BAREKENG: Journal of Mathematics and Its Applications
June 2025 Volume 19 Issue 2 Page 1445-1456
P-ISSN: 1978-7227 E-ISSN: 2615-3017
https://doi.org/10.30598/barekengvol19iss2pp1445-1456

PMC-LABELING OF SOME CLASSES OF GRAPHS CONTAINING CYCLES

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Article History:

Received: 11th November 2024
Revised: 8th February 2025
Accepted: 8th March 2025
Published: 1st April 2025

Keywords:

Circulant Graph;
Closed Web Graph;
Djembe Graph;
Ice Cream Graph;
Origami Graph;
Pentagonal Circular Ladder;
Quadrilateral Friendship Graph;
Zig-Zag Chord Graph.

ABSTRACT

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be a graph with p vertices and q edges. We have introduced a new graph labeling method using integers and cordial-related works and investigated some graphs for this labeling technique. Using this labeling concept, we have examined the graphs like path, cycle, star, complete graph, comb, and wheel graph. The first research paper on graph theory was published by Leonhard Euler. However, he did not use the word 'graph' in his work. In the early stages of the development of the subject, the vertices of a graph were specified as $\boldsymbol{v_1},, \boldsymbol{v_2}, \dots\dots$, and the edges were denoted by,$\boldsymbol{e_1}, \boldsymbol{e_2}, \dots.$ In recent times, several researchers have attempted to provide different types of labeling to the vertices and edges of a graph by identifying the relevant mathematical properties. The present paper provides a novel method of labeling by employing integers, which may form a foundation for future research work. In this paper, we investigate the pair mean cordial labeling behavior of some cycle-related graphs like the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph.

How to cite this article:
R. Ponraj, S. Prabhu and M.Sivakumar, "PMC-LABELING OF SOME CLASSES OF GRAPHS CONTAINING CYCLES",*BAREKENG: J. Math. & App.*, vol. 19, iss. 2, pp. 1445-1456, June, 2025.

1. INTRODUCTION

In this paper, we consider only finite, simple, and undirected graphs. Let $G = (V, E)$ be a graph with p vertices and q edges. The number $|V(G)|$ and $|E(G)|$, respectively are denote by the order of G and the size of G. A lot of researchers are currently interested in labeled graphs because of their wide range of applications. Graph labeling is a strong relation between numbers and the structure of graphs. Graph labeling is one of the important branches of graph theory in which the vertices or edges or both are assigned integral values with certain conditions. Graph labeling techniques are helpful in many fields, including coding theory, database administration, X-ray crystallography, radar, astronomy, circuit design, and communication networks. Graph labeling was first introduced by Rosa in 1967 [1] in the name of graceful labeling.

We follow Harary [2] for all other terminology and notations in graph theory. Gallian updates a dynamic survey on graph labeling on a regular basis [3]. I. Cahit first introduced the concept of cordial labeling in 1987 [4] as a weaker version of graceful and harmonious graphs. The harmonious, odd harmonious, and even harmonious labeling was discussed in [5]. N. Inayah et al. have examined the total product and total edge product cordial labeling of dragonfly graph in [6]. Total product cordial labeling of some graphs has been investigated in [7]. M. Prajapati and N. M. Patel have investigated the edge product cordial labeling of some cycle-related graphs [8]. N. B. Rathod and K. K. Kanani [9] proved that the triangular belt, Alternate Triangular belt, braid graph and $Z - P_n$ are k-cordial graphs. Edge irregular reflexive labeling on alternate triangular snake and double alternate quadrilateral snake were examined in [10].

P. Sumathi, and S. Kavitha [11] have computed the quotient 4-cordial labeling of some ladder graphs. On product cordial graphs have been investigated in [12]. A study on combination of shell graph and path P_2 graph was examined in [13]. U. M. Prajapati and S. J. Gajjar [8] have proved that the generalized prism graph $Y_{m,n}$ is a prime cordial graph. S. N. Daoud and N. Elsawy [14] studied the edge of even graceful labeling of a new family of graphs. V. Sharon Philomena et al. [15] have proved that the extended kusudama flower graph is a square difference graph and a cube difference graph.

We have introduced the new graph labeling method called pair mean cordial labeling (PMC-labeling) in [16] and PMC-labeling is defined as follows: Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a graph with p vertices and q edges. Define $\rho = \begin{cases} \frac{p}{2} & \text{p is even} \\ \frac{p-1}{2} & \text{p is odd,} \end{cases}$ and $\mathrm{M} = \{\pm 1, \pm 2, \dots, \pm \rho\}$. Consider a mapping $\lambda: \mathrm{V} \to \mathrm{M}$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p - 1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there is a pair mean cordial labeling (PMC-labeling) is called a pair mean cordial graph (PMC-graph). The PMC-labeling is the most interesting category of graph labeling with various applications. We have investigated the pair mean cordiality of some special graphs in [17], [18], [19]. In this paper, we investigate the pair mean cordial labeling of some graphs like the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph.

2. RESEARCH METHODS

Assign the label to the vertices of the graph $\boldsymbol{G}$ by the integers and define some conditions on the edge label, which depends on the vertex labels. Also, define vertex label conditions and edge label conditions. We investigate some cycle-related graphs for the specified conditions. The labelling-related papers, journals, books, surveys, proceedings, and articles are the resources of information employed in this research study. The literature reviews related to PMC graphs of various families of graphs are used to present this research paper. The PMC-labelling of characteristics of some graphs, such as the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph, have been investigated using the study approach.

3. RESULTS AND DISCUSSION

This study computes the PMC-labeling of the behavior of some special cycle-related graphs, including the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph.

3.1 Preliminaries

In this section, we give the basic definition relevant to the upcoming section of this paper.

Definition 1. [13] *A shell graph is the graph obtained from a cycle $C_n$with $n-3$ chords sharing a common end point called the apex.*

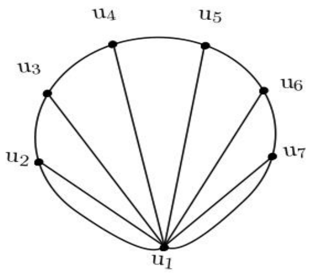

Figure 1. **Shell Graph S_7**

Definition 2. [13] *An ice cream graph I_n is the graph obtained by combing a shell graph and P_2 keeping v_1 and v_{n-1} common where $n \geq 4$ sharing common end point called the apex vertex v_0.*

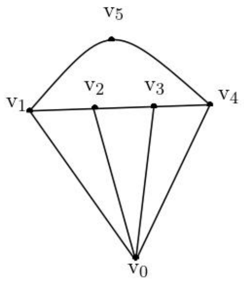

Figure 2. **Ice Cream Graph I_5**

Definition 3. [20] *The web graph Wb_n is the graph obtained from closed helm graph by attaching a single pendant edge to each of the outer cycle*.

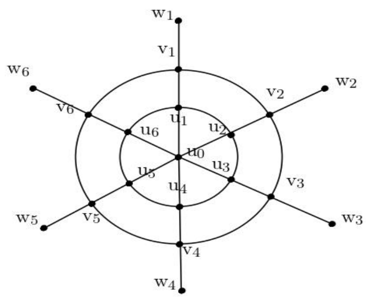

Figure 3. **Web Graph Wb_6**

Definition 4. [20] *The closed web graph CWb_n is the graph obtained from a web graph Wb_n by joining each of the outer pendant vertices consecutively to form a cycle.*

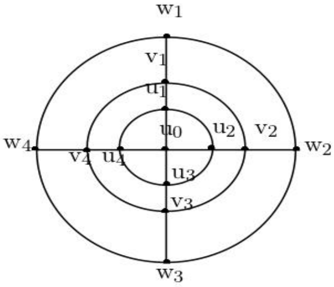

Figure 4. **Closed Web Graph CWb_4**

Definition 5. **[3]** *The circulant graph $C_n(1,m)$ is a graph with the vertex set $V(C_n(1,m)) = \{v_i | 1 \leq i \leq n\}$ and the edge set $E(C_n(1,m)) = \{v_i v_{i+1}, v_n v_1 | 1 \leq i \leq n-1\} \cup \{v_i v_{i+m} | 1 \leq i \leq n-m\} \cup \{v_{n-m+i} v_i | 1 \leq i \leq m\}$.*

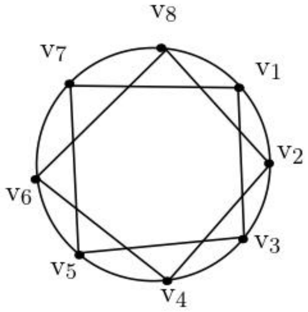

Figure 5. **Circulant Graph $C_8(1,2)$**

Definition 6. **[3]** *The zig-zag chord graph Z_n is a connected graph with the vertex set $V(Z_n) = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E(Z_n) = \{v_i v_{i+1}, v_n v_1 | 1 \leq i \leq n-1\} \cup \{v_i v_{n-i}, v_{j+1} v_{n-j} \mid 1 \leq i \leq \left\lceil \frac{n-2}{2} \right\rceil \,\&\, 1 \leq j \leq \left\lceil \frac{n-4}{2} \right\rceil\}$.*

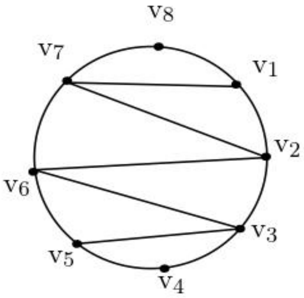

Figure 6. **Zig-Zag Graph Z_8**

Definition 7. **[11]** *The circular ladder graph CL_n is the graph obtained by using Cartesian product of cycle graph C_n with n vertices and the path graph P_2. That is $C_n \times P_2$.*

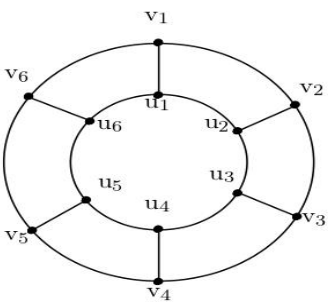

Figure 7. **Circular Ladder Graph CL_6**

Definition 8. **[11]** *The pentagonal circular ladder graph PCL_n is the graph obtained from a circular ladder graph CL_n by connecting the vertices v_1 and v_n by a new vertex w_n and connecting v_1 and v_n by an edge.*

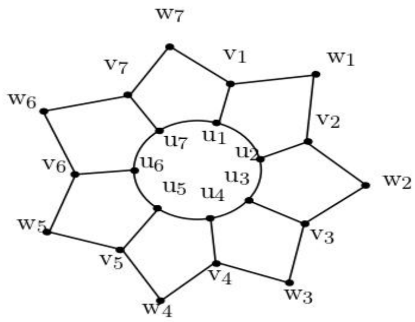

Figure 8. **Pentagonal Circular Ladder Graph PCL_7**

Definition 9. **[3]** *The djembe graph Dj_n is the graph obtained by joining the corresponding vertices of two cycles of same order n and joining all the vertices of the two cycles to a central vertex.*

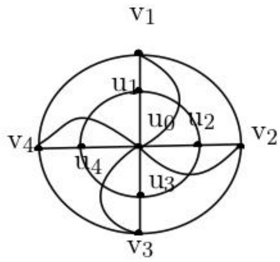

Figure 9. **Djembe Graph Dj_4**

Definition 10. **[3]** *The quadrilateral friendship graph QF_n is a graph constructed by joining $n$copies of C_4 with a common vertex.*

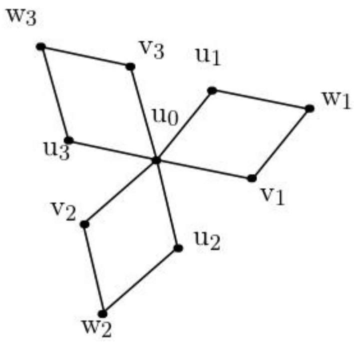

Figure 10. **Quadrilateral Friendship Graph QF_3**

Definition 11. **[3]** *The origami graph O_n is a graph with the vertex set $V(O_n) = \{u_i, v_i, w_i |\ 1 \le i \le n\}$ and the edge set $E(O_n) = \{u_iu_{i+1}, u_nu_1, w_iv_{i+1}, w_nv_1 |\ 1 \le i \le n-1\} \cup \{u_iv_i, u_iw_i, v_iw_i |\ 1 \le i \le n\}$. Then it has $3n$ vertices and $5n$edges.*

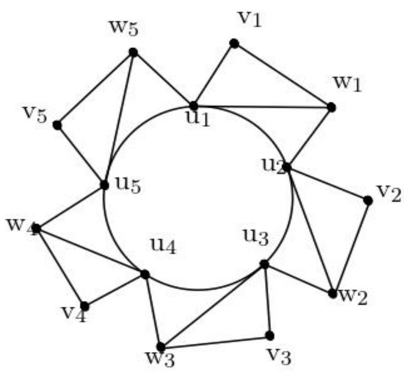

Figure 11. **Origami Graph O_5**

Theorem 1. [16] *If G is a (p,q) pair mean cordial graph (PMC-graph), then* $q \leq \begin{cases} 2p-5 & if\ p\ is\ even\ and\ p=5 \\ 2p-3 & if\ p\ is\ odd \end{cases}$.

Theorem 2. [16] *The path P_n is a pair mean cordial for all n.*

Theorem 3. [16]*The cycle C_n is a pair mean cordial for all values of n except for $n = 4$.*

Theorem 4. [16] *The complete graph K_n is pair mean cordial if and only if $n \geq 3$.*

3.2. Main Results

In this section, we have investigated the PMC-labeling of certain graphs like the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph.

Theorem 1. *The ice cream graph I_n is a PMC-graph if and only if n is even and $n \geq 6$.*

Proof. Consider the ice cream graph I_n, for $n \geq 4$. Denote by $V(I_n) = \{v_0, v_i \mid 1 \leq i \leq n\}$ and $(I_n) = \{v_iv_{i+1}, v_0v_i, v_nv_1 \mid 1 \leq i \leq n-1\}$ respectively, the vertex and edge set of the ice cream graph I_n. Then, it has $n+1$ vertices and $2n-1$ edges. Define $\lambda : V \to M = \{\pm 1, \pm 2, \dots, \pm\rho\}$, where $\rho = \begin{cases} \frac{n+1}{2}, for\ \ n+1\ is\ even \\ \frac{n}{2}, for\ \ n+1\ is\ odd, \end{cases}$ by assigning different labels in M to the different elements of V when $n+1$ is even and different labels in M to n elements of $V$and repeating a label for the remaining one vertex when $n+1$ is odd. We consider two cases:

Case (i): $n = 4, 5$

Suppose that I_n for $n = 4, 5$ is a PMC-graph. To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-2$ as the maximum number of edges labeled with 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-2$. Consequently, $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(n-2) = n+1$. Thus, $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+1-(n-2) = 3 > 1$, a contradiction.

Case (ii): $n \geq 6$

Subcase (i): n is even

Let us now assign the labels $2, 3, \dots, \frac{n}{2}$ and $-1, -2, \dots, \frac{-n}{2}$ to the corresponding vertices $v_1, v_3, \dots, v_{n-3}$ and $v_2, v_4, \dots, v_n$ respectively. Then assign the labels $1, 3$ to the corresponding vertices v_{n-1}, v_0. Consequently, $\bar{\mathbb{S}}_{\lambda_1} = n-1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$.

Subcase (ii): n is odd

To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-2$ as the maximum number of edges labeled with 1. So, the proof is same as in case (i). ■

Example 1. Figure 12 illustrates a PMC-labeling example of the ice cream graph I_6.

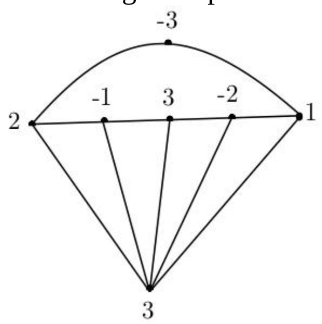

Figure 12. **The PMC-Labeling of the Ice Cream Graph I_6.**

Theorem 2. *The closed web graph CWb_n is not PMC-graph for all $n \geq 3$.*

Proof. Consider the closed web graph CWb_n, for $n \geq 3$. Denote by $V(CWb_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(CWb_n) = \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, u_nu_1, v_nv_1, w_nw_1 \mid 1 \leq i \leq n-1\} \cup \{u_0u_i, u_iv_i, v_iw_i \mid 1 \leq i \leq n\}$ respectively, the vertex and edge set of the closed web graph CWb_n. Then, it has $3n+1$ vertices and

$6n$ edges. Define $\lambda : V \to M = \{\pm 1, \pm 2, \dots, \pm\rho\}$, where $\rho = \begin{cases} \frac{3n+1}{2}, & for\ 3n+1\ is\ even \\ \frac{3n}{2}, & for\ 3n+1\ is\ odd, \end{cases}$ by assigning different labels in M to the different elements of V when $3n+1$ is even and different labels in M to $3n$ elements of V and repeating a label for the remaining one vertex when $3n+1$ is odd. Suppose that the closed web graph CWb_n is a PMC-graph. We consider two cases:

Case (i): n is odd

To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $3n-2$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq 3n-2$. Consequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q - (3n-2) = 3n+2$. Therefore, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq 3n+2-(3n-2) = 4 > 1$,

a contradiction.

Case (ii): n is even

To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $3n-1$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq 3n-1$. Subsequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q - (3n-1) = 3n+1$. Thus, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq 3n+1-(3n-1) = 2 > 1$, a contradiction. ■

Theorem 3. *The circulant graph $C_n(1,2)$ is not PMC-graph for all $n \geq 3$.*

Proof. Consider the circulant graph $C_n(1,2)$, for $n \geq 3$. Denote by $V(C_n(1,2)) = \{v_i \mid 1 \leq i \leq n\}$ and $E(C_n(1,2)) = \{v_i v_{i+1}, v_n v_1 \mid 1 \leq i \leq n-1\} \cup \{v_i v_{i+2} \mid 1 \leq i \leq n-2\} \cup \{v_{n-2+i} v_i \mid 1 \leq i \leq 2\}$ respectively, the vertex and edge set of the circulant graph $C_n(1,2)$. Then, $C_n(1,2)$ has n vertices and $2n$ edges. Suppose that $C_n(1,2)$ is a PMC-graph. Define $\lambda: V \to M = \{\pm 1, \pm 2, \dots, \pm\rho\}$, where $\rho = \begin{cases} \frac{n}{2}\ if\ n\ is\ even \\ \frac{n-1}{2}\ if\ n\ is\ odd, \end{cases}$ by assigning different labels in M to the different elements of V when n is even and different labels in M to $n-1$ elements of V and repeating a label for the remaining one vertex when n is odd. We consider two cases:

Case (i): n is even

To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-3$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq n-3$. Consequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q - (n-3) = n+3$. Thus, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq n+3-(n-3) = 6 > 1$, a contradiction.

Case (ii): n is odd

Let $n = 5$. To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-3$ as the maximum possible number of edges labeled with 1. So, the result is same as in case (i). Let $n \geq 7$. To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-2$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq n-2$. Subsequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q - (n-2) = n+2$. Thus, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq n+2-(n-2) = 4 > 1$, This is a contradiction.

■

Theorem 4. *The zig-zag chord graph Z_n is a PMC-graph for all odd values of $n \geq 7$.*

Proof. Consider the zig-zag chord graph Z_n for $n \geq 4$. Denote by $V(Z_n) = \{v_i \mid 1 \leq i \leq n\}$ and $E(Z_n) = \{v_i v_{i+1}, v_n v_1 \mid 1 \leq i \leq n-1\} \cup \{v_i v_{n-i}, v_{j+1} v_{n-j} \mid 1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \& 1 \leq j \leq \left\lceil \frac{n-4}{2} \right\rceil\}$ respectively, the vertex and edge set of the zig-zag chord graph Z_n. Then, Z_n has n vertices and $2n-3$ edges. Define $\lambda : V \to M = \{\pm 1, \pm 2, \dots, \pm\rho\}$, where $\rho = \begin{cases} \frac{n}{2}\ if\ n\ is\ even \\ \frac{n-1}{2}\ if\ n\ is\ odd, \end{cases}$ by assigning different labels in M to the different elements of V when n is even and different labels in M to $n-1$ elements of V and repeating a label for the remaining one vertex when n is odd. We consider two cases:

Case (i): $4 \leq n \leq 6$

Suppose the zig-zag chord graph Z_n is a PMC-graph. To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $n-3$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq n-3$. Consequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q-(n-3)=n$. Thus, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq n-(n-3) = 3 > 1$, a contradiction.

Case (ii): n $\geq$ 7

If n is odd, let us now assign the labels $-1, -2, \ldots, \frac{-n+5}{2}$ and $\frac{n-1}{2}, \frac{-n+1}{2}$ to the corresponding vertices $v_1, v_3, \ldots, v_{\frac{n-5}{2}}$ and $v_{\frac{n-3}{2}}, v_{\frac{n-1}{2}}$. Then, assign the labels $1, \frac{-n+3}{2}$ and $\frac{n-1}{2}, \frac{n-3}{2}, \ldots, 2$ to the corresponding vertices $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}$ and $v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \ldots, v_n$. Consequently, $\overline{\mathbb{S}}_{\lambda_1} = n-2$ and $\overline{\mathbb{S}}_{\lambda_1^c} = n-1$. Let n be even. To get the edge uv with label one, then the sum of the vertex label $\lambda(u)+\lambda(v) = 1$ or 2. These forces $n-3$ as the maximum possible number of edges labeled with 1. So, the proof is similar as in case (i). ■

Example 2. Figure 13 illustrates a PMC-labeling example of the zig-zag chord graph Z_9

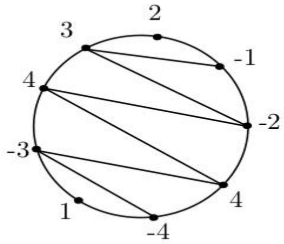

Figure 13. **The PMC-Labeling of the Zig-Zag Chord Graph Z_9**

Theorem 5. *The pentagonal circular ladder graph PCL_n is a PMC-graph for all $n \geq 3$.*

Proof. Consider the pentagonal circular ladder graph PCL_n, for $n \geq 3$. Denote by $V(PCL_n) = \{u_n, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(PCL_n) = \{u_i u_{i+1}, u_n u_1, w_i v_{i+1}, w_n v_1 \mid 1 \leq i \leq n-1\} \cup \{u_i v_i, v_i w_i \mid 1 \leq i \leq n\}$ respectively, the vertex and edge set of the pentagonal circular ladder graph PCL_n. Then, PCL_n has $3n$ vertices and $4n$ edges. We consider two cases:

Case (i): n is odd

Define $\lambda: V \to M = \left\{\pm 1, \pm 2, \ldots., \pm \frac{3n-1}{2}\right\}$ by assigning different labels in M to $3n-1$ elements of V and repeating a label for the remaining one vertex. Let us now assign the labels $2, 3, \ldots, n+1$ and $-1, -2, \ldots, -n$ to the corresponding vertices $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$. Then, assign the labels $-n-1, -n-2$ and $n+2, n+3, \ldots, \frac{3n-1}{2}$ to the corresponding vertices u_1, u_2 and $u_3, u_5, \ldots, u_{n-2}$. Next, assign the labels $-n-3, -n-4, \ldots, \frac{-3n+1}{2}$ and $1, 1$ to the corresponding vertices $u_4, u_6, \ldots, u_{n-3}$ and u_{n-1}, u_n.

Case (ii): n is even

Define $\lambda: V \to M = \left\{\pm 1, \pm 2, \ldots., \pm \frac{3n}{2}\right\}$ by assigning different labels in M to the different elements of V. Now, assign the labels to the vertices v_i and w_i, $1 \leq i \leq n$ as in case (i). Also, assign the labels $n+2, -n-1, -n-2$ and $-n-3, -n-4, \ldots, \frac{-3n}{2}$ to the corresponding vertices u_1, u_2, u_3 and $u_4, u_6, \ldots, u_{n-2}$. Assign the labels $n+3, n+4, \ldots, \frac{3n}{2}$ and 1 to the corresponding vertices $u_5, u_7, \ldots, u_{n-1}$ and u_{n-1}, u_n. In both cases, $\overline{\mathbb{S}}_{\lambda_1} = 2n = \overline{\mathbb{S}}_{\lambda_1^c}$. ■

Example 3. Figure 14 illustrates a PMC-labeling example of the pentagonal circular ladder graph PCL_7.

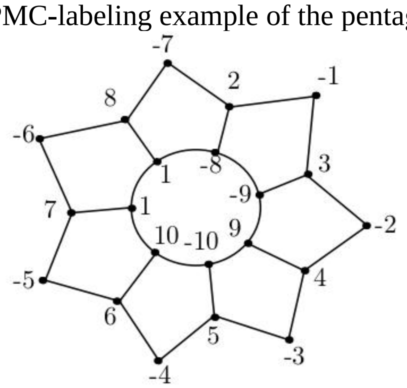

Figure 14. **The PMC-Labeling of the Pentagonal Circular Ladder Graph PCL_7**

Theorem 6. *The djembe graph Dj_n is not PMC-graph for all $n \geq 3$.*

Proof. Consider the djembe graph Dj_n for $n \geq 3$. Denote by $V(Dj_n) = \{u_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(Dj_n) = \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\} \cup \{u_0u_i, v_iu_i, u_0v_i \mid 1 \leq i \leq n\}$ respectively, the vertex and edge set of the djembe graph Dj_n. Then, Dj_n has $2n+1$ vertices and $5n$ edges. Define $\lambda: V \to M = \{\pm 1, \pm 2, \dots, \pm n\}$, by assigning different labels in M to $2n$ elements of V and repeating a label for the remaining one vertex. Suppose that Dj_n is a PMC-graph. To get the edge uv with label one, then the sum of the vertex label $\lambda(u) + \lambda(v) = 1$ or 2. These forces $2n-1$ as the maximum possible number of edges labeled with 1. That is $\overline{\mathbb{S}}_{\lambda_1} \leq 2n-1$. Consequently, $\overline{\mathbb{S}}_{\lambda_1^c} \geq q-(2n-1) = 3n+1$. Thus, $\overline{\mathbb{S}}_{\lambda_1^c} - \overline{\mathbb{S}}_{\lambda_1} \geq 3n+1-(2n-1) = n+2 \geq 5 > 1$, a contradiction. ■

Theorem 7. *The quadrilateral friendship graph QF_n is a PMC-graph for all of $n \geq 2$.*

Proof. Consider the quadrilateral friendship graph QF_n, for $n \geq 2$. Denote by $V(QF_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(QF_n) = \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1 \mid 1 \leq i \leq n-1\} \cup \{u_0u_i, v_iu_i, u_0v_i, v_iw_i \mid 1 \leq i \leq n\}$ respectively, the vertex and edge set of the quadrilateral friendship graph QF_n. Then, QF_n has $3n+1$ vertices and $4n$ edges. Note that $QF_1 \cong C_4$, the cycle C_4 is not PMC-graph [8]. We consider two cases:

Case (i): n is odd

Define $\lambda : V \to M = \left\{\pm 1, \pm 2, \dots, \pm \frac{3n+1}{2}\right\}$ by assigning different labels in M to the different elements of V. Let $\lambda(u_0) = \frac{3n+1}{2}$. Let us assign the labels $2, 5, \dots, \frac{3n-5}{2}$ and $-2, -5, \dots, \frac{-3n+5}{2}$ to the corresponding vertices $u_1, u_3, \dots, u_{n-2}$ and $u_2, u_4, \dots, u_{n-1}$. Fix the label $\frac{-3n+3}{2}$ to the vertex u_n. Then, assign the labels $3, 6, \dots, \frac{3n-3}{2}$ and $-3, -6, \dots, \frac{-3n+3}{2}$ to the corresponding vertices $v_1, v_3, \dots, v_{n-2}$ and $v_2, v_4, \dots, v_{n-1}$. Fix the label $\frac{-3n-1}{2}$ to the vertex v_n. Also, assign the labels $-1, -4, \dots, \frac{-3n+7}{2}$ and $4, 7, \dots, \frac{3n-1}{2}$ to the corresponding vertices $w_1, w_3, \dots, w_{n-2}$ and $w_2, w_4, \dots, w_{n-1}$. Fix the label 1 to the vertex w_n.

Case (ii): n is odd

Define $\lambda : V \to M = \left\{\pm 1, \pm 2, \dots, \pm \frac{3n}{2}\right\}$ by assigning different labels in M to $3n$ elements of V and repeating a label for the remaining one vertex. Let $\lambda(u_0) = -1$. So, assign the labels $2, 5, \dots, \frac{3n-2}{2}$ and $-2, -5, \dots, \frac{-3n+2}{2}$ to the corresponding vertices $u_1, u_3, \dots, u_{n-1}$ and $u_2, u_4, \dots, u_n$. Thus, assign the labels $3, 6, \dots, \frac{3n}{2}$ and $-3, -6, \dots, \frac{-3n}{2}$ to the corresponding vertices $v_1, v_3, \dots, v_{n-1}$ and $v_2, v_4, \dots, v_n$. Also, assign the labels $-1, -4, \dots, \frac{-3n+4}{2}$ and $4, 7, \dots, \frac{3n-4}{2}$ to the corresponding vertices $w_1, w_3, \dots, w_{n-1}$ and $w_2, w_4, \dots, w_{n-2}$. Fix the label 1 to the vertex w_n. In both cases, $\overline{\mathbb{S}}_{\lambda_1} = 2n = \overline{\mathbb{S}}_{\lambda_1^c}$. ■

Example 4. Figure 15 illustrates a PMC-labeling example of the quadrilateral friendship graph QF_4.

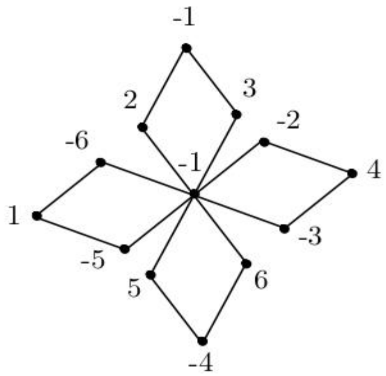

Figure 15. The PMC-Labeling of the Quadrilateral Friendship Graph QF_4.

Theorem 8. *The origami graph O_n is a PMC-graph for all $n \geq 3$ (except for $n = 4$).*

Proof. Consider the origami graph O_n, for $n \geq 3$. Denote by $V(O_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(O_n) = \{u_iu_{i+1}, u_nu_1, w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\} \cup \{u_iv_i, u_iw_i, v_iw_i \mid 1 \leq i \leq n\}$ respectively, the vertex and edge set of the origami graph O_n. Then, O_n has $3n$ vertices and $5n$ edges. Define $\lambda : V \to M =$

$\{\pm1, \pm2, \dots, \pm\rho\}$, where $\rho = \begin{cases} \frac{3n}{2} & if\ 3n\ is\ even \\ \frac{3n-1}{2} & if\ 3n\ is\ odd, \end{cases}$ by assigning different labels in M to the different elements of V when $3n$ is even and different labels in M to $3n-1$ elements of V and repeating a label for the remaining one vertex when $3n$ is odd. We consider four cases:

Case (i): $n \equiv 0\ (mod\ 4)$

Let us assign the labels $2, 5, \dots, \frac{3n-2}{2}$ and $-2, -5, \dots, \frac{-3n+2}{2}$ to the corresponding vertices $u_1, u_3, \dots, u_{n-1}$ and $u_2, u_4, \dots, u_n$. Now, assign the labels $-1, -4, \dots, \frac{-3n-4}{4}$ and $4, 7, \dots, \frac{3n+16}{4}$ to the corresponding vertices $v_1, v_3, \dots, v_{\frac{n+2}{2}}$ and $v_2, v_4, \dots, v_{\frac{n+4}{2}}$. Also, assign the labels $\frac{3n+24}{4}, \frac{3n+36}{4}, \dots, \frac{3n}{2}$ and $\frac{-3n-24}{4}, \frac{-3n-36}{4}, \dots, \frac{-3n}{2}$ to the corresponding vertices $v_{\frac{n+6}{2}}, v_{\frac{n+10}{2}}, \dots, v_{n-1}$ and $v_{\frac{n+8}{2}}, v_{\frac{n+12}{2}}, \dots, v_n$. Moreover, assign the labels $3, 6, \dots, \frac{3n+12}{4}$ and $-3, -6, \dots, \frac{-3n-12}{4}$ to the corresponding vertices $w_1, w_3, \dots, w_{\frac{n+2}{2}}$ and $w_2, w_4, \dots, w_{\frac{n+4}{2}}$. Assign the labels $\frac{-3n-16}{4}, \frac{-3n-28}{4}, \dots, \frac{-3n+4}{2}$ and $\frac{3n+28}{4}, \frac{3n+40}{4}, \dots, \frac{3n-4}{2}$ to the corresponding vertices $w_{\frac{n+6}{2}}, w_{\frac{n+10}{2}}, \dots, w_{n-1}$ and $w_{\frac{n+8}{2}}, w_{\frac{n+12}{2}}, \dots, w_{n-2}$. Fix the label 1 to the vertex w_n. Consequently, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n}{2} = \overline{\mathbb{S}}_{\lambda_1^c}$.

Case (ii): $n \equiv 1\ (mod\ 4)$

Now, assign the labels $2, 5, \dots, \frac{3n-5}{2}$ and $-2, -5, \dots, \frac{-3n+5}{2}$ to the corresponding vertices $u_1, u_3, \dots, u_{n-2}$ and $u_2, u_4, \dots, u_{n-1}$. Fix the label $\frac{-3n+1}{2}$ to the vertex u_n. So, assign the labels $-1, -4, \dots, \frac{-3n-1}{4}$ and $4, 7, \dots, \frac{3n+3}{4}$ to the corresponding vertices $v_1, v_3, \dots, v_{\frac{n+1}{2}}$ and $v_2, v_4, \dots, v_{\frac{n+3}{2}}$. Also, assign the labels $\frac{3n+21}{4}, \frac{3n+33}{4}, \dots, \frac{3n-3}{2}$ and $\frac{-3n-21}{4}, \frac{-3n-33}{4}, \dots, \frac{-3n+3}{2}$ to the corresponding vertices $v_{\frac{n+5}{2}}, v_{\frac{n+9}{2}}, \dots, v_{n-2}$ and $v_{\frac{n+7}{2}}, v_{\frac{n+11}{2}}, \dots, v_{n-1}$. Fix the label 1 to the vertex v_n.Further, assign the labels $3, 6, \dots, \frac{3n+9}{4}$ and $-3, -6, \dots, \frac{-3n-9}{4}$ to the corresponding vertices $w_1, w_3, \dots, w_{\frac{n+1}{2}}$ and $w_2, w_4, \dots, w_{\frac{n+3}{2}}$. Assign the labels $\frac{-3n-13}{4}, \frac{-3n-25}{4}, \dots, \frac{-3n+1}{2}$ and $\frac{3n+25}{4}, \frac{3n+37}{4}, \dots, \frac{3n-1}{2}$ to the corresponding vertices $w_{\frac{n+5}{2}}, w_{\frac{n+9}{2}}, \dots, w_n$ and $w_{\frac{n+7}{2}}, w_{\frac{n+11}{2}}, \dots, w_{n-1}$. If $n = 5$, $\lambda(v_n) = \frac{3n-3}{2}$. Consequently, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n-1}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{5n+1}{2}$.

Case (iii): $n \equiv 2\ (mod\ 4)$

So, assign the labels to the vertices $u_i, 1 \le i \le n$ as in case (i). Next, assign the labels $-1, -4, \dots, \frac{-3n-10}{4}$ and $4, 7, \dots, \frac{3n+10}{4}$ to the corresponding vertices $v_1, v_3, \dots, v_{\frac{n+4}{2}}$ and $v_2, v_4, \dots, v_{\frac{n+2}{2}}$. Also, assign the labels $\frac{-3n-18}{4}, \frac{-3n-30}{4}, \dots, \frac{-3n}{2}$ and $\frac{3n+30}{4}, \frac{3n+42}{4}, \dots, \frac{3n}{2}$ to the corresponding vertices $v_{\frac{n+6}{2}}, v_{\frac{n+10}{2}}, \dots, v_n$ and $v_{\frac{n+8}{2}}, v_{\frac{n+12}{2}}, \dots, v_{n-1}$. Further, assign the labels $3, 6, \dots, \frac{3n+18}{4}$ and $-3, -6, \dots, \frac{-3n-6}{4}$ to the corresponding vertices $w_1, w_3, \dots, w_{\frac{n+4}{2}}$ and $w_2, w_4, \dots, w_{\frac{n+2}{2}}$. Assign the labels $\frac{3n+22}{4}, \frac{3n+34}{4}, \dots, \frac{3n-4}{2}$ and $\frac{-3n-22}{4}, \frac{-3n-34}{4}, \dots, \frac{-3n+4}{2}$ to the corresponding vertices $w_{\frac{n+6}{2}}, w_{\frac{n+10}{2}}, \dots, w_{n-2}$ and $w_{\frac{n+8}{2}}, w_{\frac{n+12}{2}}, \dots, w_{n-1}$. Fix the label 1 to the vertex w_n. Consequently, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n}{2} = \overline{\mathbb{S}}_{\lambda_1^c}$.

Case (iv): $n \equiv 3\ (mod\ 4)$

Further, assign the labels to the vertices $u_i, 1 \le i \le n$ as in case (ii). So, assign the labels $-1, -4, \dots, \frac{-3n-7}{4}$ and $4, 7, \dots, \frac{3n+7}{4}$ to the corresponding vertices $v_1, v_3, \dots, v_{\frac{n+3}{2}}$ and $v_2, v_4, \dots, v_{\frac{n+1}{2}}$. Now, assign the labels $\frac{-3n-15}{4}, \frac{-3n-27}{4}, \dots, \frac{-3n+3}{2}$ and $\frac{3n+27}{4}, \frac{3n+39}{4}, \dots, \frac{3n-3}{2}$ to the corresponding vertices $v_{\frac{n+5}{2}}, v_{\frac{n+9}{2}}, \dots, v_{n-1}$ and $v_{\frac{n+7}{2}}, v_{\frac{n+11}{2}}, \dots, v_{n-2}$. Fix the label 1 to the vertex v_n.Next, assign the labels $3, 6, \dots, \frac{3n+15}{4}$ and

$-3,-6,\dots,\frac{-3n-3}{4}$ to the corresponding vertices $w_1, w_3, \dots, w_{\frac{n+3}{2}}$ and $w_2, w_4, \dots, w_{\frac{n+1}{2}}$. Thus, assign the labels $\frac{3n+19}{4},\frac{3n+31}{4},\dots,\frac{3n-1}{2}$ and $\frac{-3n-19}{4},\frac{-3n-31}{4},\dots,\frac{-3n+1}{2}$ to the corresponding vertices $w_{\frac{n+5}{2}}, w_{\frac{n+9}{2}},\dots,w_{n-1}$ and $w_{\frac{n+7}{2}}, w_{\frac{n+11}{2}},\dots,w_n$. Consequently, $\overline{\mathbb{S}}_{\lambda_1} = \frac{5n-1}{2}$ and $\overline{\mathbb{S}}_{\lambda_1^c} = \frac{5n+1}{2}$. ■

Example 5. Figure 16 illustrates a PMC-labeling example of the origami graph O_5.

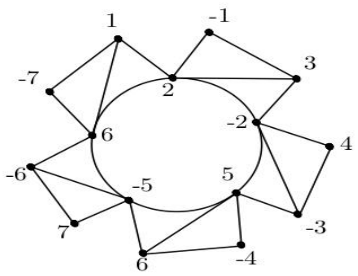

Figure 16. **The PMC-Labeling of the Origami Graph O_5.**

4. CONCLUSIONS

In this paper, we have examined the PMC-labeling of some cycle-related graphs: the ice cream graph, closed web graph, circulant graph, zig-zag chord graph, pentagonal circular ladder, djembe graph, quadrilateral friendship graph, and origami graph. Our future work is to determine whether this labeling exists on more graphs like the olive tree, step ladder graph, shadowgraph, shackle graph, tensor product graph, bull graph, scorpion graph, generalized Heawood graph, cubic diamond k-chain graph, Swastik graph, broken wheel graph, and n-cube graph.

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