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THE UTILIZATION OF TRANSITION MATRIX IN BONUS-MALUS SCHEME FOR DETERMINING MOTOR VEHICLE INSURANCE PREMIUMS

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ABSTRACT

Motor vehicle usage in Indonesia ranks among the highest globally, reaching approximately 141,992,573 units. The growing variety and number of automobiles contribute significantly to traffic congestion and heightened risks to public safety. Given the inherent dangers associated with motorized transportation, including auto theft and accidents, efforts to shift these risks to insurance companies have become crucial. The fundamental idea of insurance is to establish a pool in which policyholders can manage their risk, with premiums determined by the amount of risk that each participant adds to the group. Actuaries in the field of motor vehicle insurance must generate a reasonable premium rate utilizing a variety of methodologies, including the Bonus-Malus approach. The latter, a widely utilized approach, classifies policyholders based on their claims history, incentivizing safe driving. Examining the internal dynamics of the Bonus-Malus system necessitates studying mathematics, particularly algebra, and the use of linear algebra in transition matrices is critical in anticipating changes in bonus-malus rates over time. This research is a quantitative descriptive analysis that explores the implementation of the Bonus-Malus system using a transition matrix framework. It aims to investigate the collaboration of algebra and actuarial science in a real-world application of the Bonus-Malus scheme for motor vehicle insurance, focusing on the use of the transition matrix in premium computation, utilizing secondary data from PT. Jasa Raharja Kota Semarang for the years 2021–2022. The transition matrix analysis shows that Model 2 allows for smoother class transition, lowers the possibility of high-risk class recurrence, and provides more consistent premium adjustments. This demonstrates the model's ability to create a balanced incentive structure while interpreting claim trends. Furthermore, Model 2 has a greater expected value of Loimaranta efficiency than Model 1, supporting findings that added status improves Bonus-Malus system efficiency.

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1. INTRODUCTION

Indonesia is one of the countries with the highest number of motor vehicle users in the world. According to the data provided by the Central Agency of Statistics (BPS) on the development of motorized vehicles number according to type, there are 141,992,573 units of motor vehicles. Pointing out that as of 2020, there were 275,773,800 Indonesian citizens, this number is huge. The growing diversity and quantity of motor vehicles on the roads have led to increased traffic congestion, complicating the risks faced by individuals. Potential hazards in motorized transportation, such as accidents and vehicle loss due to various factors, are certainly possible. Risk, in this context, refers to the potential for loss arising from unforeseen dangers, making it uncertain when or if such events will transpire [1]. One strategy to mitigate these risks involves transferring them to external entities beyond human control. Currently, the insurance company is the recipient of the risk and is deemed capable of managing it

Fundamentally, insurance aims to establish a container for policyholders to manage their risks. If these risks are not all equal, it is appropriate to ask each member to pay a premium that is commensurate to the risk he or she places on the pool [2][3]. When determining rates, it is critical to quantify the underlying risk for each insured party so that claim costs are equitably compensated. In this scenario, one of the challenges that actuaries encounter worldwide in motor vehicle insurance is developing a reasonable tariff structure for policyholders [4]. There are numerous methods for calculating motor vehicle insurance premiums. Premium rates can be determined using the Bayesian technique, as [5] has done. Furthermore, [6] uses the GLM approach to calculate motor vehicle insurance premiums. On the other hand, [7] introduced a bonus-malus system in calculating premiums by using a threshold value that differentiates claim sizes into small and large based on claim frequency and claim severity distribution.

Insurance companies utilize the Bonus-Malus system as a methodology for determining the premiums for motor vehicle insurance. This method implements premium class classifications that are determined by the annual number of claims filed by policyholders [8]. In the Bonus-Malus system, policyholders who have filed one or more claims will face a premium increase known as Malus, whereas policyholders who do not file a claim will receive a reward in the form of a lower premium known as a bonus in the next premium payment period. The Bonus-Malus System (BMS), often known as an experience rating system, is based on historical insurance data. Thus, the Bonus-Malus system is expected to incentivize policyholders to drive more cautiously. Based on these rules, the bonus-malus system can be adjusted to the specific situations and conditions within the insurance environment of a given country. For example, traditional BMS often relies solely on claim frequency, which can be unfair to policyholders with small claims [9][10]. To address this, some systems are being adapted to consider both the frequency and severity of claims, making the system more equitable [11][12]. Adjustments to BMS can also be made by considering additional risk factors such as the responsibility in accidents and the number of claims [13]. This flexibility allows BMS to be fairer and more reflective of actual risk.

Exploring the internal mechanisms of insurance, particularly the Bonus-Malus system, requires studying the fields of mathematics, which is remarkably diverse, encompassing a broad range of specialized areas that explore and deepen our understanding of mathematical concepts. Some main areas of study in mathematics include algebra, mathematical analysis, geometry, number theory, probability and statistics, discrete mathematics, and financial mathematics [14][15][16]. While all areas of study in mathematics are interconnected and mutually supportive, the extensive complexity and diversity within these fields can make it challenging to perceive their interrelationships. With the ongoing development of mathematics, there is a tendency to compartmentalize and separate one area of study from another. This compartmentalization can create difficulties in identifying and comprehending the underlying connections among different branches of study in mathematics [17]. Nevertheless, recognizing the integration and interdependence among these fields becomes crucial. Appreciating the interconnections allows for a more profound and comprehensive understanding of mathematics as a whole. Opening up perspectives on these relationships can lead to a richer and more applicable comprehension of the contributions each field makes in understanding the mathematical world as a unified entity [18][19][20].

Several articles have reviewed the Bonus-Malus system in motor vehicle insurance, including the use of bonus malus in premium calculations by taking into account the frequency of claims with a geometric distribution and the number of claims with a Truncated Weibull distribution, as explained by [21]. [8] applied Bonus-Malus to calculate the amount of motor vehicle insurance premiums based on the type of accident and severity. While [22] and [19] have explored how the Bonus-Malus system is used to calculate motor vehicle insurance prices in numerous countries, including Sweden and Tunisia. However, none have presented a discussion with a focus on the application of algebra within it. Nevertheless, the application of the Bonus-Malus system in motor vehicle insurance necessitates a conceptual understanding of linear algebra, specifically concerning transition matrices.

The transition matrix is one of some key components in understanding and predicting how an individual's Bonus-Malus level may change from one period to the next [22][23]. Nevertheless, when individuals study this material in linear algebra, it is often not emphasized that transition matrices can be applied in everyday life, especially in financial or actuarial mathematics. Therefore, in this research, we are going to show the collaboration between algebra and actuarial science in a real Bonus-Malus scheme for motor vehicle insurance, focusing on how to implement a transition matrix on the scheme for premium calculation and then generating some models in a Bonus-Malus scheme that best suits the data of motor vehicle insurance's policyholder of PT Jasa Raharja Semarang.

2. RESEARCH METHODS

This research is a quantitative descriptive analysis. This research aims to systematically and factually define and comprehend the research object, specifically the use of the transition matrix inside the Bonus-Malus system, utilizing field data. This research employs quantitative analysis of supporting data, including the frequency and number of claims from PT. Jasa Raharja Kota Semarang. This study employs secondary data. The data acquired is a summary of compensation disbursements made by PT Jasa Raharja Kota Semarang from 2021 to 2022. The collection of data was conducted through documentation and literature reviews. Moreover, the phases of data analysis conducted in this research are as follows:

1. Determine the distribution of claim frequency and the claim numbers of PT. Jasa Raharja.
2. Determine the Bonus-Malus Model

   This study employs two Bonus-Malus models developed by [24]: one is the simplest model, while the other is the most efficient model. [24] employed five models in his analysis, with the highest model number delineating various levels characterized by complex regulation. Model 1 will describe the most basic model, while Model 2 will define the most effective model. Both models are unique regarding rules and the number of layers. Model 1 has a rule that determines whether the level moves up or down one level. Policyholders who file a claim in a year will either move down or stay at the lowest level. Policyholders, on the other hand, can move up one level or remain at the top level. New policyholders will be allocated to level 1, which indicates the probability of filing a claim (p) and not filing a claim $(q = 1 - p)$. If the insured makes no claims that year, he or she will be advanced to level 2. However, if one or more claims are submitted, the policyholder will stay at level one. If no claims are lodged in the current year, the policyholder will advance to the next highest level, level 3. If one or more claims are submitted, the policyholder will be moved to level 1. If no claims are filled, the policyholder will remain at level 3, but if one or more claims are filed, the policyholder will be reduced to level 2. Furthermore, model 2 employs a 5-level bonus-malus system with a total of 7 levels. The probability of making a claim is p, while the probability of not making a claim is $1 - p$, which equals q. To minimize misunderstanding, the terms status and level are used in reverse in this model because status 3 and status 4 shares the same level, which is level 3, however, status 5 and status 6 reflect level 4. A new policyholder will be assigned to level 1, specifically status 1. If no claims are lodged, the policyholder will advance to level 2 (status 2). Otherwise, he or she will remain at the lowest level. The policyholder will graduate from level 2 (status 2) to level 3 (status 4) if no claims are lodged in the current year. And so on until level 5 (status 7).
3. Determine the transition matrix and transition graph

   A transition graph is a graphical representation used in stochastic processes, particularly in Markov chains, to illustrate the possible transitions between different levels in a system and the probabilities associated with those transitions. It serves as a visual tool to map the dynamics of a system that moves between discrete levels over time [25][26]. For example, let us consider that we have a Bonus-Malus system where there are three states of premium levels, denoted as vertices 1, 2, and 3. Each state represents a specific premium level based on the customer's claim history:

Vertex 1 represents the initial state (lowest premium level); Vertex 2 represents an intermediate state (medium premium level); Vertex 3 represents the highest premium level.

The transition table provided indicates how the states change based on whether a claim was filed or not:

Table 1. The States Change Based on Whether A Claim Was Filed or Not

Current State	State After No Claims	State After Claim(s) Filed
1	2	1
2	3	1
3	3	2

Using **Table 1**, we can construct the transition graph as follows:

a. From State 1
   i. If no claims are filed, the customer moves to State 2.
   ii. If a claim is filed, the customer remains in State 1.

b. From State 2
   i. If no claims are filed, the customer moves to State 3.
   ii. If a claim is filed, the customer reverts to State 1.

c. From State 3
   i. If no claims are filed, the customer remains in State 3.
   ii. If a claim is filed, the customer moves to State 2.

The sketch of the transition graph with numbered vertices is given below:

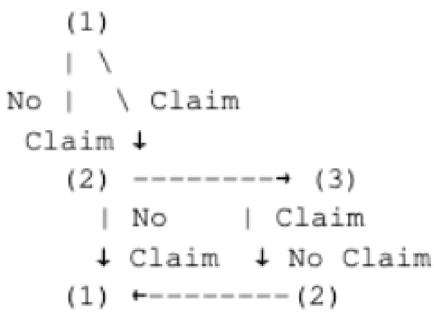

Figure 1. The Transition Graph with Numbered Vertices

In this graph on **Figure 1**, edges represent transitions between premium levels, arrows (edges) show the direction of transition based on customer actions (filing a claim or not filing a claim) and the labels on the edges indicate conditions for each transition.

However, a Bonus-Malus System modifies insurance premiums based on claims history. In this approach, a transition matrix models the likelihood of individuals moving premium classes (or "Bonus-Malus levels") based on claim filing over a given period. The matrix shows the probability of switching premium levels, with a bonus for no claims resulting in reduced premiums and a malus (penalty) for claims resulting in higher premiums [8]. In other words, the matrix lets insurance firms model and anticipate the probability of transitioning from one level to another within a particular period. Let $S = \{s_1, s_2, \cdots, s_n\}$ represent Bonus-Malus system levels or premium levels. P_{ij} reflects the likelihood that an insured individual will move from level s_i to level s_j during a particular period (typically one year) in this system's transition matrix P. The transition matrix is [27][28]:

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{11} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \quad (1)$$

where p_{ii} is probability that the insured stays in the same premium class, and p_{ij}, where $i \neq j$ is probability of moving from one class to another, usually higher if a claim is filed (malus) and lower if no claim is filed (bonus).

4. Checking the stationary distribution of the Bonus-Malus model.

The stationary distribution signifies that the Markov chain model will achieve long-term stability. For a model to achieve a stationary distribution, it must possess irreducibility and aperiodicity. Key principles in Markov chains regarding irreducibility and aperiodicity, as elucidated by [29], can be articulated as follows [30]:

Definition 1. A vector $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \cdots, \pi_t)$ is said to be the stationary distribution of a Markov chain with s level if:

a. $\pi_j \geq 0$ for all j and $\sum_{j=1}^{s} \pi_j = 1$

b. $\vec{\pi} = \vec{\pi}\boldsymbol{P}$ and $\boldsymbol{P}$ is a transition matrix. For instance:

$$[\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} p & q & 0 \\ p & 0 & q \\ 0 & p & q \end{bmatrix} = [\pi_1 \quad \pi_2 \quad \pi_3] \quad (2)$$

Theorem 1. Consider $\boldsymbol{P}$ as an irreducible and aperiodic transition matrix with finitely many levels. The transition matrix $\boldsymbol{P}$ then has a unique stationary distribution $\vec{\pi}$. Furthermore, the unique stationary distribution $\vec{\pi}$ is also called the limiting distri- bution of the Markov chain.

5. Calculate the efficiency of the Bonus-Malus model using Loimaranta efficiency

In this research, two Bonus-Malus systems were constructed, and their efficiency was determined using the Loimaranta efficiency method [31][32]. Generally, [32] define the Loimaranta efficiency as

$$e(\lambda) = \frac{\lambda}{b(\lambda)} \frac{db(\lambda)}{d\lambda} = \frac{dlog\, b(\lambda)}{dlog\, \lambda} \quad (3)$$

With $b(\lambda)$ representing the steady-level premium,

$$b(\lambda) = \sum_{j=1}^{n} \pi_j\, b_j \quad (4)$$

Equation (4) will subsequently serve as the basis for determining the asymptotic premium for all models. Then, the steps to determine Loimaranta efficiency are as follows:

6. Selecting premium level

[32] applied the premium calculation principle, calculating the average present value of total premium payments for policyholders under condition l and frequency claim for the negative binomial distribution θr, $V_l(\theta, r)$ is defined as follows:

$$V_l(\theta, r) = b_l + v \sum_{k=0}^{\infty} \binom{x-1}{r-1} \theta^r (1-\theta)^{x-r} V_{T_k}(l)(\theta) \quad (5)$$

where b_l is premium payment made by a policyholder in the current level l, v is discount factor, $\binom{x-1}{r-1}\theta^r(1-\theta)^{x-r}$ is the negative binomial density function utilized to compute the claim frequency θ. **Equation (5)** can be simplified in vector-matrix form as follows:

$$V(\theta,r) = b + v\,\boldsymbol{P}(\theta,r)^T V(\theta,r)$$

That is, the present value of the total premium payments $V(\theta,r)$ is the sum of the current premium b, plus the present value of future premiums (discount v) multiplied by the transition probability to the new level (via the transition matrix $\boldsymbol{P^T}$) and the vector V itself. Then move the right side to the left and get:

$$V(\theta,r) - v.\boldsymbol{P}(\theta,r)^T V(\theta,r) = b$$

Factorize $V(\theta,r)$, there is

$$[I - v\,\boldsymbol{P}(\theta,r)^T]\,V(\theta,r) = b$$

And obtain

$$V(\theta,r) = [I - v\,\boldsymbol{P}(\theta,r)^T]^{-1}\,b$$

Then the equation above can be simplified as

$$b = V(\theta,r)[I - v\,\boldsymbol{P}(\theta,r)^T] \tag{6}$$

7. Determining the asymptotic premium and its derivatives

The next essential step in calculating Loimaranta efficiency is determining the amount of the asymptotic premium derivative. Using **Definition 1**, we can write

$$\pi(\theta,r).\boldsymbol{P}(\theta,r) = \pi(\theta,r) \tag{7}$$

We refer to the equation

$$b(\theta,r) = \sum_{j=1}^{n} \pi_j(\theta,r)\,b_j \tag{8}$$

thus, we will have derivate the asymptotic premium is

$$\frac{db(\theta,r)}{d(\theta)} = \sum_{j=1}^{n} \pi'_j(\theta,r) b_j \tag{9}$$

where $\pi'_j(\theta,r)$ denotes the derivative of the limiting distribution $\pi_j(\theta,r)$. The subsequent step is differentiating **Equation (7)**, resulting in the equation simplifying to

$$\pi'_j(\theta,r) = \pi(\theta,r)P'^{(\theta,r)[I-P(\theta,r)]^{-1}} \tag{10}$$

$P'(\theta,r)$ is the derivative of asymptotic premium.

8. Calculating the expected value of Loimaranta efficiency

When the claim frequency (λ) is assumed to follow a Negative Binomial distribution, the Loimaranta efficiency is determined as the expected value of the Loimaranta efficiency $E(\lambda)$ for each λ in the distribution. The formula is as follows:

$$E[E_L(\lambda)] = \int_0^\infty f(\lambda;r,\theta)\,E_L(\lambda)\,d\lambda \tag{11}$$

with $E_L(\lambda)$ represents the Loimaranta efficienct for claim frequency λ and $f(\lambda; r, \theta)$ represents the probability of the Gamma distribution (distribution of parameter λ with Negative Binomial). Thus, the expected value of Loimaranta's efficiency can be determined using the following equation:

$$E[E_L(\lambda)] = \int_0^\infty \frac{1}{\Gamma(r)\mu^r} \lambda^{r-1} exp\left(-\frac{\lambda}{\mu}\right) d\lambda \tag{12}$$

3. RESULTS AND DISCUSSION

3.1 Data Distribution

This research used data on the frequency and number of claims at PT Jasa Raharja (a company founded on January 1, 1960, with the ratification of Law No. 19 PRP of 1960 regulating Level-Owned Enterprises whose full capital is the wealth of the Republic of Indonesia, operates in the Social Insurance sector). Data were collected from 2018 to 2022. **Figure 2** (a) describes the data on claims frequency at PT Jasa Raharja Kota Semarang from the year 2018 to 2022, and respectively, **Figure 2** (b) illustrates the number of claims at PT Jasa Raharja from 2018 to 2022.

(a) **(b)**

Figure 2. (a) Frequency and (b) Number of Claims at PT Jasa Raharja from 2018 - 2022

The claim frequency distribution was tested using EasyFit 5.6 Software. The results of the test indicate that the claim frequency data exhibits a specific distribution pattern. According to **Table 2**, the Negative Binomial distribution for 0.0962 and 0.7158 exhibits the lowest values for the Kolmogorov-Smirnov and Anderson-Darling test statistics. Consequently, the most suitable distribution for characterizing the claim frequency data at PT Jasa Raharja is the Negative Binomial.

Table 2. Output of Claim Frequency Distribution Test

Distribution	Kolmogorov-Smirnov		Anderson-Darling		Parameters
	Statistics	Rank	Statistics	Rank	
Uniform	0.12597	2	15.586	2	$\alpha = 62, b = 147$
Geometric	0.46327	4	16.622	3	$p = 0.00949$
Logarithmic	0.70854	5	39.352	5	$\theta = 0.99853$
Neg. Binomial	0.09620	1	0.7158	1	$n = 21, \theta = 0.17071$
Poisson	0.29093	3	28.328	4	$\lambda = 104.4$

Based on the **Table 2**, the claim frequency data from PT Jasa Raharja follows the most suitable distribution which is the Negative Binomial distribution a claim occurrence parameter (θ) of 0.17071. Analogue to this, the Gamma distribution is the best fit for the claim amount data based on the Kolmogorov-Smirnov and Anderson-Darling tests, with values of 0.06597 and 0.29548. Additionally, the parameter estimates for each distribution for the number of claims data from PT Jasa Raharja are presented in **Table 3**.

Table 3. Output of Claim Number Distribution Test

Distribution	Kolmogorov-Smirnov		Anderson-Darling		Parameters
	Statistics	Rank	Statistics	Rank	
Fatigue Life (3P)	0.00800	3	0.30099	3	$\alpha = 0.29587, \beta = 1.5565 \times 10^9, \gamma = 7.1797 \times 10^8$
Gamma	0.06597	1	0.29548	1	$\alpha = 5.1553, \beta = 2.1423 \times 10^8$
Gen. Extreme Value	0.06823	2	0.31774	7	$k = 0.06619, \sigma = 5.1468 \times 10^8, \mu = 2.1288 \times 10^9$
Inv. Gaussian (3P)	0.06918	4	0.31005	5	$\lambda = 1.9219 \times 10^{10}, \mu = 1.653 \times 10^9, \gamma = 6.896 \times 10^8$
Lognormal (30)	0.07871	5	0.32993	8	$\sigma = 0.1996, \mu = 21.554$

Table 3 indicates that the distribution of the claim number data follows a Gamma distribution with parameter $\alpha = 5.1553, \beta = 2.1423 \times 10^8$.The Gamma and Negative Binomial distributions were chosen in this study based not only on data fit tests performed with EasyFit software, but also on relevant theoretical considerations in the context of insurance claim data modeling. The Negative Binomial distribution was chosen to describe claim frequency because it is suitable for overdispersion conditions, which occur when the variance of data exceeds its mean value. This is frequently observed in real claim data and is not well handled by the Poisson distribution. According to [33], the Negative Binomial distribution is a typical actuarial model for claim number because it is more flexible than the Poisson or Geometric distributions. Meanwhile, the Gamma distribution is employed to characterize claim size because it is consistent with the properties of claim data, which are continuous, positive, and often asymmetric. This distribution has long been utilized in risk theory because to its adaptability and capacity to handle real-world data distribution. According to [34], the Gamma distribution is one of the most used choices in the collective approach to insurance risk, especially in computing premiums and reserves. As a result, while alternative distributions such as Poisson and Geometric were evaluated, the Gamma and Negative Binomial distributions were chosen because they are more statistically acceptable and theoretically relevant to the Bonus-Malus system.

3.2 Bonus-Malus (B-M) Modeling

The research will employ two Bonus-Malus models built by [24]. According to the regulations of each model described in the previous section, the transition graph for model 1 and model 2 can be seen in the **Figure 3** and **Figure 4** as follows:

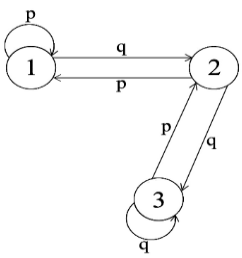

Figure 3. Model 1's Transition Graph

Figure 3 presents a transition graph for Model 1 based on the rules discussed earlier.

Table 4. Summary of Model 1's Transition Graph

Current Level	Level After no Claims	Level After Claim(s) Filled
1	2	1
2	3	1
3	3	2

Furthermore, the transition graph and summary of the transition graph for Model 2 can be seen in the following **Figure 4** and **Table 5**.

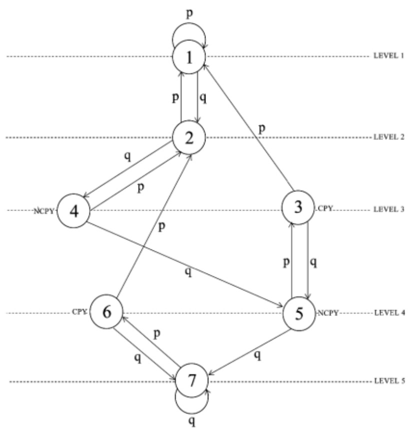

Figure 4. **Model 2's Transition Graph**

The transition graph for Model 2 presented in the **Figure 4** is also based on the rules discussed earlier.

Table 5. **Summary of Model 2's Transition Graph**

Current Level	Level After no Claims	Level After Claim(s) Filled
1	2	1
2	4	1
3	5	1
4	5	2
5	7	3
6	7	2
7	7	6

3.3 Transition Matrix of The Bonus-Malus (B-M) Models

3.3.1 Model 1

Based on the results of checking the distribution of research data, which show that the claim frequency distribution is Negative Binomial with parameters $n = 21$, $\theta = 0.17071$, and the bonus malus model that will be formed in **Figure 3**, the transition matrix of model 1 is as follows.

The transition matrix then can be written as follows,

$$\boldsymbol{P(\theta, r)} = \begin{bmatrix} (1-\theta)^r & 1-(1-\theta)^r & 0 \\ (1-\theta)^r & 0 & 1-(1-\theta)^r \\ 0 & (1-\theta)^r & 1-(1-\theta)^r \end{bmatrix} \quad (13)$$

With $n = 21$ and $\theta = 0.17071$, the transition matrix then can be written as follows,

$$\boldsymbol{P(\theta, r)} = \begin{bmatrix} 0.0196 & 0.9804 & 0.0000 \\ 0.0196 & 0.0000 & 0.9804 \\ 0.0000 & 0.0196 & 0.9804 \end{bmatrix}$$

In summary, a policyholder's probability of moving down one level or remaining at the lowest level is denoted by p, where $p = (1-\theta)^r$. Conversely, q represents the probability of moving up one level or staying at the highest level, where $q = 1-(1-\theta)^r$ and $p + q = 1$. At level one, the probability of staying at the same level is $(1-\theta)^r$. If no claims are filed, the policyholder advances to level 2, with this transition occurring with a probability of $1-(1-\theta)^r$. This straightforward model applies similarly to policyholders at levels 2 or 3.

3.3.2 Model 2

For model 2, the transition matrix is

$$\boldsymbol{P} = \begin{bmatrix} (1-\theta)^r & 1-(1-\theta)^r & 0 & 0 & 0 & 0 & 0 \\ (1-\theta)^r & 0 & 0 & 1-(1-\theta)^r & 0 & 0 & 0 \\ (1-\theta)^r & 0 & 0 & 0 & 1-(1-\theta)^r & 0 & 0 \\ 0 & (1-\theta)^r & 0 & 0 & 1-(1-\theta)^r & 0 & 0 \\ 0 & 0 & (1-\theta)^r & 0 & 0 & 0 & 1-(1-\theta)^r \\ 0 & (1-\theta)^r & 0 & 0 & 0 & 0 & 1-(1-\theta)^r \\ 0 & 0 & 0 & 0 & 0 & (1-\theta)^r & 1-(1-\theta)^r \end{bmatrix} \quad (14)$$

With $n = 21$ and $\theta = 0.17071$, the transition matrix can then be written as follows,

$$\boldsymbol{P} = \begin{bmatrix} 0.0196 & 0.9804 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0196 & 0.0000 & 0.0000 & 0.9804 & 0.0000 & 0.0000 & 0.0000 \\ 0.0196 & 0.0000 & 0.0000 & 0.0000 & 0.9804 & 0.0000 & 0.0000 \\ 0.0000 & 0.0196 & 0.0000 & 0.0000 & 0.9804 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0196 & 0.0000 & 0.0000 & 0.0000 & 0.9804 \\ 0.0000 & 0.0196 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.9804 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0196 & 0.9804 \end{bmatrix}$$

Seven transition probabilities exist. The probability of making a claim is denoted as p, and the probability of not making a claim is denoted as q. Depending on the claims record of the previous year, a policyholder with a claim-free year will advance one level, while a policyholder with a non-claim-free year will slide down one or two levels.

3.4 Stationary Distribution of B-M Models

The last step in Bonus-Malus modeling is to ensure that the model employed in the study exhibits long-term stability. As mentioned before, key indicator of long-term model stability is the stationarity of the model distribution, which must exhibit irreducible and aperiodic properties. Using **Equation (2)**, the $\vec{\pi}$ value for each model is calculated as follows:

$$\overrightarrow{\pi_1} = [0.00039169 \quad 0.01959232 \quad 0.98001599] \quad (15)$$

The vector in **Equation (15)** indicates that in the long term, the probability of a policyholder being at level 1 is 0.00039169; 0.01959232 for level 2; and 0.98001599 for level 3, depending on the initial distribution. The policyholder has the lowest long-term probability of being at level 1 and the highest probability of being at level 3. The identical process is applied for the transition matrix of Model 2, we will have

$$\overrightarrow{\pi_2} = \begin{bmatrix} 0.00000783 \\ 0.00000798 \\ 0.00038385 \\ 0.00000783 \\ 0.01958434 \\ 0.00000767 \\ 0.98000049 \end{bmatrix}$$

The above vector indicates that in the long run, the probability of a policyholder failing at level 1 is 0.00000783; 0.00000798 at level 2; 0.00038385 at level 3; 0.00000783 at level 4; 0.01958434 at level 5; 0.00000767; and 0.98000049 in level 7. The policy-holder has the largest chance of falling at level 7 regardless of the initial distribution; level 2 offers the lowest possibility of falling.

3.5 The B-M Model's Efficiency

The efficiency of the Bonus-Malus model will be calculated using Loimaranta efficiency. The processes in determining Loimaranta efficiency are as follows:

3.5.1 Selecting Premium Level

At this stage, the premiums for each level will be set because the policyholder's premium amount is unknown in this study, the premium level is determined using an assumption. The premium level used in this study is based on the percentage of premium value mentioned by [31]. According to them, the basic premium is calculated using rating parameters such as weight, selling price or car capacity, type of use (personal or business), and type of coverage (comprehensive, third-party only, or mixed). This is the premium that a driver with no previous claims history would pay. A Bonus-Malus scale is used to assign bonuses and penalties based on positive and negative claims experience. After one claim-free year, a person advances up one step and earns a bigger benefit, whereas filing one or more claims moves them down several steps. In theory, the new insured begins that phase with a 100% premium rate.

Table 6. Bonus-Malus System Transition Rules and Premium Percentages

Step	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Percentage	**120**	**100**	**90**	**80**	**70**	**60**	**55**	**50**	**45**	**40**	**37.5**	**35**	**32.5**	**30**
0 claims	2	3	4	5	6	7	8	9	10	11	12	13	14	14
1 claim	1	1	1	1	2	3	4	5	6	7	7	8	8	9
2 claims	1	1	1	1	1	1	1	1	2	3	3	4	4	5
3 claims	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Assume b is the premium amount paid by the policyholder at the lowest level, 1. In this instance, there is no discount at level 1. As a result, the policyholder's premium at level 1 is 1 per unit. At level 2, a 10% discount is applied, resulting in a premium payment of 0.9 for the insured. At level 3, we applied a 20% discount, resulting in a premium payment of 0.8. Based on **Table 6**, the premium level in model 1 can be represented in matrix form as follows:

$$b = \begin{bmatrix} 1 \\ 0.9 \\ 0.8 \end{bmatrix}$$

Furthermore, the premium level for model 2 can be expressed as

$$b = [1 \quad 0.9 \quad 0.8 \quad 0.8 \quad 0.7 \quad 0.7 \quad 0.6]^T$$

3.5.2 Determining the Asymptotic Premium and its Derivatives

The next step is calculating asymptotic premium, $b(\theta)$, using the limiting distribution of each model. Using **Equation (6)**, the asymptotic premium for model 1 is 0.80203757 and model 2 is 0.60204306. The derivative of the asymptotic premium for each model can following be computed using **Equation (7)**. The derivative of the asymptotic premium for Model 1 is

$$\frac{db(\theta,r)}{d\theta} = -0.0479215$$

and for model 2 is

$$\frac{db(\theta,r)}{d\theta} = -0.1635777$$

3.5.3 Calculating the Expected Value of Loimaranta Efficiency

Before determining the expected efficiency of Loimaranta, a graph of Loimaranta efficiency can be created. Using the R program, the combined Loimaranta efficiency graphs for models 1 and 2 can be expressed as follows:

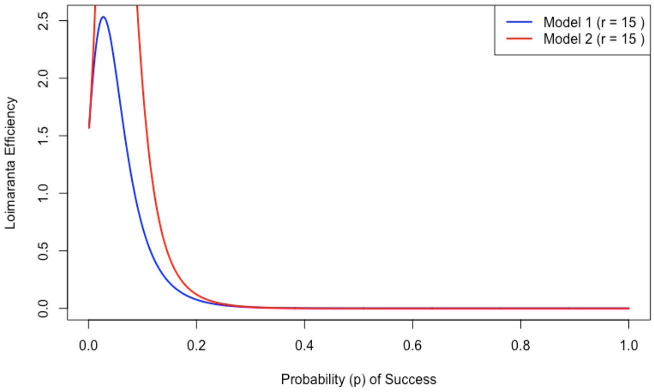

Figure 5. The Combination Graph of Loimaranta Efficiency for Model 1 and Model 2

In **Figure 5**, the efficacy of Loimaranta is compared between Model 1 and Model 2. Model 1 is a straightforward model that employs only three levels, whereas Model 2 is a model that employs seven levels. The plot depicted in the Figure above demonstrates that the Loimaranta efficiency of model 2 is greater than that of model 1, provided that the r and p values of the model are identical. The parameter r specifies the number of claim-free periods desired to be attained, while the parameter p represents the probability of success for policyholders who refrain from filing claims within a specific period. In the same period (r = 15), the efficiency value for model 2 increases significantly, resulting in a relatively significant gap with the efficiency of model 1, as the probability of the policy-holder not filing a claim increase when p = 0. The maximal efficiency of Model 1 is 2.5 when the probability of no claim is approximately 0.1, and its value decreases as the probability of no claim approaches 1. Similarly, the efficiency value of model 2 increases beyond 2.5 and then diminishes until the probability approaches 1. Model 2 is more efficient than Model 1 in light of these findings. If model 2 is the most effective, this study reinforces the research findings conducted by **[24]**. Furthermore, the hypothesis that Bonus-Malus' efficiency increases as the number of rules (status) increases is supported by the fact that the system rules remain consistent **[24]**.

Currently, insufficient data exists to reach a conclusive determination regarding the best model, as it is solely based on graphical observations. The efficiency metrics for models 1 and 2 of Loimaranta indicate that they are not much different. The Loimaranta efficiency value of Model 1 exceeds that of Model 2 when the chance of a policyholder refraining from filing a claim is (θ = 0). As θ rises, the Loimaranta efficiency of Model 2 surpasses that of Model 1. Consequently, the expected value of the Loimaranta efficiency must be computed. Utilizing **Equation (10)** and the R software, the expected value of the Loimaranta efficiency for each model is derived as follows:

Table 7. Expected Loimaranta Efficiency for Model 1 and Model 2

Model	Expected Loimaranta Efficiency
1	0.2089693
2	0.4772151

Table 7 indicates that Model 2 has the greatest efficiency value for Loimaranta, measuring 0.47721511, in comparison to Model 1. Despite Model 1 exhibiting a higher efficiency value compared to Model 2 at the chance of policyholders refraining from submitting a claim, it is ultimately less efficient than Model 2, which achieves a better efficiency value at a more rapid rate. An elevated efficiency rate indicates that the model can accurately represent the frequency of claims. The model can be deemed capable of achieving greater efficiency than other models in a brief period. This attribute is more significant than merely achieving a higher peak efficiency value while being slower in attaining a superior efficiency level. Consequently, we may ascertain that Model 2 is the superior model since it exhibits the highest anticipated Loimaranta efficiency based on the research completed by **[24]**.

4. CONCLUSION

This study obtains several findings, including:

1. The transition matrix is crucial in managing the bonus-malus system. Under the Bonus-Malus system, the transition matrix simulates an insurance policyholder's movement between different levels of risk class or premium category based on their claims history. The transition matrix enables insurance companies to assess efficiency by examining the movement of insureds across classes and its influence on premium revenue.
2. In this study, the Bonus-Malus system is constructed from two models proposed by **[24]** and utilized in vehicle insurance. The Bonus-Malus system was applied using monthly claim frequency data acquired from PT Jasa Raharja between 2018 and 2022, which is distributed Negative Binomial. The Bonus-Malus system's calculation phases yield estimated parameter values, which are used to assess model efficiency. The study found that model 2 is more efficient than model 1. This study's findings support prior research by **[24]**, which found that having additional statuses or levels improves system efficiency.

AUTHOR CONTRIBUTIONS

Seftina Diyah Miasary: Conceptualization, Data Curation, Writing-Review and Editing. Ayus Riana Isnawati: Conceptualization, Formal Analysis, Writing-Review and Editing. Hanna Zhafira Marmu'asyifa: Project Administration. All authors discussed the results and contributed to the final manuscript.

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