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THE RAINBOW VERTEX-CONNECTION NUMBERS OF WHEEL-SHIELD GRAPHS

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ABSTRACT

Let G be a nontrivial simple connected graph, ab be an edge of G and m be an integer greater than or equal to 3. A path of order m, denoted by P_m, is a graph whose vertices can be labelled $v_1, v_2, \ldots, v_m$ such that $E(P_m) = \{v_1v_2, v_2v_3, \ldots, v_{m-1}v_m\}$. A G_{ab}^m-shield graph is a graph obtained by P_m and $m-1$ copies of G such that the ab edge of i-th G embedded to i-th edge of P_m by embedding a to v_i and b to v_{i+1}. A path in a vertex-colored graph is said to be rainbow-vertex path if every internal vertex in the path has different color. A vertex-colored graph is said to be rainbow-vertex connected if for every pair of vertices there exists a rainbow-vertex path connecting them. The rainbow- vertex connection number of G, denoted by $rvc(G)$, is the minimum colors needed to make G rainbow-vertex connected. In this paper, we determine the rainbow-vertex connection numbers of of wheel-shield graphs $(W_n)_{ab}^m$, specifically finding that the number ranges from $m-2$ to $m+1$ depending on the order of the wheel.

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1. INTRODUCTION

Mathematics is a fundamental discipline that holds significant importance due to its extensive and close application in everyday life [1]. One area of mathematics that has extensive applications in everyday life is graph theory. Graph theory garners significant attention due to its practical applications in everyday life, such as scheduling transportation departures [2], organizing course timetables [3], determining the shortest routes [4] cyber security [5] and many other uses [6].

Diestel [7] defines a *graph* G as a pair of two sets $(V(G), E(G))$ where $V(G)$ is called the set of vertices in G and $E(G) \subseteq [V(G)]^2 = \{\{x,y\}|x,y \in V(G), x \neq y\}$ is the set of edges in G. An edge $e = \{x,y\}$ can be written as xy. The number of vertices of a graph G, denoted by $|V(G)|$, is its order and the number of edges, denoted by $|E(G)|$, is its *size*.

The distance from vertex u to vertex v, denoted by $d(u,v)$, is the length of the shortest path connecting u and v if there exists a path connecting them, or infinity if there are no paths connecting u and v. The diameter of G, denoted by $diam(G)$, is max $\{d(u,v)|u,v \in V(G)\}$.

One of the topics in graph theory is graph colorings, which involves associating vertices, edges, or faces with a set of natural numbers [8]. This problem is known as an NP-complete problem and has various real-world applications [9] such as register allocation, scheduling, and frequency assignment in telecommunications, where it is used to model and solve problems involving the assignment of limited resources to conflicting tasks or entities, often represented as nodes and edges in a graph [10].

There are many types of graph colorings. One of which is a rainbow-vertex coloring of graph. It was introduced by [11]. Rainbow coloring, a concept in graph theory where each edge of a graph is assigned a distinct color, has found applications in the study of Hamilton cycles in edge-colored graphs. A Hamilton cycle is a cycle that visits each vertex of the graph exactly once [12]. Recent research has shown that under certain conditions, randomly colored graphs almost surely admit rainbow Hamilton cycles, which are Hamilton cycles where each edge has a different color. These findings provide insights into the structural properties and characteristics of complex networks and graphs [12].

In addition to its theoretical applications, rainbow coloring has also been applied in the domain of data visualization, particularly in the design of effective colormaps for graphical inference tasks [13]. Studies have demonstrated that colormaps featuring a wide range of uniquely nameable colors, such as those found in rainbow colormaps, can enhance cognitive performance in tasks that require model-based judgments and decision-making. This suggests that leveraging the principles of rainbow coloring in the creation of data visualizations can facilitate more accurate interpretation and understanding of complex datasets [13]. By assigning distinct colors to different elements or categories within a visualization, rainbow coloring can help users quickly identify patterns, relationships, and anomalies, ultimately leading to improved comprehension and decision-making based on the visualized information.

A path in a vertex-colored graph is said to be rainbow-vertex path if every internal vertex in the path has different color [14]. A vertex-colored graph is said to be rainbow-vertex connected if for every pair of vertices there exists a rainbow-vertex path. The rainbow-vertex connection number of G, denoted by $rvc(G)$, is the minimum colors needed to make G rainbow-vertex connected. In this paper, we determine the rainbow-vertex connection numbers of G_{ab}^m-shield graphs where G are wheels.

2. RESEARCH METHODS

This research employs a literature study approach (library research), the axiomatic deductive method, and pattern recognition. The axiomatic deductive method involves the use of deductive proofs that are applicable in mathematical logic [15]. The study involves examining books, textbooks, journals, and scientific articles on the number of rainbow vertex connections.

We used several definitions, lemmas, and observations in order to prove the theorems. The definitions, lemmas, and observations are as follows.

Definition 1. [16] Let $\boldsymbol{n}$ be a natural number with $\boldsymbol{n} \geq \mathbf{3}$. A wheel with $\boldsymbol{n} + \mathbf{1}$ vertices, denoted by $\boldsymbol{W_n}$, is a graph which vertices and edges can be defined by

$V(W_n) = \{v, v_i \mid i \in [1, n]\}$ and

$E(W_n) = \{vv_i \mid i \in [1, n]\} \cup \{v_i v_{i+1} \mid i \in [1, n-1]\} \cup \{v_1 v_n\}$, respectively,

The vertex v is called the hub of W_n, see **Figure 1**.

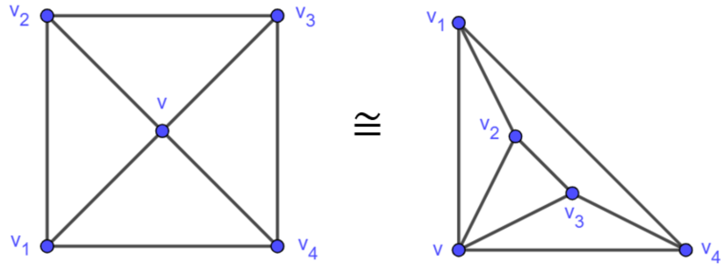

Figure 1. A Wheel with Order 4 (W_4)

Lemma 1. [17] Let G be a non-trivial connected graph with order n and $diam(G)$ be the diameter of G, then $diam(G) - 1 \leq rvc(G) \leq n - 2$.

Lemma 2. [17] [18] Let G be a non-trivial connected graph and c be the number of cut vertices in G, then $rvc(G) \geq c$

Theorem 1. [17] Let $n \geq 3$ and C_n be a cycle graph with order n, then

$$rvc(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 2, & \text{if } n \in \{5, 9\}; \\ \left\lceil \frac{n}{2} \right\rceil - 1, & \text{if } n \in \{3, 4, 6, 7, 8, 10, 11, 12, 13, 15\}; \\ \left\lceil \frac{n}{2} \right\rceil, & \text{if } n = 14 \text{ } or \text{ } n \geq 16. \end{cases}$$

3. RESULTS AND DISCUSSION

Graph operation is a technique for generating a new graph by merging two existing graphs [19]. There are various types of operations, such as joint, corona, comb, shackle, and amalgamation [20] In this paper, the shield graph is obtained by combining two graphs in different way from those operations.

Let G be a nontrivial simple connected graph, ab be an edge of G, and m be an integer at least 3. A *path* of order m, denoted by P_m, is a graph whose vertices can be labelled by $v_1, v_2, \ldots, v_m$ such that $E(P_m) = \{v_1v_2, v_2v_3, \ldots, v_{m-1}v_m\}$. A G_{ab}^m-shield graph is a graph obtained by P_m and $m - 1$ copies of G such that the ab edge of i-th G embedded to i-th edge of P_m by embedding a to v_i and b to v_{i+1}.

3.1 Wheel-Shield Graphs

Let m and n be two natural numbers at least 3, W_n be a wheel with order $n + 1$, and $ab \in E(W_n)$. There are two types of wheel-shield graphs, namely:

1. A wheel-shield graph where neither a nor b is the hub of W_n, denoted by $Sh(m, W_n, ab)$, see **Figure 2** for the example;

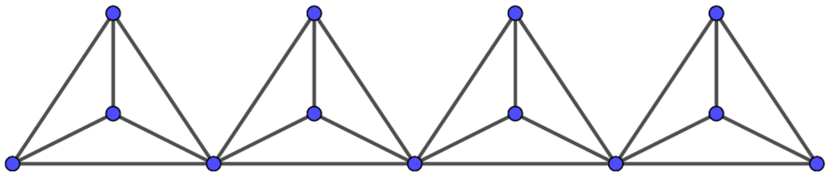

Figure 2. A Wheel-Shield Graph $Sh(5, W_3, ab)$

2. A wheel-shield graph where a is the hub of W_n, denoted by $Sh(m, W_n, \bar{a}b)$, see **Figure 3** for the example.

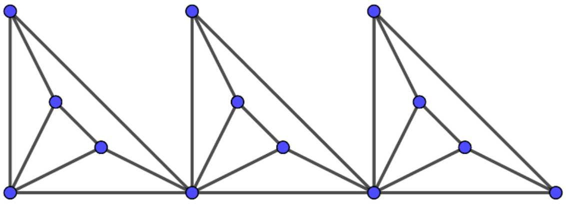

Figure 3. **A Wheel-Shield Graph $Sh(4, W_4, \overline{a}b)$**

Definition 2. Let m and n be two natural numbers at least 3. A *wheel-shield graph* $Sh(m, W_n, ab)$ as shown in **Figure 4** is a graph with the vertex set and the edge set defined as follows, respectively.

$V\big(Sh(m, W_n, ab)\big) = \{p_i | i \in [1, m]\} \cup \{v_j | j \in [1, m-1]\} \cup \{v_{k,j} | k \in [1, n-2], j \in [1, m-1]\}$ and

$$\begin{aligned} E\big(Sh(m, W_n, ab)\big) &= \{p_i p_{i+1} | i \in [1, m-1]\} \cup \{v_j p_i, v_j p_{i+1} \,| i \in [1, m-1], j \in [1, m-1], i = j\} \\ &\cup \{v_j v_{k,j} | k \in [1, n-2], j \in [1, m-1]\} \cup \{v_{k,j} v_{k+1,j} | k \in [1, n-3], j \in [1, m-1]\} \\ &\cup \{v_{1,j} p_i, v_{n-2,j} p_{i+1} | i \in [1, m-1], j \in [1, m-1], i = j\}. \end{aligned}$$

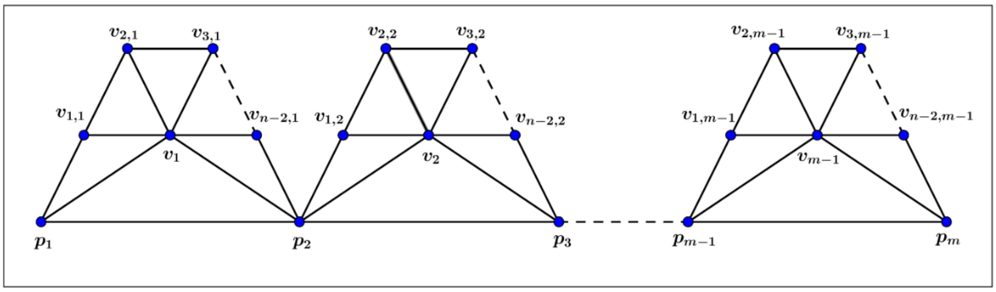

Figure 4. **A Wheel-Shield Graph $Sh(m, W_n, ab)$**

Theorem 2. Let m and n be two natural numbers at least 3, then the rainbow-vertex connection number of $Sh(m, W_n, ab)$, a wheel-shield graph where neither a nor b is the hub of W_n, is

$$rvc(Sh(m, W_n, ab)) = \begin{cases} m-2, & \text{for } n = 3; \\ m+1, & \text{for (even } n \geq 8 \text{ and } m \in [3, n-4]\text{) or} \\ & \text{(odd } n \geq 9 \text{ and } m \in [3, n-5]); \\ m, & \text{for others } m \text{ and } n. \end{cases}$$

Proof. We divide the proof into nine cases as follows.

Case 1. For $n = 3$

(i) It will be shown that $rvc\big(Sh(m, W_3, ab)\big) \geq m-2$. Based on **Lemma 1**, $rvc\big(Sh(m, W_3, ab)\big) \geq diam\,\big(Sh(m, W_3, ab)\big) - 1 = m - 1 - 1 = m - 2$.

(ii) Conversely, it will be shown that $rvc\big(Sh(m, W_3, ab)\big) \leq m-2$. Define a vertex $(m-2)$-coloring $c: V\big(Sh(m, W_3, ab)\big) \to [1, m-2]$ as follows:

$$c(x) = \begin{cases} i-1, & \text{if } x = p_i \text{ for every } i \in [2, m-1]; \\ 1, & \text{otherwise.} \end{cases}$$

For every $x, y \in V\big(Sh(m, W_3, ab)\big)$ there exist $p_i, p_j \in V\big(Sh(m, W_3, ab)\big)$ with $xp_i, p_j y \in E\big(Sh(m, W_3, ab)\big)$ so that the path $x, p_i, p_{i+1}, p_{i+2}, \ldots, p_j, y$ be a vertex rainbow path. Based on (i) and (ii), $rvc\big(Sh(m, W_3, ab)\big) = m-2$. An illustration of this proof can be seen in **Figure 5**.

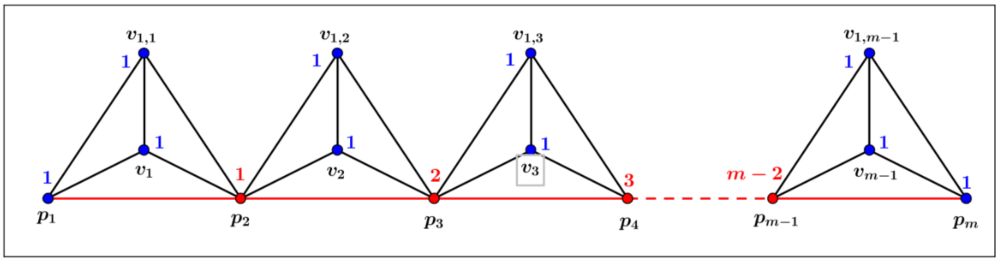

Figure 5. **A rainbow-vertex $(m-2)$-coloring of $Sh(m, W_3, ab)$**

Case 2. For $n = 4$

(i) It will be shown that $rvc(Sh(m, W_4, ab)) \geq m$. Based on **Lemma 1**, $rvc(Sh(m, W_4, ab)) \geq diam\,(Sh(m, W_4, ab)) - 1 = m + 1 - 1 = m$.

(ii) Conversely, it will be shown that $rvc(Sh(m, W_4, ab)) \leq m$. Define a vertex m-coloring $c: V(Sh(m, W_4, ab)) \rightarrow [1, m]$ as follows:

$$c(x) = \begin{cases} i, & \text{if } x = p_i \text{ for every } i \in [1, m]; \\ 1, & \text{otherwise.} \end{cases}$$

For every $x, y \in V(Sh(m, W_4, ab))$ there exist $p_i, p_j \in V(Sh(m, W_4, ab))$ with $xp_i, p_jy \in E(Sh(m, W_4, ab))$ so that the path $x, p_i, p_{i+1}, p_{i+2}, \dots, p_j, y$ be a rainbow-vertex path. Based on (i) and (ii), $rvc(Sh(m, W_4, ab)) = m$. The illustration of this proof can be seen in **Figure 6**.

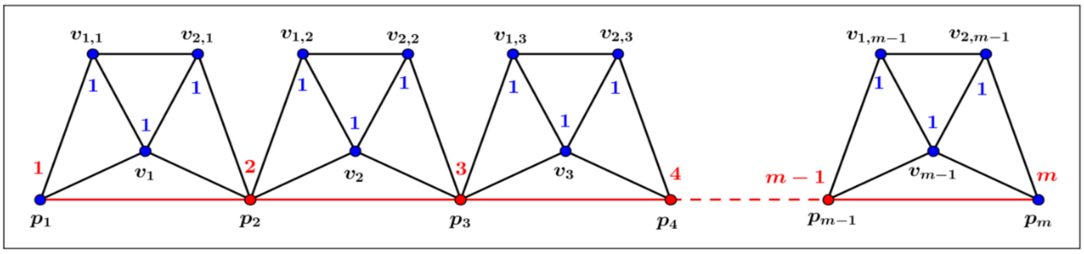

Figure 6. **A Rainbow-Vertex m-Coloring of $Sh(m, W_4, ab)$**

Case 3. For $n = 5$

(i) It will be shown that $rvc(Sh(m, W_5, ab)) \geq m$. Based on Lemma 1, $rvc(Sh(m, W_5, ab)) \geq diam\,(Sh(m, W_5, ab)) - 1 = m + 1 - 1 = m$.

(ii) Conversely, it will be shown that $rvc(Sh(m, W_5, ab)) \leq m$. Define a vertex m-coloring $c: V(Sh(m, W_5, ab)) \rightarrow [1, m]$ as follows:

$$c(x) = \begin{cases} i, & \text{if } x = p_i \text{ for every } i \in [1, m]; \\ j+1, & \text{if } x = v_{1,j} \text{ for every } j \in [1, m-1]; \\ j, & \text{if } x = v_{3,j} \text{ for every } j \in [1, m-1]; \\ 1, & \text{otherwise.} \end{cases}$$

By this coloring, for every two vertices in $Sh(m, W_5, ab)$, there exists a rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 1**.

Table 1. **$x - y$ Rainbow-Vertex Paths in $Sh(m, W_5, ab)$**

Cases	Rainbow Vertex Paths
$xp_i, p_jy \in E(Sh(m, W_5, ab))$	$x, p_i, p_{i+1}, p_{i+2}, \dots, p_j, y$
$xp_i, p_jy \notin E(Sh(m, W_5, ab))$ $\{x, p_i\} \subseteq N(a)$ and $\{p_j, y\} \subseteq N(b)$	$x, a, p_i, p_{i+1}, p_{i+2}, \dots, p_j, b, y$

For another case, there exists a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 1**. Based on (i) and (ii), $rvc(Sh(m, W_5, ab)) = m$.

Case 4. For $n = 6$

(i) It will be shown that $rvc(Sh(m, W_6, ab)) \geq m$. Based on Lemma 1, $rvc(Sh(m, W_6, ab)) \geq diam\ (Sh(m, W_6, ab)) - 1 = m + 1 - 1 = m$.

(ii) Conversely, it will be shown that $rvc(Sh(m, W_6, ab)) \leq m$. Define a vertex m-coloring $c: V(Sh(m, W_6, ab)) \to [1, m]$ as follows:

$$c(x) = \begin{cases} 1, & \text{if } x = v_1; \\ m-1, & \text{if } x = v_i \text{ for every } i \in [2, m-2]; \\ m, & \text{if } x = v_{m-1}; \\ i\,, & \text{if } x = p_i \text{ for every } i \in [1, m]; \\ (i+j) \bmod m, & \text{if } x = v_{i,j} \text{ for every } i \in [1,2] \text{ and } j \in [1, m-1]; \\ (j-1) \bmod m, & \text{if } x = v_{3,j} \text{ for every } j \in [2, m-1]; \\ j, & \text{if } x = v_{4,j} \text{ for every } j \in [1, m-1]. \end{cases}$$

By this coloring, for every two vertices in $Sh(m, W_6, ab)$, there exists a rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 2**.

Table 2. $x - y$ Rainbow-Vertex Paths in $Sh(m, W_6, ab) \cong H$

Cases	Notes	Rainbow Vertex Paths
$d(x, y) = diam\ H$		$x, v_1, p_2, p_3, \dots, p_{m-1}, v_{m-1}, y$
$xp_i, p_jy \in E(H)$		$x, p_i, p_{i+1}, p_{i+2}, \dots, p_j, y$
$xp_i, p_jy \notin E(H)$	$a_1a_2, b_1b_2, xa_1, a_2p_i, p_jb_1, b_2y \in E(H)$	

For another cases, there exist a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 2**. Based on (i) and (ii), $rvc(Sh(m, W_6, ab)) = m$.

Case 5. For $n = 7$

(i) It will be shown that $rvc(Sh(m, W_7, ab)) \geq m$. Based on **Lemma 1**, $rvc(Sh(m, W_7, ab)) \geq diam\ (Sh(m, W_7, ab)) - 1 = m + 1 - 1 = m$.

(ii) Conversely, it will be shown that $rvc(Sh(m, W_7, ab)) \leq m$. Define a vertex m-coloring $c: V(Sh(m, W_7, ab)) \to [1, m]$ as follows:

$$c(v_{i,j}) = \begin{cases} i+j, & \text{for every } i \in [1,2] \text{ and } j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ j-i+7, & \text{for every } i \in [4,5] \text{ and } j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ j-i, & \text{for every } i \in [1,2] \text{ and } j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]; \\ j+i-5, & \text{for every } i \in [4,5] \text{ and } j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]; \\ 1, & \text{for every } i = 3 \text{ and } j \in [1, m-1]. \end{cases}$$

$$c(v_i) = \begin{cases} i, & \text{for every } i \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ i+1, & \text{for every } i \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]. \end{cases}$$

$$c(p_i) = i, \quad \text{for every } i \in [1, m].$$

By this coloring, for every two vertices in $Sh(m, W_7, ab)$, there exists a rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 3**.

Table 3. $v_{i,j} - v_{k,l}$ **Rainbow-Vertex Paths in** $Sh(m, W_7, ab)$

Cases	Rainbow Vertex Paths
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], k \in [1,3], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_{1,l}, v_{2,l}, v_{3,l}$
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], k \in [4,5], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, p_{l+1}, v_{5,l}, v_{4,l}$
$j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], i \in [1,2], j < l$	$v_{2,j}, v_{1,j}, p_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], i \in [3,5], j < l$	$v_{3,j}, v_{4,j}, v_{5,j}, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$

For another case, there exists a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 3**. Based on (i) and (ii), $rvc\big(Sh(m, W_7, ab)\big) = m$.

Case 6. For even number $n \geq 8$ and $3 \leq m \leq n-4$

(i) It will be shown that $rvc\big(Sh(m, W_n, ab)\big) \geq m+1$. Suppose $rvc\big(Sh(m, W_n, ab)\big) \leq m$. Without loss of generality, give a color for $v_1, p_2, p_3, \dots, p_{m-1}, v_{m-1}$ with $1, 2, \dots, m$, respectively, so the $v_{\frac{n-2}{2},1} - v_{\frac{n}{2},m-1}$-path becomes a rainbow-vertex path as shown in **Figure 7**.

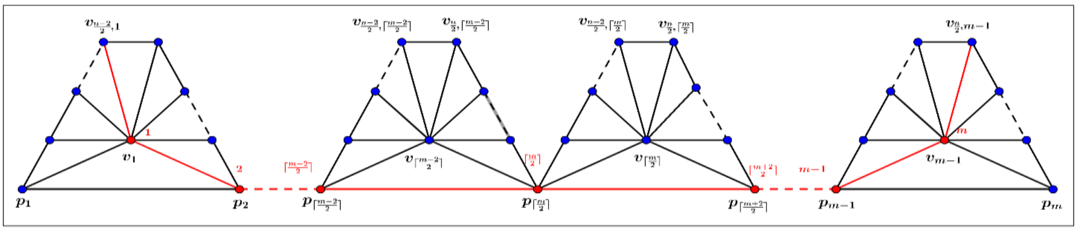

Figure 7. $Sh(m, W_n, ab)$ **with even $n \geq 8$ and $3 \leq m \leq n-4$**

A $v_{\frac{n-2}{2},1} - v_{\frac{n}{2},\lceil\frac{m}{2}\rceil}$-path with $d(v_{\frac{n-2}{2},1} - v_{\frac{n}{2},\lceil\frac{m}{2}\rceil}) \leq m$ must go through $v_{\lceil\frac{m}{2}\rceil}$. Because the colors $1, 2, \dots, \left\lceil\frac{m}{2}\right\rceil$ have been used, the colors that can be used to color $v_{\lceil\frac{m}{2}\rceil}$ are $\left\lceil\frac{m+2}{2}\right\rceil, \left\lceil\frac{m+4}{2}\right\rceil, \dots, m$ as shown in **Figure 8**.

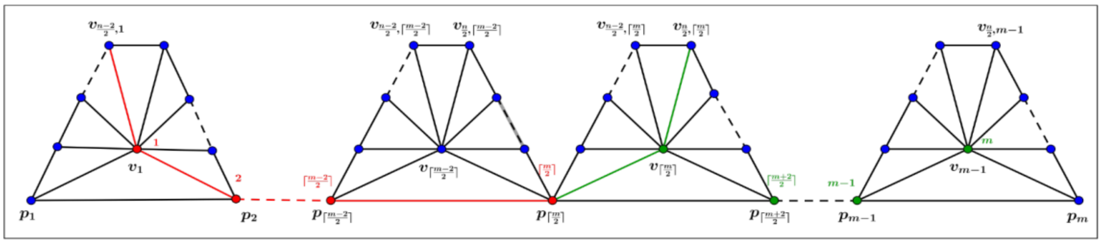

Figure 8. $Sh(m, W_n, ab)$ **with even $n \geq 8$ and $3 \leq m \leq n-4$**

Now, the $v_{\frac{n-2}{2},\lceil\frac{m}{2}\rceil} - v_{\frac{n}{2},m-1}$-path with $d(v_{\frac{n-2}{2},\lceil\frac{m}{2}\rceil} - v_{\frac{n}{2},m-1})$ also must go through $v_{\lceil\frac{m}{2}\rceil}$, as shown in **Figure 9**. Since the colors $\left\lceil\frac{m+2}{2}\right\rceil, \left\lceil\frac{m+4}{2}\right\rceil, \dots, m$ have been used, then the colors that can be used to color $v_{\lceil\frac{m}{2}\rceil}$ are $1, 2, \dots, \left\lceil\frac{m}{2}\right\rceil$. This means there is no color that can be used to color $v_{\lceil\frac{m}{2}\rceil}$. So, $rvc\big(P(m, W_n, ab)\big) \geq m+1$.

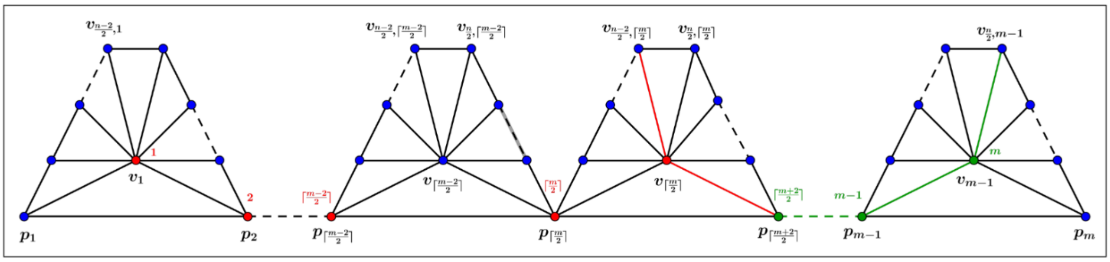

Figure 9. $\boldsymbol{Sh(m, W_n, ab)}$ **with even** $\boldsymbol{n \geq 8}$ **and** $\boldsymbol{3 \leq m \leq n-4}$

(ii) Conversely, it will be shown that $rvc\big(Sh(m, W_n, ab)\big) \leq m+1$. Define a vertex $(m+1)$-coloring $c: V\big(Sh(m, W_3, ab)\big) \to [1, m+1]$ as follows:

$$c(v_{i,j}) = \begin{cases} [(i+j) \bmod m] + 1, & \text{for every } i \in \left[1, \dfrac{n-2}{2}\right] \text{ and } j \in \left[1, \left\lceil \dfrac{m}{2} \right\rceil\right]; \\ [(j-i+n) \bmod m] + 1, & \text{for every } i \in \left[\dfrac{n}{2}, n-2\right] \text{ and } j \in \left[1, \left\lceil \dfrac{m}{2} \right\rceil\right]; \\ [(j-i) \bmod m] + 1, & \text{for every } i \in \left[1, \dfrac{n-2}{2}\right] \text{ and } j \in \left[\left\lceil \dfrac{m+2}{2} \right\rceil, m-1\right]; \\ j+i+2-n, & \text{for every } i \in \left[\dfrac{n}{2}, n-2\right] \text{ and } j \in \left[\left\lceil \dfrac{m+2}{2} \right\rceil, m-1\right]. \end{cases}$$

$$c(v_i) = \begin{cases} i, & \text{for every } i \in \left[1, \left\lceil \dfrac{m-2}{2} \right\rceil\right]; \\ m+1, & \text{for every } i = \left\lceil \dfrac{m}{2} \right\rceil; \\ i+1, & \text{for every } i \in \left[\left\lceil \dfrac{m+2}{2} \right\rceil, m-1\right]. \end{cases}$$

$c(p_i) = i$, for every $i \in [1, m]$.

By this coloring, for every two vertices in $Sh(m, W_n, ab)$ there exists a rainbow-vertex path that connects the two vertices. The rainbow- vertex path can be seen in Table 4.

Table 4. $\boldsymbol{v_{i,j} - v_{k,l}}$ **Rainbow-Vertex Paths in** $\boldsymbol{Sh(m, W_n, ab)}$ **with even** $\boldsymbol{n \geq 8}$ **and** $\boldsymbol{3 \leq m \leq n-4}$

Cases	Rainbow Vertex Path
$j \in \left[1, \left\lceil \frac{m-2}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-2}{2} \right\rceil\right], k \in \left[1, \frac{n}{2}\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_{1,l}, v_{2,l}, \dots, v_{k,l}$
$j \in \left[1, \left\lceil \frac{m-2}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-2}{2} \right\rceil\right], k \in \left[\frac{n+2}{2}, n-2\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, p_{l+1}, v_{n-2,l}, v_{n-3,l}, \dots, v_{\frac{n+2}{2},l}$
$j \in \left[\left\lceil \frac{m+2}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+2}{2} \right\rceil, m-1\right], i \in \left[1, \frac{n-2}{2}\right], j < l$	$v_{\frac{n-2}{2},j}, v_{\frac{n-4}{2},j}, \dots, v_{1,j}, p_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[\left\lceil \frac{m+2}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+2}{2} \right\rceil, m-1\right], i \in \left[\frac{n}{2}, n-2\right], j < l$	$v_{\frac{n}{2},j}, v_{\frac{n+2}{2},j}, \dots, v_{n-2,j}, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[1, \left\lceil \frac{m-2}{2} \right\rceil\right], l \in \left[\left\lceil \frac{m}{2} \right\rceil, m-1\right]$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$

For another cases, there exist a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in Table 4. Based on (i) and (ii), $rvc\big(Sh(m, W_n, ab)\big) = m+1$.

Case 7. For even number $\boldsymbol{n \geq 8}$ **and** $\boldsymbol{m \geq n-3}$

(i) It will be shown that $rvc\big(Sh(m, W_n, ab)\big) \geq m$. Based on Lemma 1, $rvc\big(Sh(m, W_n, ab)\big) \geq diam\,\big(Sh(m, W_n, ab)\big) - 1 = m+1-1 = m$.

(ii) Conversely, it will be shown that $rvc\big(Sh(m, W_n, ab)\big) \leq m$. Define a vertex m-coloring $c: V\big(Sh(m, W_m, ab)\big) \to [1, m]$ as follows:

$$c(v_{i,j}) = \begin{cases} i+j, & \text{for every } i \in \left[1, \frac{n-2}{2}\right] \text{ and } j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ (j-i+n) \bmod m, & \text{for every } i \in \left[\frac{n}{2}, n-2\right] \text{ and } j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ (j-i) \bmod m, & \text{for every } i \in \left[1, \frac{n-2}{2}\right] \text{ and } j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]; \\ j+i+2-n, & \text{for every } i \in \left[\frac{n}{2}, n-2\right], j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]. \end{cases}$$

$$c(v_i) = \begin{cases} i, & \text{for every } i \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right]; \\ i+1, & \text{for every } i \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]. \end{cases}$$

$$c(p_i) = i, \quad \text{for every } i \in [1, m].$$

By this coloring, for every two vertices in $P(m, W_n, ab)$ there exist rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 5**.

Table 5. $v_{i,j} - v_{k,l}$ **Rainbow-Vertex Paths in** $Sh(m, W_n, ab)$ **with even** $n \geq 8$ **and** $m \geq n-3$

Cases	Rainbow Vertex Paths
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], k \in \left[1, \frac{n}{2}\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_{1,l}, v_{2,l}, \dots, v_{\frac{n}{2},l}$
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], k \in \left[\frac{n+2}{2}, n-2\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, p_{l+1}, v_{n-2,l}, v_{n-3,l}, \dots, v_{\frac{n+2}{2},l}$
$j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], i \in \left[1, \frac{n-2}{2}\right], j < l$	$v_{\frac{n-2}{2},j}, v_{\frac{n-4}{2},j}, \dots, v_{1,j}, p_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right], i \in \left[\frac{n}{2}, n-2\right], j < l$	$v_{\frac{n}{2},j}, v_{\frac{n+2}{2},j}, \dots, v_{n-2,j}, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[1, \left\lceil \frac{m-1}{2} \right\rceil\right], l \in \left[\left\lceil \frac{m+1}{2} \right\rceil, m-1\right]$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$

For another cases, there exist a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 5**. Based on (i) and (ii), $rvc(Sh(m, W_n, ab)) = m$.

Case 8. For odd number $n \geq 9$ and $3 \leq m \leq n-5$

(i) It will be shown that $rvc(Sh(m, W_n, ab)) \geq m+1$. Suppose $rvc(Sh(m, W_n, ab)) \leq m$. Without loss of generality, give a color for $v_1, p_2, p_3, \dots, p_{m-1}, v_{m-1}$ with $1, 2, \dots, m$, respectively, so the $v_{\frac{n-1}{2},1} - v_{\frac{n-1}{2},m-1}$-path becomes rainbow vertex path as shown in **Figure 10**.

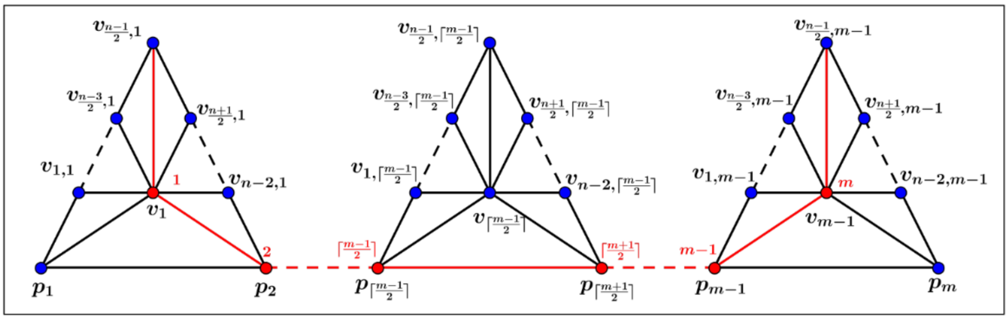

Figure 10. $Sh(m, W_n, ab)$ **with odd** $n \geq 9$ **and** $3 \leq m \leq n-5$

The $v_{\frac{n-1}{2},1} - v_{\frac{n-1}{2},\left\lceil\frac{m-1}{2}\right\rceil}$-path with $d\left(v_{\frac{n-1}{2},1} - v_{\frac{n-1}{2},\left\lceil\frac{m-1}{2}\right\rceil}\right) \le m$ must go through $v_{\left\lceil\frac{m-1}{2}\right\rceil}$. Because the colors $1, 2, \dots, \left\lceil\frac{m-1}{2}\right\rceil$ have been used, then the colors that can be used to color $v_{\left\lceil\frac{m}{2}\right\rceil}$ are $\left\lceil\frac{m+1}{2}\right\rceil, \left\lceil\frac{m+3}{2}\right\rceil, \dots, m$ as shown in **Figure 11**.

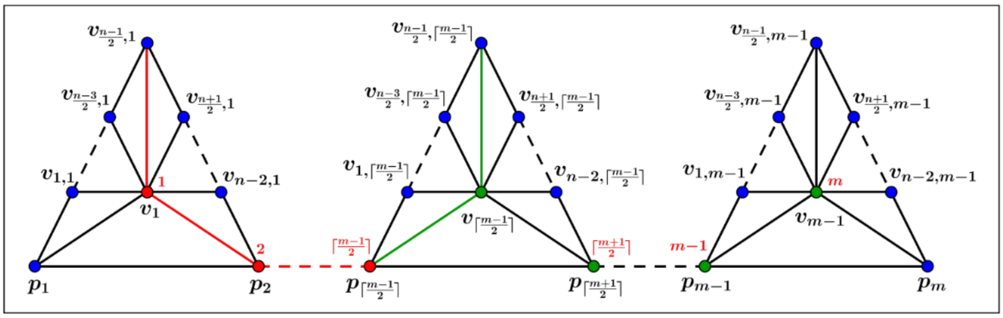

Figure 11. $\boldsymbol{Sh(m, W_n, ab)}$ **with odd** $\boldsymbol{n \ge 9}$ **and** $\boldsymbol{3 \le m \le n-5}$

Now, the $v_{\frac{n-1}{2},\left\lceil\frac{m-1}{2}\right\rceil} - v_{\frac{n-1}{2},m-1}$-path with $d\left(v_{\frac{n-1}{2},\left\lceil\frac{m-1}{2}\right\rceil} - v_{\frac{n-1}{2},m-1}\right)$ also must go through $v_{\left\lceil\frac{m-1}{2}\right\rceil}$ as shown in **Figure 12**. Since the colors $\left\lceil\frac{m+1}{2}\right\rceil, \left\lceil\frac{m+3}{2}\right\rceil, \dots, m$ have been used, then the colors that can be used to color $v_{\left\lceil\frac{m}{2}\right\rceil}$ are $1, 2, \dots, \left\lceil\frac{m-1}{2}\right\rceil$. This means there is no color that can be used to color $v_{\left\lceil\frac{m-1}{2}\right\rceil}$. So, $rvc\big(Sh(m, W_n, ab)\big) \ge m+1$.

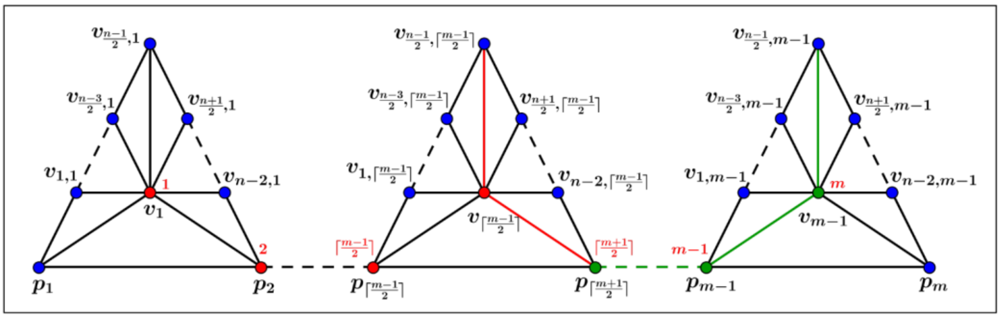

Figure 12. $\boldsymbol{Sh(m, W_n, ab)}$ **with odd** $\boldsymbol{n \ge 9}$ **and** $\boldsymbol{3 \le m \le n-5}$

(ii) Conversely, it will be shown that $rvc\big(Sh(m, W_n, ab)\big) \le m+1$. Define a vertex $(m+1)$-coloring $c: V\big(Sh(m, W_3, ab)\big) \to [1, m+1]$ as follows:

$$c(v_{i,j}) = \begin{cases} (i+j) \bmod m + 1, & \text{for every } i \in \left[1, \frac{n-3}{2}\right] \text{ and } j \in \left[1, \left\lceil\frac{m}{2}\right\rceil\right]; \\ (j-i+n) \bmod m + 1, & \text{for every } i \in \left[\frac{n+1}{2}, n-2\right] \text{ and } j \in \left[1, \left\lceil\frac{m}{2}\right\rceil\right]; \\ (j-i) \bmod m + 1, & \text{for every } i \in \left[1, \frac{n-3}{2}\right] \text{ and } j \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right]; \\ j+i+2-n, & \text{for every } i \in \left[\frac{n+1}{2}, n-2\right] \text{ and } j \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right]; \\ 1, & \text{for every } i = \frac{n-1}{2} \text{ and } j \in [1, m-1]. \end{cases}$$

$$c(v_i) = \begin{cases} i, & \text{for every } i \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right]; \\ m+1, & \text{for every } i = \left\lceil\frac{m}{2}\right\rceil; \\ i+1, & \text{for every } i \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right]. \end{cases}$$

$c(p_i) = i,$ for every $i \in [1, m]$.

By this coloring, for every two vertices in $Sh(m, W_n, ab)$ there exists a rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 6**.

Table 6. $v_{i,j} - v_{k,l}$ **Rainbow-Vertex Paths in** $Sh(m, W_n, ab)$ **with odd** $n \geq 9$ **and** $3 \leq m \leq n-5$

Cases	Rainbow Vertex Paths
$j \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right], l \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right], k \in \left[1, \frac{n-1}{2}\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_{1,l}, v_{2,l}, \dots, v_{\frac{n-1}{2},l}$
$j \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right], l \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right], k \in \left[\frac{n+1}{2}, n-2\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, p_{l+1}, v_{n-2,l}, v_{n-3,l}, \dots, v_{\frac{n+1}{2},l}$
$j \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right], l \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right], i \in \left[1, \frac{n-1}{2}\right], j < l$	$v_{\frac{n-1}{2},j}, v_{\frac{n-3}{2},j}, \dots, v_{1,j}, p_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right], l \in \left[\left\lceil\frac{m+2}{2}\right\rceil, m-1\right], i \in \left[\frac{n+1}{2}, n-2\right], j < l$	$v_{\frac{n+1}{2},j}, v_{\frac{n+3}{2},j}, \dots, v_{n-2,j}, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[1, \left\lceil\frac{m-2}{2}\right\rceil\right], l \in \left[\left\lceil\frac{m}{2}\right\rceil, m-1\right]$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$

For another cases, there exist a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 6**. Based on (i) and (ii), $rvc\big(Sh(m, W_n, ab)\big) = m + 1$.

Case 9. For odd number $n \geq 9$ and $m \geq n-4$

(i) It will be shown that $rvc\big(Sh(m, W_n, ab)\big) \geq m$. Based on **Lemma 1**, $rvc\big(Sh(m, W_n, ab)\big) \geq diam\,\big(Sh(m, W_n, ab)\big) - 1 = m + 1 - 1 = m$.

(ii) Conversely, it will be shown that $rvc\big(Sh(m, W_n, ab)\big) \leq m$. Define a vertex m-coloring $c: V\big(Sh(m, W_m, ab)\big) \to [1, m]$ as follows:

$$c(v_{i,j}) = \begin{cases} (i+j) \bmod m, & \text{for every } i \in \left[1, \frac{n-3}{2}\right] \text{ and } j \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right]; \\ (j-i+n) \bmod m, & \text{for every } i \in \left[\frac{n+1}{2}, n-2\right] \text{ and } j \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right]; \\ (j-i) \bmod m, & \text{for every } i \in \left[1, \frac{n-3}{2}\right] \text{ and } j \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right]; \\ j+i+2-n, & \text{for every } i \in \left[\frac{n+1}{2}, n-2\right] \text{ and } j \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right]; \\ 1, & \text{for every } i = \frac{n-1}{2} \text{ and } j \in [1, m-1]. \end{cases}$$

$$c(v_i) = \begin{cases} i, & \text{for every } i \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right]; \\ i+1, & \text{for every } i \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right]. \end{cases}$$

$c(p_i) = i, i \in [1, m]$.

By this coloring, for every two vertices in $Sh(m, W_n, ab)$ there exist a rainbow-vertex path that connects the two vertices. The rainbow-vertex path can be seen in **Table 7**.

Table 7. $v_{i,j} - v_{k,l}$ **Rainbow-Vertex Paths in** $P(m, W_n, ab)$ **with odd** $n \geq 9$ **and** $m \geq n-4$

Cases	Rainbow Vertex Paths
$j \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right], l \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right], k \in \left[1, \frac{n-1}{2}\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_{1,l}, v_{2,l}, \dots, v_{\frac{n-1}{2},l}$
$j \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right], l \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right], k \in \left[\frac{n+1}{2}, n-2\right], j < l$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, p_{l+1}, v_{n-2,l}, v_{n-3,l}, \dots, v_{\frac{n+1}{2},l}$

Cases	Rainbow Vertex Paths
$j \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right], l \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right], i \in \left[1, \frac{n-1}{2}\right], j < l$	$v_{\frac{n-1}{2},j}, v_{\frac{n-3}{2},j}, \dots, v_{1,j}, p_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right], l \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right], i \in \left[\frac{n+1}{2}, n-2\right], j < l$	$v_{\frac{n+1}{2},j}, v_{\frac{n+3}{2},j}, \dots, v_{n-2,j}, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$
$j \in \left[1, \left\lceil\frac{m-1}{2}\right\rceil\right], l \in \left[\left\lceil\frac{m+1}{2}\right\rceil, m-1\right]$	$v_{i,j}, v_j, p_{j+1}, p_{j+2}, \dots, p_l, v_l, v_{k,l}$

For another case, there exists a rainbow-vertex path which is a subpath of one of rainbow-vertex paths in **Table 6**. Based on (i) and (ii), $rvc(Sh(m, W_n, ab)) = m$.■

Definition 3. Let m and n be two natural numbers at least 3. A *wheel-shield graph* $Sh(m, W_n, \bar{a}b)$ as shown in **Figure 13** is a graph with the vertex set and the edge set defined as follows, respectively.

$V(Sh(m, W_n, \bar{a}b)) = \{p_i | i \in [1, m]\} \cup \{v_{k,j} | k \in [1, n-1], j \in [1, m-1]\}$ and

$$E(Sh(m, W_n, \bar{a}b)) = \{p_i p_{i+1} \,| i \in [1, m-1]\} \cup \{v_{k,j}, p_i | i \in [1, m-1], j \in [1, m-1], k \in [1, n-1]\}$$
$$\cup \{v_{1,j}p_i, v_{n-1,j}p_{i,} | i = j+1, i \in [2, m], j \in [1, m-1]\}.$$

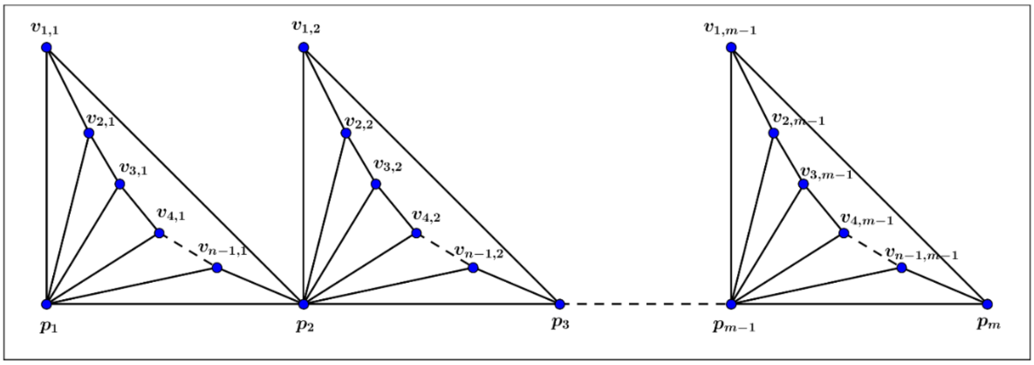

Figure 13. A wheel-Shield Graph $Sh(m, W_n, \bar{a}b)$

Theorem 3. Let m and n be two natural numbers at least 3, then the rainbow-vertex connection number of $Sh(m, W_n, \bar{a}b)$, a wheel-shield graph where a is the hub of W_n, is $rvc(Sh(m, W_n, \bar{a}b)) = m - 1$.

Proof.

(i) It will be shown that $rvc(Sh(m, W_n, \bar{a}b)) \geq m - 1$. Based on **Lemma 1**, $rvc(Sh(m, W_n, \bar{a}b)) \geq diam\,(Sh(m, W_n, \bar{a}b)) - 1 = m - 1$.

(ii) Conversely, it will be shown that $rvc(Sh(m, W_n, \bar{a}b)) \leq m - 1$. Define a vertex $(m-1)$-coloring $c: V(Sh(m, W_n, \bar{a}b)) \to [1, m-1]$ as follows:

$$c(x) = \begin{cases} i, & \text{if } x = p_i \text{ for every } i \in [1, m-1]; \\ 1, & \text{otherwise.} \end{cases}$$

For every $x, y \in V(Sh(m, W_n, \bar{a}b))$, there exist $p_i, p_j \in V(Sh(m, W_n, \bar{a}b))$ with $xp_i, p_j y \in E(Sh(m, W_n, \bar{a}b))$ such that $x, p_i, p_{i+1}, p_{i+2}, \dots, p_j, y$ is a rainbow-vertex path, see **Figure 14**. Based on (i) and (ii), $rvc(Sh(m, W_n, \bar{a}b)) = m - 1$.■

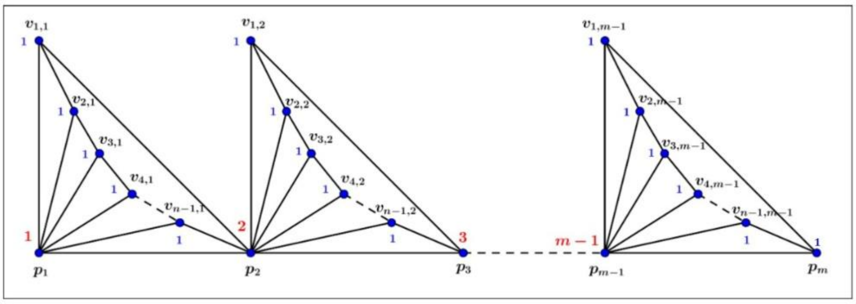

Figure 14. **A rainbow-vertex $(m-1)$-coloring of $Sh(m, W_n, \bar{a}b)$**

For example, $rvc\big(Sh(4, W_4, \bar{a}b)\big) = 3$, see **Figure 15**.

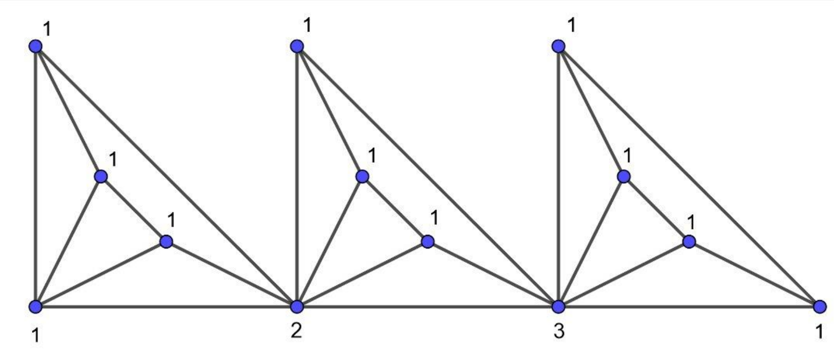

Figure 15. **A Rainbow-Vertex 3-coloring of $Sh(4, W_4, \bar{a}b)$**

4. CONCLUSION

Based on the results and discussion, it can be concluded that the rainbow-vertex connection number for wheel-shield graphs varies from $m-2$ to $m+1$ depending on the order of the wheel.

AUTHOR CONTRIBUTIONS

Ratnaning Palupi: Conceptualization, Data Curation, Formal Analysis, Investigation, Writing-Review and Editing. M. Salman AN: Project Administration, Resources, Supervision, Validation, Writing-Review and Editing. All authors discussed the results and contributed to the final manuscript.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest to report study.

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