discourse.

Analytically, the research applies algebraic techniques to determine the set of idempotent elements in $\mathbb{Z}_n$, followed by constructing the triple idempotent graph based on defined adjacency rules. Graph-theoretical tools are then used to examine and prove various structural properties. Logical deduction and formal mathematical proof serve as the primary methods for validating the results and drawing conclusions. Through this approach, the study seeks not only to synthesize insights from the existing literature but also to discover and establish new properties of $TI(\mathbb{Z}_n)$, contributing to the deeper understanding of algebraic structures through a graph-theoretic lens.

2.1 Basic Concepts of Graphs

Here are some basic concepts in graph theory as referenced by Chartrand *et al.* [26].

Definition 1. *A graph G consists of a finite, non-empty set of vertices $V(G)$ and a collection $E(G)$ containing pairs of vertices, known as edges. The total count of vertices in $V(G)$ is called the order of the graph, while the total count of edges in $E(G)$ is referred to as the size.*

An example of a graph G can be seen in **Figure 1**. In **Figure 1**, the graph G has an order of 5, represented by the vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, and a size of 6, represented by the edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

Definition 2. *A $u-v$ walk in a graph G is defined as a sequence of vertices starting at u and ending at v. A $u-v$ trail is a $u-v$ walk where no edge is repeated. A circuit is a $u-v$ trail of length 3 or more that starts and finishes at the same vertex. A cycle is a circuit that does not revisit any vertex, except the starting and ending vertex.*

For example, in **Figure 1**, graph G contains a $v_1 - v_3$ walk: $v_1, e_1, v_2, e_6, v_4, e_4, v_3$. A $v_1 - v_3$ trail example is: $v_1, e_1, v_2, e_2, v_5, e_3, v_3$. An example of a $v_2 - v_2$ circuit is: $v_2, e_6, v_4, e_5, v_1, e_1, v_2$. A cycle in graph G is the $v_2 - v_2$ cycle: $v_2, e_2, v_5, e_3, v_3, e_4, v_4, e_6, v_2$.

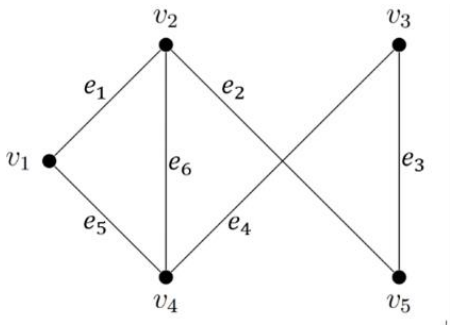

Figure 1. Graph G

Definition 3. *A graph G is considered connected if there exists a path connecting every pair of vertices in G.*

Figure 1 illustrates a connected graph since all pairs of vertices in G are connected.

Definition 4. *A Hamiltonian cycle in a graph G is a cycle that passes through every vertex of G exactly once. A graph containing a Hamiltonian cycle is called a Hamiltonian graph.*

In **Figure 1**, graph G is a Hamiltonian graph as it contains such a cycle. The Hamiltonian cycle is $v_2, e_1, v_1, e_5, v_4, e_4, v_3, e_3, v_5, e_2, v_2$.

Definition 5. *The girth of a graph G, denoted $gr(G)$, refers to the length of its shortest cycle.*

In **Figure 1**, $\mathrm{gr}(G) = 3$, corresponding to the $v_2 - v_2$ cycle: $v_2, e_6, v_4, e_5, v_1, e_1, v_2$.

Definition 6. *The diameter of a graph G denoted $\mathrm{diam}(G)$, is the greatest distance between any pair of vertices in G.*

In **Figure 1**, $\mathrm{diam}(G) = 2$, as the longest shortest path between any two vertices is at most 2.

Definition 7. *The degree of a vertex v in graph G, denoted $deg_G(v)$ is the number of edges incident to v.*

For example, in **Figure 1**, $\deg_G(v_1) = 2$, $\deg_G(v_2) = 3$, $\deg_G(v_3) = 2$, $\deg_G(v_4) = 3$, and $\deg_G(v_5) = 2$.

Theorem 1. **[26]** *(Dirac's Theorem) If G is a graph with order $k \geq 3$ and the degree of each vertex v is at least $k/2$, then G is a Hamiltonian graph.*

Definition 8. *A circuit C in a connected graph G that traverses every edge exactly once is called a Eulerian circuit.*

Definition 9. *A connected graph G that contains a Eulerian circuit is known as a Eulerian graph.*

Theorem 2. **[26]** *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

2.2 The Triple Idempotent Graph of Commutative Ring

Here is the definition of the triple idempotent graph along with several properties obtained by Kurniawan et al. **[25]**.

Definition 10. *Let R be a commutative ring, and let $I(R)$ represent the set of idempotent elements within R. The triple idempotent graph of the commutative ring R, denoted as $TI(R)$, is a graph whose the set of vertices $R^* = R - \{0,1\}$. Two distinct vertices u and v are connected by an edge if there exist an element $w \in R - \{0,1\}$ where $w \neq u$ and $w \neq v$ such that $uv \notin I(R), uw \notin I(R), vw \notin I(R)$, and $uvw \in I(R)$.*

Example 1. Here is an example of $TI(\mathbb{Z}_7)$ formed by ring $R = \mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$. Based on the definition of the triple idempotent graph of the commutative ring R given in **Definition 10**, the vertex set of $TI(\mathbb{Z}_7)$ is determined to be $\{2,3,4,5,6\}$. According to the adjacency criteria in $TI(R)$, when $u = 2, v = 3$, we find an element $w = \bar{6}$ such that $uv \neq 1, vw \neq 1, uw \neq 1$, and $uvw = 1$. Therefore, an edge exists

between 2 and 3. In the same way, the set of all edges in $TI(\mathbb{Z}_7)$ is obtained as $E(TI(\mathbb{Z}_7)) = \{(2,3),(3,6),(2,6),(4,6),(4,5),(5,6)\}$. The $TI(\mathbb{Z}_7)$ is shown in **Figure 2**.

Lemma 1. **[25]** *Given ring $\mathbb{Z}_n$ where n is prime and $n \geq 7$. For every $u, v \in V(TI(\mathbb{Z}_n))$, u is not adjacent to v if $uv = 1$ or $uv = x \neq 1$ where $x = u^{-1}$ or $x = v^{-1}$.*

Theorem 3. **[25]** *Let $\mathbb{Z}_n$ be a ring of integers modulo n. If n prime and $n \geq 7$, then $TI(\mathbb{Z}_n)$ is a connected graph.*

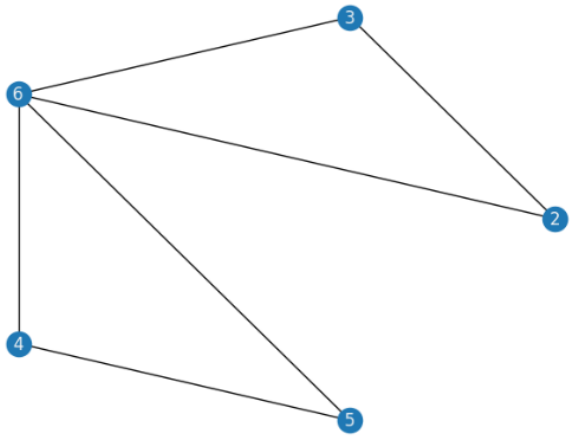

Figure 2. Triple Idempotent Graph of $\mathbb{Z}_7$

3. RESULTS AND DISCUSSION

The first theorem indicates that if the triple idempotent graph of the ring of integers modulo n is connected, then its girth is 3.

Lemma 2. *If p is a prime, then 0 and 1 are the only two idempotent elements of $\mathbb{Z}_p$.*

Proof. Suppose that there exists another idempotent element call it a such that $a^2 = a$. Let $x \equiv a \pmod p$ such that $mp = x - a$ for some $m \in \mathbb{Z}$. On the other hand, $x \equiv a^2 \pmod p$ such that $np = x - a^2$ for some $n \in \mathbb{Z}$. It follows that $mp - np = (m-n)p = a^2 - a$ such that $p|(a^2 - a)$ or $p|a(a-1)$. By Euclid's lemma, we get $p|a$ and $p|(a-1)$, in other words $a = 0 \pmod p$ or $a = 1 \pmod p$. Thus 0 or 1 are the only two idempotent elements in $\mathbb{Z}_p$.■

Lemma 3. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is a prime and $n \geq 7$, then there exists a vertex $u \in V(TI(\mathbb{Z}_n))$ with $u = n - 1$ that is adjacent to any vertex $v \in V(TI(\mathbb{Z}_n))$.*

Proof. Given $\mathbb{Z}_n$ with n is a prime and $n \geq 7$. Take any vertex $v \in V(TI(\mathbb{Z}_n))$, we will show that v is adjacent to the vertex $u = n - 1$. First, it is shown that the inverse of $u = n - 1$ is itself. Consider $u.u = (n-1)(n-1) = n^2 - 2n + 1 = 1$. With **Lemma 2**, it is clear that the idempotents of $\mathbb{Z}_p$ are only 0 and 1. Since $u \neq v$ then $uv \neq 1$ so $uv \notin I(\mathbb{Z}_n)$. Since every non-zero element in $\mathbb{Z}_n$ has an inverse, we can find another vertex $w = (uv)^{-1}$ such that $uvw = uv(uv)^{-1} = 1$. Since $uv \neq 1$ and there exists w such that $uvw = 1 \in I(\mathbb{Z}_n)$, then u and v are adjacent. Thus, it is proven vertex $u = n - 1$ is adjacent to any vertex $v \in V(TI(\mathbb{Z}_n))$. ■

Theorem 4. *If $TI(\mathbb{Z}_n)$ is a connected graph, then $diam(TI(\mathbb{Z}_n)) = 2$.*

Proof. Based on **Lemma 3**, we know that the vertex $u = n - 1$ is connected to any vertex $v \in V(TI(\mathbb{Z}_n))$. Thus, the distance between any two vertices $a, b \in TI(\mathbb{Z}_n)$ only has two possibilities, either 1 or 2. If a and b are adjacent then $d(a,b) = 1$. If a and b are not adjacent then we have a path $a - u - b$ such that the distance between a and b is 2. Thus, the largest distance between any two distinct vertices in $TI(\mathbb{Z}_n)$ is 2, so $diam(TI(\mathbb{Z}_n)) = 2$. ■

Theorem 5. *If $TI(\mathbb{Z}_n)$ is a connected graph, then $gr(TI(\mathbb{Z}_n)) = 3$.*

Proof. Because $TI(\mathbb{Z}_n)$ is a connected graph, there must exist at least two vertices a and b that are mutually adjacent. Therefore, there exists a vertex $c \in V(TI(\mathbb{Z}_n))$ such that $ab \neq e$, $ac \neq e$, $bc \neq e$, and $abc = e$, for $e \in I(\mathbb{Z}_n)$. Furthermore, it is clear that a is adjacent to b and c, b is adjacent to a and c, and c is adjacent to a and b. Thus, a cycle $a - b - c - a$ is obtained, implying that the girth of the graph is equal to 3, i.e., $gr(TI(\mathbb{Z}_n)) = 3$. ■

To illustrate **Theorem 5**, here is an example regarding the girth $TI(\mathbb{Z}_7)$. In **Figure 2**, it can be seen that the shortest cycle in the graph $TI(\mathbb{Z}_7)$ has at least 3 vertices, one of which is the cycle $2 - 3 - 6 - 2$.

The following lemmas address the adjacency between vertices in $TI(\mathbb{Z}_n)$, and will be utilized in proving the theorems regarding the vertex degree and the minimum degree in $TI(\mathbb{Z}_n)$.

Lemma 4. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is a prime and $n \geq 11$, then the vertex 4 is always non-adjacent to the vertex 4^{-1} and $4 \neq 4^{-1}$.*

Proof. According to **Lemma 1**, the vertex 4 is not adjacent to 4^{-1}. Next, assume for contradiction $4 = 4^{-1}$, then $4.4 = 4(4^{-1}) = 1$. This implies that $16 = kp + 1$ or $15 = kp$ where $k \in \mathbb{Z}$ and p is a prime number greater than or equal to 11. However, the only prime divisors of 15 are 3 and 5, and no such prime $p \geq 11$ divides 15. Thus, $4^2 \neq 1$ and $4 \neq 4^{-1}$. ■

Lemma 5. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is a prime and $n \geq 11$, then the vertex 4 is not adjacent to the vertex $(4^2)^{-1}$. Furthermore, $(4^2)^{-1} \neq 4$ and $(4^2)^{-1} \neq 4^{-1}$.*

Proof. Note that $4.(4^2)^{-1} = 4.4^{-2} = 4^{-1}$. According to **Lemma 1**, 4 and $(4^2)^{-1}$ are not adjacent.

Now suppose $(4^2)^{-1} = 4$, then $4^3 = 1$. This implies that $64 = kp + 1$ or $63 = kp$. But 63 has prime factors $\{3,7\}$ and none of them satisfy $p \geq 11$, a contradiction. Therefore, $4^3 \neq 1$ and $(4^2)^{-1} \neq 4$. Next, suppose $(4^2)^{-1} = 4^{-1}$. By the uniqueness of inverses, it must be that $4^2 = 4$, which implies 4 is an idempotent element. But the only idempotent elements in $\mathbb{Z}_P$ with p prime are 0 and 1. Thus, $(4^2)^{-1} \neq 4^{-1}$. ■

Lemma 6. *Given $\mathbb{Z}_n$ with n prime and $n \geq 11$, the vertex 4 is not adjacent to the vertices 2^{-1} and $(n-2)^{-1}$. Furthermore, $2^{-1} \neq (n-2)^{-1}$., and neither of them is equal to the vertices $4, 4^{-1}$, or $(4^2)^{-1}$.*

Proof. In $\mathbb{Z}_n$ with n prime and $n \geq 11$, the element 4 always has two square roots, namely 2 and $(n-2)$. The vertices 2^{-1} and $(n-2)^{-1}$ are the inverses of these two vertices, respectively. We will show that 4 is not adjacent to the vertex 2^{-1}. Since $2.2 = 4$, we have $2^{-1}.4 = 2^{-1}.2.2 = 2$. According to **Lemma 1**, 4 is not adjacent to 2^{-1}. Similarly, 4 is also not adjacent to $(n-2)^{-1}$. Next, we will show that the vertices 2^{-1} and $(n-2)^{-1}$ are distinct. Suppose $(n-2)^{-1} = 2^{-1}$, then by the uniqueness of inverses, $(n-2) = 2$. In $\mathbb{Z}_n$, this holds only for $n = 4$. Thus, $2^{-1} = (n-2)^{-1}$ in $\mathbb{Z}_n$ with n prime and $n \geq 11$. Next, it will be shown that neither 2^{-1} nor $(n-2)^{-1}$ is equal to the vertices $4, 4^{-1}$, or $(4^2)^{-1}$. Suppose that $2^{-1} = 4$ then $1 = 2^3$. This implies $kp + 1 = 8$ or $kp = 7$. Since the only prime factor that satisfies this equation is $p = 7$, this contradicts $p \geq 11$. Thus, $2^{-1} \neq 4$. Next, suppose that $2^{-1} = 4^{-1}$, then $2 = 4$. In $\mathbb{Z}_n$, this holds only in $\mathbb{Z}_2$. Therefore, $2^{-1} \neq 4^{-1}$. Next, suppose $2^{-1} = (4^2)^{-1}$, then $2 = 4^2$. This implies $kp + 2 = 16$ 0r $kp = 14$. Since the prime factors that satisfy this equation are 2 and 7, this contradicts $p \geq 11$. Thus, $2^{-1} \neq (4^2)^{-1}$. Suppose that $(n-2)^{-1} = 4$, then $1 = (n-2)^3$. This implies $-8(mod\ n) = 1$. In $\mathbb{Z}_n$, this holds only for $\mathbb{Z}_3$ and $\mathbb{Z}_9$. Thus, it is proven that$(n-2)^{-1} \neq 4$ in $\mathbb{Z}_n$ with n prime $-8(\text{mod } n) = 1$ and $n \geq 11$. Next, suppose $(n-2)^{-1} = 4^{-1}$, then $(n-2) = 4$. In $\mathbb{Z}_n$, this holds only for $\mathbb{Z}_3$ and $\mathbb{Z}_6$. Thus, $(n-2)^{-1} \neq 4^{-1}$. Next, suppose $(n-2)^{-1} = (4^2)^{-1}$, then $(n-2).4 = 1$. By the uniqueness of inverses, $(n-2)^{-1} = 4$. This is a contradiction because it has already been proven that $(n-2)^{-1} \neq 4$. Thus, $(n-2)^{-1} \neq (4^2)^{-1}$. ■

Several of these lemmas are subsequently used to calculate the degree of vertex 4 in the graph $TI(\mathbb{Z}_n)$.

Lemma 7. *Let $\mathbb{Z}_n$ be the ring of integers modulo n. If n is a prime and $n \geq 11$, then $deg(4) = n - 7$.*

Proof. Consider the vertex $4 \in V(TI(\mathbb{Z}_n))$. We will show that the degree of vertex 4 in $TI(\mathbb{Z}_n)$ is $n - 7$. Since $|V(TI(\mathbb{Z}_n))| = n - 2$, we must show that $\deg(4) = |V(TI(\mathbb{Z}_n))| - 5$. We will demonstrate that there are 5 vertices in $TI(\mathbb{Z}_n)$ that are not adjacent to vertex 4. The first vertex is 4 itself, as $TI(\mathbb{Z}_n)$ is a simple graph, so a vertex is not adjacent to itself. Next, according to **Lemma 4**, **Lemma 5**, and **Lemma 6**, there are 4 other vertices, namely $\{4^{-1}, (4^2)^{-1}, 2^{-1}, (n-2)^{-1}\}$. Next, consider any vertex $u \in V(TI(\mathbb{Z}_n))$ with n prime and $n \geq 11$. We will show that aside from the 5 vertices mentioned above, u is always adjacent to vertex 4. By **Lemma 3**, there exists a w such that w can be expressed as $w = 4u$. Since $\mathbb{Z}_n$ with n prime is a

field, there exists a w^{-1} such that $4uw^{-1} = ww^{-1} = 1$. Hence, it follows that 4 is adjacent to both u and w^{-1}. Next, suppose u is not adjacent to 4. Then, u must be one of the 5 vertices mentioned earlier. By **Lemma 1**, there are two cases where u is not adjacent to 4. First, if $u.4 = 1$, by the uniqueness of inverses, u must be 4^{-1}. This implies $u = v_2$. Second, if $u.4 = u^{-1}$or if $u.4 = 4^{-1}$, assuming if $u.4 = u^{-1}$, then if $4 = (u^{-1})^2$. This implies $u = v_4$ or $u = v_5$. Next, assuming $u.4 = 4^{-1}$, then $u.4^2 = 1$. By the uniqueness of inverses, u must be $(4^2)^{-1}$. This implies $u = v_3$. Thus, if u is not adjacent to 4, it must be one of the 5 vertices mentioned earlier. Therefore, it is proven that for $\mathbb{Z}_n$ with n prime and $n \geq 11$, vertex 4 has a degree of $n - 7$. ■

Lemma 8. *Let $\mathbb{Z}_n$ be the ring of integers modulo n where n is a prime and $n \geq 3$. If SQ is the set of elements that have a square root, then $|SQ| = \frac{n-1}{2}$.*

Proof. Let $\mathbb{Z}_n^*$ be the set of nonzero elements of $\mathbb{Z}_n$, i.e., $\mathbb{Z}_n^* = \{1,2, \dots, n-2, n-1\}$. Since n is a prime, $\mathbb{Z}_n^*$ form a multiplicative group of order $n - 1$, which is always even for $n \geq 3$. An element $a \in \mathbb{Z}_n^*$ is said to have a square root if there exists some $x \in \mathbb{Z}_n$ such that $x^2 = a \pmod n$. These elements are called quadratic residues modulo n. Now, in $\mathbb{Z}_n^*$, each nonzero square x^2 has exactly two distinct square roots: x and $-x$, and $x^2 = (-x)^2 \pmod n$, but $x = -x \pmod n$ unless $x = 0$, which is not in $\mathbb{Z}_n^*$. Thus, all quadratic residues come in distinct pairs. Therefore, the number of distinct elements in $\mathbb{Z}_n^*$ that are quadratic residues is $|SQ| = \frac{|\mathbb{Z}_n^*|}{2} = \frac{n-1}{2}$. ■

Lemma 9. *Let $\mathbb{Z}_n$ be the ring of integers of modulo n. If n is a prime and $n \geq 11$, then $deg(y) \geq deg(4)$ for any vertex $y \in V(TI(\mathbb{Z}_n))$.*

Proof. Consider any $y \in V(TI(\mathbb{Z}_n))$. We will show that $\deg(y) \geq \deg(4)$. According to **Lemma 8**, not all vertices have square roots. The number of vertices with 1 square root is $\frac{n-2}{2}$. For vertices y that do have square roots, we will show that the degree of y will always be equal to or greater than the degree of vertex 4. According to **Lemma 1**, there are two cases for any vertex u not to be adjacent to y. First, if $u.y = 1$. By the uniqueness of inverses, u must be y^{-1}. Second, if $u.y = u^{-1}$ or $u.y = y^{-1}$. Suppose $u.y = u^{-1}$, then $y = (u^{-1})^2$. Next, suppose $u.y = y^{-1}$, then $u.y^2 = 1$. By the uniqueness of inverses, u must be $(y^2)^{-1}$. So, at most 3 vertices that are not adjacent to y. Therefore, it is proven that for any vertex $y \in V(TI(\mathbb{Z}_n))$ with n prime and $n \geq 11$, we have $\deg(y) \geq \deg(4)$. ■

Below is a theorem regarding one of the characteristics of the triple idempotent graph of the ring $\mathbb{Z}_n$ where n is a prime and $n \geq 11$, resulting the minimum degree in the graph $TI(\mathbb{Z}_n)$ is $n - 7$.

Theorem 6. *Given the ring $\mathbb{Z}_n$. If n is a prime and $n \geq 11$, then $\delta(TI(\mathbb{Z}_n)) = n - 7$.*

Proof. By **Lemma 9,** it is concluded that 4 is the vertex with the minimum degree. Furthermore, by **Lemma 7**, we have $\deg(4) = n - 7$. Therefore, the minimum degree in $TI(\mathbb{Z}_n)$ where n is a prime and $n \geq 11$ is $n - 7$. ■

Consider the following example to illustrate **Theorem 6** on the minimal degree of $TI(\mathbb{Z}_{13})$

Example 2. Given ring $\mathbb{Z}_{13}$, we have $V(TI(\mathbb{Z}_{13})) = \{2,3,4, \dots ,12\}$ and the set of idempotent elements $I(\mathbb{Z}_{13}) = \{0,1\}$. We aim to show that 4 has a degree of $n - 7$. Since $|V(TI(\mathbb{Z}_n))| = n - 2$, it is equivalent to $\deg(4) = |V(TI(\mathbb{Z}_{13}))| - 5 = 11 - 5 = 6$. We identify five vertices that are not adjacent to 4, namely $\{4,6,7,9,10\}$. This means that every vertex $u \in V(TI(\mathbb{Z}_n))\backslash\{4,6,7,9,10\}$ is adjacent to 4. For example, take $u = 2$. Then $4.2 = 8$ and there exists $8^{-1} = 5$ such that $4.2.5 = 1$. Thus, 4 is adjacent to both 2 and 5. Similarly, 4 adjacent to 3,5,8,11,12. Therefore, the neighbors of 4 are $\{2,3,5,8,11,12\}$. Hence, the degree of vertex 4 is 6, verifying that $\deg(4) = |V(TI(\mathbb{Z}_{13}))| - 5 = 6 = n - 7$. For all other vertices (i.e., vertices other than 4), their degrees are at least 6, as illustrated in **Figure 3**.

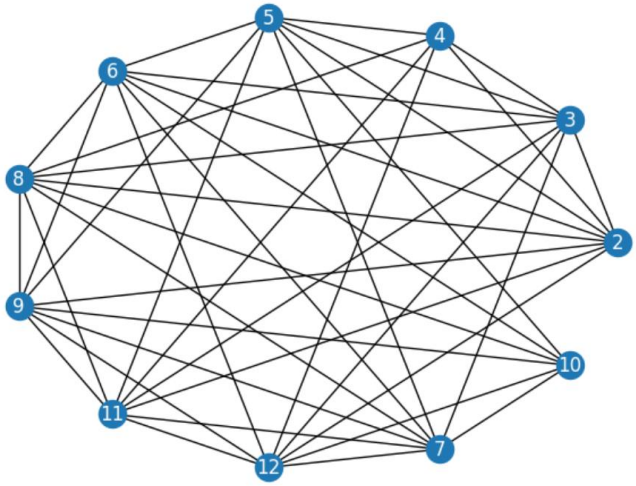

Figure 3. Triple Idempotent Graph of $\mathbb{Z}_{13}$

The following theorem demonstrates the sufficient condition for $TI(\mathbb{Z}_n)$ to be a Hamiltonian graph.

Theorem 7. *Let $\mathbb{Z}_n$ be the ring of integer of modulo n. If n is prime and $n \geq 13$, then $TI(\mathbb{Z}_n)$ is a Hamiltonian graph.*

Proof. Based on **Theorem 1**, a simple graph G with $k \geq 3$ vertices is Hamiltonian if every vertex has degree at least $k/2$. From **Theorem 6**, the minimum degree of any vertex in $TI(\mathbb{Z}_n)$ with n prime and $n \geq 13$ is $n-7$. Because the number of vertices in $TI(\mathbb{Z}_n)$ is $k = n-2$, the minimum degree becomes $k-5$. The condition $k-5 \geq \frac{k}{2}$ holds when $k \geq 10$. For $n \geq 13$ we have $k = n-2 \geq 11$, so the condition is satisfied. Therefore, the graph $TI(\mathbb{Z}_n)$ with n prime and $n \geq 13$ meets the sufficient condition to be Hamiltonian. Thus, $TI(\mathbb{Z}_n)$ is a Hamiltonian graph for all prime $n \geq 13$. ■

Example 3. Given the graph $TI(\mathbb{Z}_{13})$ with its graph shown in **Figure 3**. In $TI(\mathbb{Z}_{13})$, the number of vertices is $|V(TI(\mathbb{Z}_{13}))| = 11$, and the minimum degree in $TI(\mathbb{Z}_{13})$ is $11-5=6$. Furthermore, based on **Theorem 1**, a sufficient condition for a simple graph G with at least $k \geq 3$ vertices to be a Hamiltonian graph is if the degree of each vertex is at least $k/2$ for every vertex in G. We obtain $k/2$ in $TI(\mathbb{Z}_{13})$ to be $11/2 < 6$, then $TI(\mathbb{Z}_{13})$ satisfies the sufficient condition and $TI(\mathbb{Z}_{13})$ is a Hamiltonian graph. Furthermore, a Hamiltonian cycle in $TI(\mathbb{Z}_{13})$ is shown to be $2-3-4-5-6-7-8-9-10-12-11-2$.

The following Lemma shows that the degree of each vertex of $TI(\mathbb{Z}_n)$ with n prime and $n \geq 7$ is even.

Lemma 10. *Let $\mathbb{Z}_n$ be the ring of integer of modulo n. If n is a prime and $n \geq 7$, then every vertex in the graph $TI(\mathbb{Z}_n)$ has even degree.*

Proof. Since n is prime, the only idempotent element of $\mathbb{Z}_n$ are 0 and 1. Therefore, the adjacency condition can be reduced to $ab \neq 1, bc \neq 1, ac \neq 1$ and $abc = 1$ for $a, b, c \in V(TI(\mathbb{Z}_n))$. Let $u \in V(TI(\mathbb{Z}_n))$. If u is adjacent to the vertex v, then there exists $w \in V(TI(\mathbb{Z}_n))$ such that $uvw = 1$, which implies $w = (uv)^{-1}$. By the definition of the graph, since multiplication is commutative and associative, w is also adjacent to u and v. Note that $\mathbb{Z}_n$ is a field (as n is prime). By the uniqueness of inverses in a field, for any pair u and v such that $uvw = 1$, the element w is uniquely determined. Hence, each such triplet $\{u, v, w\}$ forms a triangle, contributing degree 2 to each involved vertex. This means that adjacency always occurs in pairs, each vertex u is adjacent to vertices in a way that its total degree is $2k$ where k is the number of such adjacency combinations. Hence, $\deg(u)$ is even for all u. Therefore, in the graph $TI(\mathbb{Z}_n)$ with n prime and $n \geq 7$, the degree of every vertex is even. ■

The following theorem shows that $TI(\mathbb{Z}_n)$ with n prime and $n \geq 7$ is a Eulerian graph.

Theorem 7. *Let $\mathbb{Z}_n$ be the ring of integer of modulo n. If n is a prime and $n \geq 7$, then $TI(\mathbb{Z}_n)$ is a Eulerian graph.*

Proof. Based on **Theorem 2**, a graph is Eulerian if it is connected and all vertices have even degree. By **Lemma 10**, every vertex in the graph $TI(\mathbb{Z}_n)$ has even degree when n is a prime and $n \geq 7$. Therefore, $TI(\mathbb{Z}_n)$ is an Eulerian graph for all prime $n \geq 7$. ■

The following example is provided to help better understand **Lemma 10** and **Theorem 7**.

Example 4. Given the ring $TI(\mathbb{Z}_{11})$. The visualization of $TI(\mathbb{Z}_{11})$ can be seen in **Figure 4**. It can be seen that vertices 3, 4, 5, and 9 have a degree of 4, vertices 2, 6, 7, and 8 have a degree of 6, and vertex 10 has a degree of 8. Furthermore, based on **Theorem 2**, $TI(\mathbb{Z}_{11})$ is an Eulerian graph. The following is an Eulerian circuit found in $TI(\mathbb{Z}_{11})$: $2-5-7-10-9-8-6-3-0-5-6-9-2-8-3-7-2-10-6-7-4-8-10-4-2$.

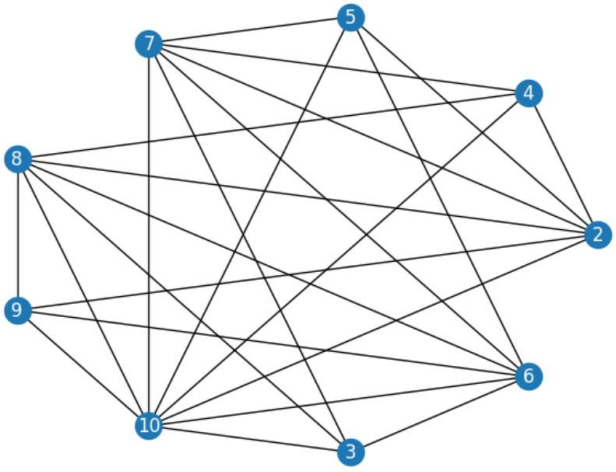

Figure 4. Triple Idempotent Graph of $\mathbb{Z}_{11}$

4. CONCLUSIONS

Based on the research conducted, it can be concluded that the triple idempotent graph $TI(\mathbb{Z}_n)$ exhibits several interesting structural properties. If the graph $TI(\mathbb{Z}_n)$ is connected, it has a diameter of 2 and a girth of 3, indicating that any two vertices are at most two steps apart and that the smallest cycle in the graph has length three. These parameters highlight the graph's strong connectivity and cyclic structure.

Furthermore, it was shown that $TI(\mathbb{Z}_n)$ is a Eulerian graph for $n \geq 7$, implying that there exists a closed trail that visits every edge exactly once under this condition. In addition, for $n \geq 7$, the graph is Hamiltonian, meaning it contains a cycle that passes through every vertex exactly once. These findings emphasize the rich and intricate nature of the graph as n increases.

Overall, the results contribute to a deeper understanding of the interplay between algebraic structures in modular rings and the combinatorial properties of their associated graphs. This study opens up further questions related to chromatic number, planarity, and spectral properties of $TI(\mathbb{Z}_n)$, which may be addressed in future research.

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REFERENCES

[1] I. Beck, "COLORING OF COMMUTATIVE RINGS," *J. Algebr.*, vol. 116, no. 1, pp. 208–226, 1988, doi: https://doi.org/10.1016/0021-8693(88)90202-5

[2] D. F. Anderson and P. S. Livingston, "THE ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING," *J. Algebr.*, vol. 217, no. 2, pp. 434–447, 1999, doi: https://doi.org/10.1006/jabr.1998.7840.

[3] R. Akhtar and L. Lee, "CONNECTIVITY OF THE ZERO-DIVISOR GRAPH FOR FINITE RINGS," *Involv. (Journal Math.)*, vol. 9, no. 3, pp. 415–422, 2016.doi: https://doi.org/10.2140/involve.2016.9.415

[4] D. Nongsiang and P. K. Saikia, "ON THE NON-NILPOTENT GRAPHS OF A GROUP," *Int. Electron. J. Algebr.*, vol. 22,

no. 09, pp. 78–96, 2017, doi: https://doi.org/10.24330/ieja.325927.
[5] F. Mahmudi and M. Soleimani, "SOME PROPERTIES OF THE MAXIMAL GRAPH OF A COMMUTATIVE RING," *Southeast Asian Bull. Math.*, vol. 43, pp. 525–536, 2019.
[6] R. Yudatama, V. Y. Kurniawan, and S. B. Wiyono, "ANNIHILATOR GRAPH OF SEMIRING OF MATRICES OVER BOOLEAN SEMIRING," *J. Phys. Conf. Ser.*, vol. 1494, no. 1, 2020, doi: https://doi.org/10.1088/1742-6596/1494/1/012009.
[7] K. F. Pawar and S. S. Joshi, "STUDY OF PRIME GRAPH OF A RING," *Thai J. Math.*, vol. 17, no. 2, pp. 369–377, 2019.
[8] M. Afkhami and M. Hassankhani, "THE CAYLEY SUM GRAPH OF IDEALS OF A LATTICE," *Gen. Algebr. Appl.*, vol. 40, pp. 217–230, 2020, doi: https://doi.org/10.7151/dmgaa.1332.
[9] I. N. Fitriani and V. Y. Kurniawan, "ALGORITHM FOR CONSTRUCTING THE TRIPLE UNIT GRAPH OF TYPE II OF RING Z_n USING PYTHON," *BAREKENG J. Ilmu Mat. dan Terap.*, vol. 18, no. 3, pp. 1639–1648, 2024, doi: https://doi.org/10.30598/barekengvol18iss3pp1639-1648.
[10] V. Y. Kurniawan and C. F. Ekasiwi, "ALGORITHM FOR CONSTRUCTING TRIPLE IDENTITY GRAPH OF RING Z_n USING PYTHON," *BAREKENG J. Ilmu Mat. dan Terap.*, vol. 18, no. 3, pp. 1629–1638, 2024, doi: https://doi.org/10.30598/barekengvol18iss3pp1629-1638.
[11] I. Dolinka *et al.*, "ENUMERATION OF IDEMPOTENTS IN DIAGRAM SEMIGROUPS AND ALGEBRAS," *J. Comb. Theory. Ser. A*, vol. 131, pp. 119–152, 2015, doi: https://doi.org/10.1016/j.jcta.2014.11.008.
[12] V. G. Bardakov, I. B. S. Passi, and M. Singh, "ZERO-DIVISORS AND IDEMPOTENTS IN QUANDLE RINGS," *Osaka J. Math.*, vol. 59, no. 3, pp. 611–637, 2022.
[13] E. S. Almotairi, M. I. Bhat, and A. M. Alghamdi, "GROUP ACTION ON THE SET OF NONUNITS IN RINGS," *J. Math.*, vol. 2023, no. ii, pp. 0–3, 2023, doi: https://doi.org/10.1155/2023/9711759.
[14] H. Su and Y. Wei, "THE DIAMETER OF UNIT GRAPHS OF RINGS," *Taiwan. J. Math.*, vol. 23, no. 1, pp. 1–10, 2019, doi: https://doi.org/10.11650/tjm/180602.
[15] A. Sharma and D. K. Basnet, "NIL CLEAN DIVISOR GRAPH," pp. 1–9, 2019, [Online]. Available: http://arxiv.org/abs/1903.02287
[16] B. A. Rather, S. Pirzada, T. A. Naikoo, and Y. Shang, "ON LAPLACIAN EIGENVALUES OF THE ZERO-DIVISOR GRAPH ASSOCIATED TO THE RING OF INTEGERS MODULO N," *Mathematics*, vol. 9, no. 5, pp. 1–17, 2021, doi: https://doi.org/10.3390/math9050482.
[17] J. East and R. D. Gray, "DIAGRAM MONOIDS AND GRAHAM–HOUGHTON GRAPHS: IDEMPOTENTS AND GENERATING SETS OF IDEALS," *J. Comb. Theory. Ser. A*, vol. 146, pp. 63–128, 2017, doi: https://doi.org/10.1016/j.jcta.2016.09.001.
[18] G. L. Litvinov, V. P. Maslov, A. Y. Rodionov, and A. N. Sobolevski, "UNIVERSAL ALGORITHMS, MATHEMATICS OF SEMIRINGS AND PARALLEL COMPUTATIONS," *Lect. Notes Comput. Sci. Eng.*, vol. 75 LNCSE, pp. 63–89, 2011, doi: https://doi.org/10.1007/978-3-642-14941-2_4.
[19] S. Pirzada and M. Aijaz, "METRIC AND UPPER DIMENSION OF ZERO DIVISOR GRAPHS ASSOCIATED TO COMMUTATIVE RINGS," *Acta Univ. Sapientiae, Inform.*, vol. 12, no. 1, pp. 84–101, 2020, doi: https://doi.org/10.2478/ausi-2020-0006.
[20] A. Li and Q. Li, "A KIND OF GRAPH STRUCTURE ON NON-REDUCED RINGS," *Algebr. Colloq.*, vol. 17, no. 1, pp. 173–180, 2010, doi: https://doi.org/10.1142/S1005386710000180.
[21] M. Schaps, "THE COARSE STRUCTURE OF THE REPRESENTATION ALGEBRA OF A FINITE MONOID," *J. Discret. Math.*, vol. 2014, pp. 1–7, 2014, doi: https://doi.org/10.1155/2014/529804.
[22] S. Chattopadhyay, K. L. Patra, and B. K. Sahoo, "LAPLACIAN EIGENVALUES OF THE ZERO DIVISOR GRAPH OF THE RING ZN," *Linear Algebra Appl.*, vol. 584, pp. 267–286, 2020, doi: https://doi.org/10.1016/j.laa.2019.08.015.
[23] C. Maldonado and D. Penazzi, "Lattices and {N}orton Algebras of {J}ohnson, {G}rassmann and {H}amming Graphs," no. 1204.1947, pp. 1–16, 2012, [Online]. Available: http://arxiv.org/abs/1204.1947
[24] H. Q. Mohammad and N. H. Shuker, "Idempotent Divisor Graph of Commutative Ring," *Iraqi J. Sci.*, vol. 63, no. 2, pp. 645–651, 2022, doi: 10.24996/ijs.2022.63.2.21.
[25] V. Y. Kurniawan, B. Purboutomo, and N. A. Kurdhi, "Connectivity of The Triple Idempotent Graph of Ring Zn," *Int. J. Comput. Sci. Appl. Math.*, vol. 10, no. 1, pp. 34–37, 2024, doi: http://dx.doi.org/10.12962/j24775401.v10i1.20266.
[26] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 6th ed. New York: CRC Pers Taylor and Francis Group, 2016. doi: 10.1201/b19731.