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ANALYSIS OF THREE SERVERS CLOSED SERIES QUEUING NETWORK WITH DELAY TIME USING MAX-PLUS ALGEBRA

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ABSTRACT

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Max-Plus Algebra, which is the union of the set of all real numbers with an infinite singleton, equipped with maximum (max) and plus (+) operations, can be used to model and analyze algebraically the dynamics of a closed queuing network. This study aims to analyze the effect of delays in the start time of service activities on a closed series queuing network with three servers. This study is a study based on literature studies, mathematical model studies and simulations assisted by the Scilab computer program. The results show that the max-plus eigenvalue of a closed series queuing network with 3 servers, which is also the periodicity of network dynamics, is the largest service time of the server in the network. Delays in servers with the largest service time will continue to propagate for subsequent schedules. Delays in servers whose service time is not the maximum can still be tolerated, as long as the delay does not exceed the size of the element in the initial max-plus eigenvector, which corresponds to its largest service time. In this case, the system will be able to return to normal according to the original schedule, after undergoing a maximum of 4 stages of the service process since the beginning of the delay. Meanwhile, delays that exceed this will cause network scheduling to be late and will continue to spread to subsequent services.

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1. INTRODUCTION

Max-plus algebra, which is the set of $\boldsymbol{R}_\varepsilon := \boldsymbol{R} \cup \{\varepsilon\}$, where $\varepsilon := -\infty$, equipped with maximum (max) and plus (+) operations, has been used well to model and analyze algebraically network problems that contain synchronization and linearization problems, such as railway networks, production system networks, project scheduling and queuing systems [1][2][3][4][5][6]. Modeling these problems with an approach using max and plus operations can provide a more integrated and unified way and the resulting equations are analogous to the results in conventional system theory [7][8][9]. By using max-plus algebra, the above network problems can be expressed in a matrix equation over max-plus algebra. Furthermore, from this matrix equation, the system dynamics can be analyzed which include input-output and system periodicity through the values and eigenvectors of its max-plus algebra.

The network modeling and analysis discussed above mostly assume that the arrival time (start of activity) and departure time (completion of activity) run on time as scheduled (planned). However, in reality in the real world we often find that the scheduled time cannot always be met, so there is a problem with delay time . The delay time that appears for one or more service units (*servers*) in a network system will certainly affect the delay of the service times of the units and the overall system time. For this reason, it is necessary to model and analyze the system with the delay time.

One of the efforts that have been made in modeling and analyzing networks with delay times with max-plus algebra for railway networks has been carried out by several researchers, such as in which [6], [10] discusses the control of delay time propagation on railway networks, in [11] which has discussed quite completely for railway networks and problems that are still relatively simple. For the problem of delay times in production networks, not many have studied it. In [12] and [13], problems related to delay times for simple production systems [14] are mentioned, but have not been discussed completely as in which [11] discusses the problem of delay times for simple railway networks.

Research on the problem of delay time in production networks using max-plus algebra and its application in everyday life, as far as researchers have studied, not many have done it. There are several variations of existing production network structures, from the simplest to the most complex, which involve a combination of several network structures [15][16]. The production network structure that is still relatively simple is a closed serial queue network with n single servers. Although it is said to be simple, in its analysis involving various possible variations in activity time between its n servers, it will provide complexity in its analysis. The existing analysis problem is to identify the periodic behavior of the queue network. Furthermore, in the initial periodic conditions, which are also said to be in normal conditions, a delay condition will be identified that can become normal by itself or not, after going through several series of activity processes.

As an initial step in the research on the problem of delay time in a closed serial queue network with n single servers, to reduce the level of complexity in the analysis, this study will discuss a closed serial queue network with 3 single servers. The discussion assumes the conditions given in the analysis using max-plus algebra, ignoring real conditions that are not relevant to those assumed. The conditions to be discussed include the delay time situation where the server service time is all the same, and the server service is not the same. Furthermore, the longest delay time will be determined where the network can return to normal or not, after starting several stages of subsequent activities. The results of this study are expected to provide an overview of the analysis results which can then be generalized for more servers and will ultimately provide an overview for research for n servers.

2. RESEARCH METHODS

This section begins with discussing the eigen value and vector of the matrices over max-plus algebra. Eigen value and vectors are used to find out the characteristics of the dynamical system over max-plus algebra. Moreover, the characteristic of closed series queuing network also discussed in this section.

2.1 Max-Plus Eigenvalues and Vectors

The following operations are defined: $\forall a, b \in \boldsymbol{R}_\varepsilon$, $a \oplus b := \max(a, b)$ and $a \otimes b := a + b$. In [1], [3], and [17] it is shown that $(\boldsymbol{R}_\varepsilon, \oplus, \otimes)$ is an idempotent semifield. The algebraic structure $\boldsymbol{R}_{\max} := (\boldsymbol{R}_\varepsilon, \oplus, \otimes)$

is called max-plus algebra, which is then simply written as $\boldsymbol{R}_{\max}$. Operation $\oplus$ and $\otimes$ on $\boldsymbol{R}_{\max}$ above can be extended to matrix operations in $\boldsymbol{R}_{\max}^{m\times n}$ [3].

Definition 1. (Eigenvalues and max-plus eigenvectors, [3][17][18][19])

Given $A \in \boldsymbol{R}_{\max}^{n\times n}$. *Scalar* $\lambda \in \boldsymbol{R}_{\max}$ *called* eigenvalues max-plus *matrix* A *if there is a vector* $\boldsymbol{v} \in \boldsymbol{R}_{\max}^{n}$ *with* $\boldsymbol{v} \neq \boldsymbol{\varepsilon}_{n\times 1}$ *so that* $A\otimes \boldsymbol{v} = \lambda \otimes \boldsymbol{v}$. *Vector* $\boldsymbol{v}$ *said called the* max-plus eigenvectors *of the matrix* A *corresponds to* λ.

Theorem 2. (Existence of Eigenvalues [1][2][3][17])

Given $A \in \boldsymbol{R}_{\max}^{n\times n}$. *Scalar* $\lambda_{\max}(A) = \bigoplus_{k=1}^{n}(\frac{1}{k}\operatorname{trace}(A^{\otimes^k}))$, *namely the maximum average weight of an elementary circuit in* $G(A)$, *is a max-plus eigenvalue of the matrix* A, *where* $trace(A^{\otimes^k}) = \bigoplus_{i=1}^{n}(A^{\otimes^k})_{ii}$.

In the proof of Theorem 2 [3] it is explained that if the points i form an arc in the critical circuit, then the i-th column of the matrix B^* is the max-plus eigenvector of the matrix A which corresponds to the eigenvalue $\lambda_{\max}(A)$, where the matrix $B = -\lambda_{\max}(A)\otimes A$, and $B^* = E\oplus B \oplus ...\oplus B^{\otimes^{n-1}}$. This eigenvector is called the max-plus fundamental eigenvector corresponding to the max-plus eigenvalue $\lambda_{\max}(A)$. It is also explained that the max-plus linear combination of the max-plus fundamental eigenvectors of matrix A is also the max-plus eigenvector corresponding to $\lambda_{\max}(A)$.

It is further [3] explained that a matrix is said to be irreducible if its weight graph is strongly connected, which then has a single max-plus algebraic eigenvalue, namely $\lambda_{\max}(A)$, with a fundamental eigenvector $\boldsymbol{v}$ where $v_i \neq \varepsilon$ for each $i \in \{1, 2, ..., n\}$. This situation corresponds to the real situation, where the initial departure time is finite. Furthermore, to be realistic, the initial departure time of the customer must be non-negative. For this reason, it is necessary to modify the fundamental eigenvector $\boldsymbol{v}$ so that all its components are non-negative. The vector is formed $\boldsymbol{v}^* = \beta\otimes \boldsymbol{v}$, with $\beta = -\min_i(v_i)$, for $i = 1, 2, ..., n$, which are called the max-plus initial eigenvectors.

2.2 Closed Series Queuing Network

Next, consider a closed series (closed tandem) queue network with n single-serves [4], [3], [5] as shown in Figure 1 below.

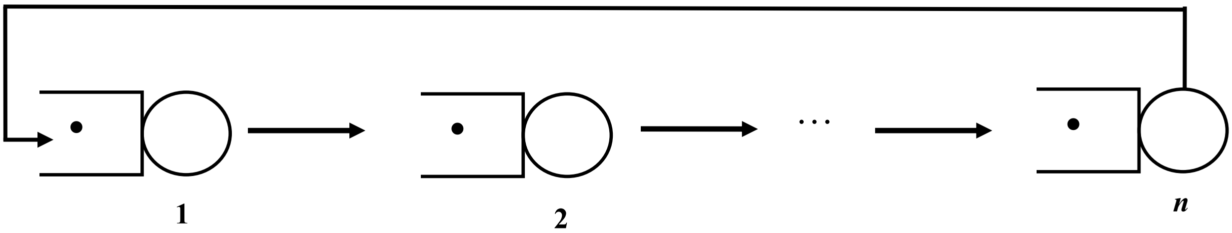

Figure 1. Closed Series Queue Network

The basic assumptions in this network are as follows:

1. The queue buffer capacity is infinite.
2. Queues work on the First-In First-Out (FIFO) principle.
3. The movement of customers from one queue to the next queue does not require time.
4. Customers must queue from start to finish in sequence to receive service from each waiter.

One cycle of network service is the process from the customer entering the 1st server buffer to leaving the nth server. After the completion of service at the n-th server, the customer returns to the first queue for a new cycle of network service. At the beginning of the observation, all servers are not providing service, where the buffer at the i-th server contains 1 customer for each $i = 1, 2, ..., n$. In Figure 1 above, gives the initial state of the closed series queue network in question, with customers denoted by "•".

Closed series queue networks can be found in assembly plant systems, such as car assembly or electronic goods. The customers in this system are pallets while the service is the assembly machine. The pallet in question is a kind of table or place where components or semi-finished goods are placed and move to visit the assembly machines. Initially, a pallet 1 enters the 1st machine support, then enters the 1st machine and the 2nd pallet enters the 1st machine support. In this 1st machine, the components are placed and prepared to be assembled on the next machine. Next, the 1st pallet enters the 2nd machine support and the 2nd pallet enters the 1st machine. And so on for the n available pallets, so that a state is achieved as in Figure 2 above, where the initial state of observation is achieved. After the assembly is completed on the nth machine, the assembled goods will leave the network, while the pallet carrying them will return to the 1st machine support, to start a new cycle of network service, and so on.

Suppose that

$a_i(k)$ = the arrival time of the k-th customer at the i-th server,

$d_i(k)$ = time of departure of the k-th customer from the i-th server,

t_i = service time at the i-th server.

for k = 1, 2, ... and i = 1, 2, ..., n. Furthermore, the queue dynamics at the server i, as discussed in [4], can be expressed in the following equation

$$d(k) = A \otimes d(k-1) \tag{1}$$

with $A = T \otimes (G \oplus E) = \begin{bmatrix} t_1 & \varepsilon & \cdots & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \ddots & \ddots & & \vdots \\ \vdots & & t_{n-1} & t_{n-1} & \varepsilon \\ \varepsilon & \cdots & \varepsilon & t_n & t_n \end{bmatrix}$.

Equation (1) above is a dynamic model of the queue network. In it [3] is discussed that if the fastest initial departure time of the customer is an eigenvector of matrix A, then the customer departure time for each service can occur periodically with a period size equal to the eigenvalue of matrix A. Furthermore, the fastest initial departure time discussed is an eigenvector of matrix A, so that the system is periodic from the beginning.

This study uses a literature study method supported by computer computing and mathematical studies. This study is a study based on literature studies that include relevant theoretical studies and conducts model analysis by conducting simulations assisted by computer computing applications [20], [21], [22]. In the simulation stage related to technical calculations, the study will use the assistance of the Scilab 5.5.2 Computer Program [23] with Toolboxes MAXPLUSV04032016 [24]. From the results of computer simulations for various situations, assumptions will then emerge that can provide an overview of the research results. The results of the study are in the form of mathematical explanations related to the answers to research problems.

The important equation in this study is the max-plus algebraic matrix equation that models the following closed serial queuing network model [4].

$$d(k) = A \otimes d(k-1)$$

with $A == \begin{bmatrix} t_1 & \varepsilon & \cdots & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \ddots & \ddots & & \vdots \\ \vdots & & t_{n-1} & t_{n-1} & \varepsilon \\ \varepsilon & \cdots & \varepsilon & t_n & t_n \end{bmatrix}$,

$d_i(k)$ = time of departure of the k-th customer *from* the i-th server,

t_i = service time at the i-th server,

ε: $= -\infty$,

for k = 1, 2, ... and i = 1, 2, ..., n .

The research steps are as follows. This research will analyze the problem of delay time in a closed serial queuing network [3][25], using max-plus algebra. The research begins by re-studying the modeling and analysis of a simple railway network with a delay time in [11], then the ideas are tried to be applied to a closed serial queuing network. Furthermore, an understanding of the problem of delay time in a closed serial queuing network is given. Then an analysis is carried out for various delay situations, as well as efforts to return to the original periodic condition. The delay situations that will be analyzed include delays when all service times are the same and not the same for the three servers. A condition will be sought where the delay will return to normal by itself or not, after the system continues its activities for several stages. Related to this, in more detail, it will be analyzed, how long is the maximum delay time so that the network will run normally again or not. Not being able to run normally means that the delay that occurs will continue to spread or continue. The step of this research can be shown as a Figure 2 below.

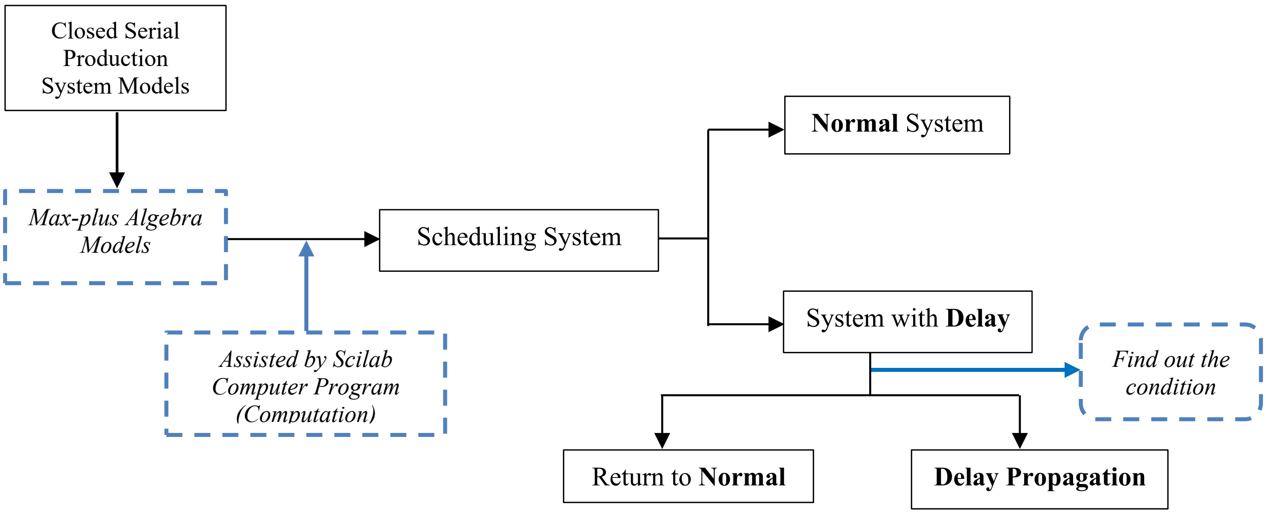

Figure 2. Procedure of the Research

Based on Figure 2 above, there are two main stages in this research. The first stage is to model the closed series model of production system using max-plus algebra. In this research, the production network used is a production network with 3 servers. The resulting model is analyzed to determine a production scheduling system. Periodicity analysis is carried out by analyzing the eigen values and eigen vectors. The scheduling system computation is assisted by the Scilab 5.5.2 computer program. The second stage is to analyze the model of the production system if delays occur. Various delay variations are simulated in the system. The impact of delays is analyzed on the original scheduling. In this research, the delay conditions are investigated which allow the scheduling system to return to normal and the system to experience delay propagation.

3. RESULTS AND DISCUSSION

The closed series queue network is discussed along with its dynamics and modeling using max-plus algebra. Furthermore, several special results of the simulation of the dynamics of the queue network with delay time are discussed by taking 3 servers and several cases of delay time. From the results of the special case simulation, it is generalized for the case of n servers and provides conjectures and discussions related to certain cases of delay time.

3.1 Eigenvalues and Eigenvectors of Max-Plus Algebra for a Queuing System with 3 Servers

Consider the closed series queue network as above by taking only 3 servers, with their service times respectively t_1, $t_2$and t_3, so that the matrix in Equation (1) above becomes

$$A = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix}.$$

According to **Theorem 2** and its explanation in it [3] can be concluded that the largest element in matrix A is the max-plus eigenvalue of matrix A. Thus, the max-plus eigenvalue of matrix A is the longest service time among the services in the queue network system above.

Next, we review four possible service time values as follows:

i) For $t_1 = t_2 = t_3$

According to the explanation above, the eigenvalue of matrix A is $\lambda_{\max}(A) = t_1 = t_2 =$ and t_3 the fundamental eigenvector is

$\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{v}^*$.

Suppose $\boldsymbol{d}(0) = \boldsymbol{v}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then

$\boldsymbol{d}(1) = A \otimes \boldsymbol{d}(0) = A \otimes \boldsymbol{v}^* = t_1 \otimes \boldsymbol{v}^*$,

$\boldsymbol{d}(2) = A \otimes \boldsymbol{d}(1) = 2\,(t_1 \otimes \boldsymbol{v}^*)$, ...,

$\boldsymbol{d}(k) = k\, t_1 \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, for $k = 1, 2, 3, \ldots$

ii) For $\max\{t_1, t_2, t_3\} = t_1$

We obtained by the eigenvalue of matrix A is $\lambda_{\max}(A) = t_1$ and the fundamental eigenvector is

$\boldsymbol{v} = \begin{bmatrix} 0 \\ -2t_1 + 2t_2 \\ -2t_1 + t_2 + t_3 \end{bmatrix}$,

Thus, the initial max-plus eigenvector is $\boldsymbol{v}^* = \begin{bmatrix} 2t_1 - t_2 - t_3 \\ t_2 - t_3 \\ 0 \end{bmatrix}$.

If we take $\boldsymbol{d}(0) = \boldsymbol{v}^* = \begin{bmatrix} 2t_1 - t_2 - t_3 \\ t_2 - t_3 \\ 0 \end{bmatrix}$,

then we get $\boldsymbol{d}(k) = k\, t_1 \otimes \begin{bmatrix} 2t_1 - t_2 - t_3 \\ t_2 - t_3 \\ 0 \end{bmatrix}$, for $k = 1, 2, 3, \ldots$.

iii) For $\max\{t_1, t_2, t_3\} = t_2$

We obtained by the eigenvalue of matrix A is $\lambda_{\max}(A) = t_2$ and the fundamental eigenvector is

$\boldsymbol{v} = \begin{bmatrix} t_1 - 2t_2 + t_3 \\ 0 \\ -2t_2 + 2t_3 \end{bmatrix}$

Thus, that the initial eigenvector max-plus is $\boldsymbol{v}^* = \begin{bmatrix} 0 \\ -t_1 + 2t_2 - t_3 \\ -t_1 + t_3 \end{bmatrix}$.

If we take $\boldsymbol{d}(0) = \boldsymbol{v}^* = \begin{bmatrix} 0 \\ -t_1 + 2t_2 - t_3 \\ -t_1 + t_3 \end{bmatrix}$,

then we get $\boldsymbol{d}(k) = k\, t_2 \otimes \begin{bmatrix} 0 \\ -t_1 + 2t_2 - t_3 \\ -t_1 + t_3 \end{bmatrix}$, for $k = 1, 2, 3, \ldots$.

iv) For $\max\{t_1, t_2, t_3\} = t_3$

We obtained by the eigenvalue of matrix A is $\lambda_{\max}(A) = t_3$ and the fundamental eigenvector is

$$\boldsymbol{v} = \begin{bmatrix} t_1 - t_3 \\ t_1 + t_2 - 2t_3 \\ 0 \end{bmatrix},$$

Thus, the initial max-plus eigenvector is $\boldsymbol{v}^* = \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix}$.

If we take $\boldsymbol{d}(0) = \boldsymbol{v}^* = \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix}$,

then we get $\boldsymbol{d}(k) = k\, t_3 \otimes \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix}$, for $k = 1, 2, 3, \ldots$

Next, we will discuss the problem of customer departure delays from the normal schedule that has been determined based on the initial eigenvector. To simplify the discussion, it is assumed that the delay occurs since the beginning of the departure, although the delay can occur at any time for a certain k.

3.2 Departure Time Delay for System with $t_1 = t_2 = t_3$

Suppose a delay occurs at the 1st server, amounting to 1 unit of time ($u.t$), then the initial departure time vector becomes

$\boldsymbol{v}^{*1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and the next departure time is

$$\boldsymbol{d}^1(1) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_1 & t_1 & \varepsilon \\ \varepsilon & t_1 & t_1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 \otimes 1 \\ t_1 \otimes 1 \\ t_1 \end{bmatrix} = t_1 \otimes \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{d}^1(2) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_1 & t_1 & \varepsilon \\ \varepsilon & t_1 & t_1 \end{bmatrix} \otimes \begin{bmatrix} t_1 \otimes 1 \\ t_1 \otimes 1 \\ t_1 \end{bmatrix} = \begin{bmatrix} 2t_1 \otimes 1 \\ 2t_1 \otimes 1 \\ 2t_1 \otimes 1 \end{bmatrix} = 2\, t_1 \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ldots,$$

$$\boldsymbol{d}^1(k) = k\, t_1 \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ for } k = 2, 3, \ldots.$$

Furthermore, it is obtained that

$$\boldsymbol{d}^1(k) - \boldsymbol{d}(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This means that starting at $k = 2$, the system will continue to be 1 unit late for all services on subsequent departures. The same can happen for delays on the 2nd and 3rd service, or on all three services simultaneously. More generally, this also applies to delays of h $u.t$.

Table 1. Simulation for $t_1 = t_2 = t_2 = 3$

Normal Condition							t_1 with 1 $u.t$ late							Late – Normal						
k	***0***	***1***	***2***	***3***	***4***	…	***k***	***0***	***1***	***2***	***3***	***4***	…	***k***	***0***	***1***	***2***	***3***	***4***	…
d_1	0	3	6	9	12	…	d_1	1	4	7	10	13	…	d_1	1	1	1	1	1	…
d_2	0	3	6	9	12	…	d_2	0	4	7	10	13	…	d_2	0	1	1	1	1	…
d_3	0	3	6	9	12	…	d_3	0	3	7	10	13	…	d_3	0	0	1	1	1	…

3.3 Departure Time Delay for System where $\max\{t_1, t_2, t_3\} = t_3$

The following will discuss the problem of network system delay where $\max\{t_1, t_2, t_3\} = t_3$. The delay for the network system where $\max\{t_1, t_2, t_3\} = t_1$ and $\max\{t_1, t_2, t_3\} = t_2$ can be done in the same way.

Suppose the delay occurs at the 3rd server of 1 *u.t*, then

$$\boldsymbol{d}^{31} = \boldsymbol{v}^{*31} = \begin{bmatrix} -t_2+t_3 \\ 0 \\ -t_1-t_2+2t_3 \otimes 1 \end{bmatrix}$$

$$\boldsymbol{d}^{31}(1) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -t_2+t_3 \\ 0 \\ -t_1-t_2+2t_3 \otimes 1 \end{bmatrix} = \begin{bmatrix} -t_2+2t_3 \otimes 1 \\ t_3 \\ -t_1-t_2+3t_3 \otimes 1 \end{bmatrix} = t_3 \otimes \begin{bmatrix} -t_2+t_3 \otimes 1 \\ 0 \\ -t_1-t_2+2t_3 \otimes 1 \end{bmatrix}$$

$$\boldsymbol{d}^{31}(2) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -t_2+2t_3 \otimes 1 \\ t_3 \\ -t_1-t_2+3t_3 \otimes 1 \end{bmatrix} = \begin{bmatrix} -t_2+3t_3 \otimes 1 \\ 2t_3 \otimes 1 \\ -t_1-t_2+3t_3 \otimes 1 \end{bmatrix} = 2t_3 \otimes \begin{bmatrix} -t_2+t_3 \otimes 1 \\ 1 \\ -t_1-t_2+2t_3 \otimes 1 \end{bmatrix}$$

$$\boldsymbol{d}^{31}(k) = k\, t_3 \otimes \begin{bmatrix} -t_2+t_3 \\ 0 \\ -t_1-t_2+2t_3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ for } k = 2, 3, 4, \ldots$$

It was further obtained that

$$\boldsymbol{d}^{31}(k) - \boldsymbol{d}(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This means that the system will continue to be 1 *u.t* late for subsequent departures. Thus, it seems that a delay on the server with the larger service time, where this larger service time is also an eigenvalue, which is also the departure period, will cause the system to continue to be late. More generally, this also applies to a delay of *h u.t* on the server with the larger time.

Table 2. **Simulation for $t_1 = 1, t_2 = 2, t_3 = 3$**

Normal Condition							t_3 with 1 *u.t* late							Late – Normal						
k	*0*	*1*	*2*	*3*	*4*	…	*k*	*0*	*1*	*2*	*3*	*4*	…	*k*	*0*	*1*	*2*	*3*	*4*	…
d_1	1	4	7	10	13	…	d_1	1	5	8	11	14	…	d_1	0	1	1	1	1	…
d_2	0	3	6	9	12	…	d_2	0	3	7	10	13	…	d_2	0	0	1	1	1	…
d_3	3	6	9	12	15	…	d_3	4	7	10	13	16	…	d_3	1	1	1	1	1	…

Then, suppose a delay occurs at the 1st and 2nd service, where the service time is smaller, with a delay of $-t_1 - t_2 + 2t_3$ *u.t*, then the initial departure time vector becomes

$$\boldsymbol{v}^{*h} = \boldsymbol{d}^h(0) = \begin{bmatrix} -t_1 + -2t_2 + 3t_3 \\ -t_1-t_2+2t_3 \\ -t_1-t_2+2t_3 \end{bmatrix}, \text{ so the next departure time is}$$

$$\boldsymbol{d}^h(1) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -t_1 + -2t_2 + 3t_3 \\ -t_1-t_2+2t_3 \\ -t_1-t_2+2t_3 \end{bmatrix} = \begin{bmatrix} -2t_2+3t_3 \\ -t_1-t_2+3t_3 \\ -t_1-t_2+3t_3 \end{bmatrix} = t_3 \otimes \begin{bmatrix} -2t_2+2t_3 \\ -t_1-t_2+2t_3 \\ -t_1-t_2+2t_3 \end{bmatrix}$$

$$\boldsymbol{d}^h(2) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -2t_2+3t_3 \\ -t_1-t_2+3t_3 \\ -t_1-t_2+3t_3 \end{bmatrix} = \begin{bmatrix} -t_2+3t_3 \\ -t_1+3t_3 \\ -t_1-t_2+4t_3 \end{bmatrix} = 2t_3 \otimes \begin{bmatrix} -t_2+t_3 \\ -t_1+t_3 \\ -t_1-t_2+2t_3 \end{bmatrix}$$

$$\boldsymbol{d}^h(3) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -t_2+3t_3 \\ -t_1+3t_3 \\ -t_1-t_2+4t_3 \end{bmatrix} = \begin{bmatrix} -t_2+4t_3 \\ -t_1+t_2+3t_3 \\ -t_1-t_2+5t_3 \end{bmatrix} = 3t_3 \otimes \begin{bmatrix} -t_2+t_3 \\ -t_1+t_2 \\ -t_1-t_2+2t_3 \end{bmatrix}$$

$$\boldsymbol{d}^h(4) = \begin{bmatrix} t_1 & \varepsilon & t_1 \\ t_2 & t_2 & \varepsilon \\ \varepsilon & t_3 & t_3 \end{bmatrix} \otimes \begin{bmatrix} -t_2 + 4t_3 \\ -t_1 + t_2 + 3t_3 \\ -t_1 - t_2 + 5t_3 \end{bmatrix} = \begin{bmatrix} -t_2 + 5t_3 \\ 4t_3 \\ -t_1 - t_2 + 6t_3 \end{bmatrix} = 4t_3 \otimes \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix}$$

...

$$\boldsymbol{d}^h(k) = k\, t_3 \otimes \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix}, \text{ for } k = 4, 5, 6, \ldots$$

Next, $\boldsymbol{d}^h(k)$ is obtained $\boldsymbol{d}(k) = \boldsymbol{0}$, for $k = 4, 5, 6, \ldots$. This means that the system will return to normal on the next departure. This also applies to delays smaller than $-t_1 - t_2 + 2t_3$ $u.t$.

Table 3. **Simulation for $t_1 = 1, t_2 = 2, t_3 = 3$**

Normal Condition							t_1, t_2 with 3 $u.t$ late							Late – Normal						
k	***0***	***1***	***2***	***3***	***4***	…	***k***	***0***	***1***	***2***	***3***	***4***	…	***k***	***0***	***1***	***2***	***3***	***4***	…
d_1	1	4	7	10	13	…	d_1	4	5	7	10	13	…	d_1	3	1	0	0	0	…
d_2	0	3	6	9	12	…	d_2	3	6	8	10	12	…	d_2	3	3	2	1	0	…
d_3	3	6	9	12	15	…	d_3	3	6	9	12	15	…	d_3	0	0	0	0	0	…

Then, suppose a delay occurs at the 1st service, which has a smaller service time, with a difference of $-t_1 - t_2 + 2t_3 \otimes 1$ $u.t$, then the initial departure time vector becomes

$$\boldsymbol{v}^* = \boldsymbol{d}^{h+1}(0) = \begin{bmatrix} -t_1 + -2t_2 + 3t_3 \otimes 1 \\ -t_1 - t_2 + 2t_3 \\ -t_1 - t_2 + 2t_3 \end{bmatrix},$$

in the same way as in the previous explanation, it can be shown that

$$\boldsymbol{d}^{h+1}(k) = k\, t_3 \otimes \begin{bmatrix} -t_2 + t_3 \\ 0 \\ -t_1 - t_2 + 2t_3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ for } k = 4, 5, 6, \ldots.$$

Next, we obtain

$$\boldsymbol{d}^{h+1}(k) - \boldsymbol{d}(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ for } k = 4, 5, 6, \ldots.$$

This means that the system will continue to be 1 $u.t$ late for subsequent departures. Likewise in general for a delay of $-t_1 - t_2 + 2t_3 \otimes h$ on the 1st and 2nd service, there will be a delay of h units on subsequent departures.

4. CONCLUSIONS

From the discussion above, the following conclusions can be drawn. The max-plus eigenvalue of a closed series queuing network of 3 servers, which is also the periodicity of network dynamics, is the largest service time of a server in the network. Delays in servers with the largest service time will continue to propagate to subsequent scheduling. Delays in servers whose service time is not the maximum can still be tolerated, if the delay does not exceed the size of the element in the initial max-plus eigenvector, which corresponds to its largest service time. In this case, the system will be able to return to normal according to the original schedule, after undergoing a maximum of 4 stages of the service process since the beginning of the delay. While delays that exceed this will cause the network scheduling to be late and will continue to propagate to subsequent services.

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REFERENCES

[1] F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *SYNCHRONIZATION AND LINEARITY : AN ALGEBRA FOR DISCRETE EVENT SYSTEMS*. Paris: Wiley, 2001. Accessed: Feb. 02, 2024. [Online]. Available: https://www.rocq.inria.fr/metalau/cohen/documents/BCOQ-book.pdf.

[2] B. Heidergott, G. J. Olsder, and J. van der Woude, *MAX PLUS AT WORK*. Princeton: Princeton University Press, 2006. [Online]. Available: http://about.jstor.org/terms.

[3] M. A. Rudhito, *ALJABAR MAX-PLUS DAN PENERAPANNYA*. Yogyakarta: Sanata Dharma University Press, 2016.

[4] N. K. Krivulin, “A MAX-ALGEBRA APPROACH TO MODELING AND SIMULATION OF TANDEM QUEUEING SYSTEMS,” 1995. doi: https://doi.org/10.48550/arXiv.1212.0895.

[5] N. K. Krivulin, “THE MAX-PLUS ALGEBRA APPROACH IN MODELLING OF QUEUEING NETWORKS,” in *Summer Computer Simulation Conference*, Portland: SCS, Jul. 1996, pp. 485–490. doi: https://doi.org/10.48550/arXiv.1212.0895.

[6] Subiono, “ALJABAR MIN-MAX PLUS DAN TERAPANNYA,” 2015.

[7] N. K. Krivulin, “ALGEBRAIC MODELLING AND PERFORMANCE EVALUATION OF ACYCLIC FORK-JOIN QUEUEING NETWORKS,” in *Advances in Stochastic Simulation Methods*, V. B. and E. S. Balakrishnan N. and Melas, Ed., Boston, MA: Birkhäuser Boston, 2000, pp. 63–81. doi: https://doi.org/10.1007/978-1-4612-1318-5_5.

[8] J. Komenda, S. Lahaye, J. L. Boimond, and T. van den Boom, “MAX-PLUS ALGEBRA AND DISCRETE EVENT SYSTEMS,” in *IFAC-PapersOnLine*, Elsevier B.V., Jul. 2017, pp. 1784–1790. doi: https://doi.org/10.1016/j.ifacol.2017.08.163.

[9] B. De Schutter, T. van den Boom, J. Xu, and S. S. Farahani, “ANALYSIS AND CONTROL OF MAX-PLUS LINEAR DISCRETE-EVENT SYSTEMS: AN INTRODUCTION,” *Discrete Event Dynamic Systems: Theory and Applications*, vol. 30, no. 1, pp. 25–54, Mar. 2020, doi: https://doi.org/10.1007/s10626-019-00294-w.

[10] M. Hoekstra, “CONTROL OF DELAY PROPAGATION IN RAILWAY NETWORKS USING MAX-PLUS ALGEBRA,” Delft University of Technology, Delft, 2020. Accessed: Feb. 02, 2024. [Online]. Available: http://resolver.tudelft.nl/uuid:b5816386-0ed9-4760-a6de-f0db0c3a5226

[11] G. Vissers, “MAX-PLUS EXTENSIONS A STUDY OF TRAIN DELAYS,” Delft University of Technology, Delft, 2022. [Online]. Available: http://repository.tudelft.nl/.

[12] H. Al Bermanei, *Applications of Max-Plus Algebra to Scheduling*. Abo: Abo Akademi University Press, 2021.

[13] C. Martínez-Olvera and J. Mora-Vargas, “A MAX-PLUS ALGEBRA APPROACH TO STUDY TIME DISTURBANCE PROPAGATION WITHIN A ROBUSTNESS IMPROVEMENT CONTEXT,” *Math Probl Eng*, vol. 2018, 2018, doi: https://doi.org/0.1155/2018/1932361.

[14] P. Majdzik, “A FEASIBLE SCHEDULE FOR PARALLEL ASSEMBLY TASKS IN FLEXIBLE MANUFACTURING SYSTEMS,” *International Journal of Applied Mathematics and Computer Science*, vol. 32, no. 1, pp. 51–63, Mar. 2022, doi: https://doi.org/0.34768/amcs-2022-0005.

[15] Z. Sya’diyah, “MAX PLUS ALGEBRA OF TIMED PETRI NET FOR MODELLING SINGLE SERVER QUEUING SYSTEMS,” *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 17, no. 1, pp. 0155–0164, Apr. 2023, doi: https://doi.org/10.30598/barekengvol17iss1pp0155-0164.

[16] B. Heidergott, “A CHARACTERISATION OF (MAX, +)-LINEAR QUEUEING SYSTEMS,” 2000. doi: https://doi.org/10.1023/A:1019102429650.

[17] B. de Schutter, “MAX-ALGEBRAIC SYSTEM THEORY FOR DISCRETE EVENT SYSTEMS,” Katholieke Universiteit Leuven, Leuven, 1996. Accessed: Feb. 02, 2024. [Online]. Available: https://www.dcsc.tudelft.nl/~bdeschutter/pub/rep/phd.pdf.

[18] S. Siswanto, “THE EXISTENCE OF SOLUTION OF GENERALIZED EIGENPROBLEM IN INTERVAL MAX-PLUS ALGEBRA,” *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 17, no. 3, pp. 1341–1346, Sep. 2023, doi: https://doi.org/10.30598/barekengvol17iss3pp1341-1346.

[19] G. Ariyanti, “A NOTE ON THE SOLUTION OF THE CHARACTERISTIC EQUATION OVER THE SYMMETRIZED MAX-PLUS ALGEBRA,” *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 4, pp. 1347–1354, Dec. 2022, doi: https://doi.org/10.30598/barekengvol16iss4pp1347-1354.

[20] A. Abate, A. Cimatti, A. Micheli, and M. Syifa’ul Mufid, “COMPUTATION OF THE TRANSIENT IN MAX-PLUS LINEAR SYSTEMS VIA SMT-SOLVING.” [Online]. Available: https://es.fbk.eu/people/amicheli/resources/formats20/.

[21] E. Altman and D. Fiems, “BRANCHING PROCESSES, THE MAX-PLUS ALGEBRA AND NETWORK CALCULUS.” doi: https://doi.org/10.1007/978-3-642-30782-9_18.

[22] G. Espindola-Winck, L. Hardouin, M. Lhommeau, and R. Santos-Mendes, “CRITERIA STOCHASTIC FILTERING OF MAX-PLUS DISCRETE EVENT SYSTEMS WITH BOUNDED RANDOM VARIABLES,” in *IFAC-PapersOnLine*, Elsevier B.V., Nov. 2022, pp. 13–18. doi: https://doi.org/10.1016/j.ifacol.2023.01.041.

[23] D. Systemes, “SCILAB,” Apr. 01, 2015, *Scilab Enterprises, France*: 5.5.2. Accessed: Oct. 09, 2024. [Online]. Available: https://www.scilab.org/news/scilab-552-release

[24] S. Subiono, D. Adzkiya, and K. Fahim, “TOOLBOX ALJABAR MAX-PLUS DAN PETRINET,” 2016, *Departemen Matematika, Institut Teknologi Sepuluh Nopember, Surabaya*: MAXPLUSV04032016. Accessed: Oct. 23, 2024. [Online]. Available: http://koncomatematika.blogspot.com/2016/05/cara-menginstall-toolbox-max-plus-scilab.html

[25] A. E. S. H. Maharani and A. Suparwanto, “APPLICATION OF SYSTEM MAX-PLUS LINEAR EQUATIONS ON SERIAL MANUFACTURING MACHINE WITH STORAGE UNIT,” *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 2, pp. 525–530, Jun. 2022, doi: https://doi.org/10.30598/barekengvol16iss2pp525-530.