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CERTAIN INDEXES OF UNIT GRAPH IN INTEGER MODULO RINGS WITH SPECIFIC ORDERS

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ABSTRACT

Topological indices quantify structural properties of graphs and find wide applications in chemistry, physics, and network analysis. This study investigates several key indices—namely the Harary Index, Wiener Index, Randić Index, Schultz Index, and the Zagreb Indices—within the context of unit graphs derived from the ring of integers modulo. General formulas for these indices are established, demonstrating how they reflect the combinatorial and algebraic characteristics of unit graphs. Each index captures distinct structural aspects: the Wiener Index evaluates global connectivity and correlates with molecular stability and boiling points; the Randić Index highlights molecular branching relevant to enzyme activity; the Harary Index models electronic interactions through distance reciprocals; and the Zagreb Indices and Schultz Index provide insights into bonding properties and molecular interactions. By linking these indices to unit graphs, this work reinforces the synergy between graph theory and algebra, offering a systematic framework to interpret algebraic structures through graph-based invariants. The results not only contribute to theoretical understanding but also suggest potential applications in modeling chemical compounds and complex networks, paving the way for further exploration of topological indices in other algebraically defined graphs.

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1. INTRODUCTION

Graph representations like coprime graphs, non-coprime graphs, and power graphs provide unique insights into algebraic structures. Coprime graphs connect group elements based on its elements order which coprime, while non-coprime graphs highlight complementary relationships [1][2][3]. Power graphs link elements when one is a power of the other, revealing structural hierarchies [4]. Other representations, such as unit graphs and nilpotent graphs, further enrich the study of algebraic systems, making these tools invaluable for visualizing and understanding complex algebraic properties [5][6].

Topological indices are numerical descriptors of graph structures, often utilized to model systems in nature and various scientific disciplines [7]. These indices have garnered significant attention due to their ability to link graph topology with real-world phenomena. In the field of chemistry, they play a crucial role in predicting physicochemical properties, chemical reactivity, and biological activity of compounds [8]. Graph theory is particularly applied in molecular chemistry to represent molecular structures, where nodes signify chemical elements, and edges denote bonds. This method is widely used across disciplines, including group representation [5]. Several key indices hold significant importance both theoretically and practically. One of the notable developments in this area is the growing interest in graph energy, a concept based on the eigenvalues of graph-related matrices. It offers deeper insight into the spectral characteristics of graphs and has proven useful in various fields, including chemistry, physics, and algebraic graph theory [9].

The Wiener Index, one of the oldest topological indices, and most frequently employed, is calculated by summing the all vertex pairs minimal-path distances [10]. In molecular chemistry, it provides essential insights into the stability, boiling points, and other physicochemical characteristics of compounds. Its capability to reflect overall molecular connectivity makes it indispensable in analyzing organic compounds and their interactions [11].

The Randić Index, introduced as a measure of molecular branching, is another influential topological index [12]. This particular index is determined by considering adjacent vertices degrees in a graph. It is particularly useful for evaluating molecular stability, enzyme activity, and the resilience of chemical structures. Its sensitivity to branching patterns makes it highly applicable to the study of hydrocarbons and complex organic molecules.

The Harary Index measures graph connectivity by summing the reciprocals of distances between vertex pairs. This index, closely related to electronic and molecular properties, is extensively used in chemistry to model electronic interactions and predict reactivity in certain compounds [13].

The First Zagreb Index, derived from vertex degrees, is linked to molecular energy and bonding properties [14]. It is instrumental in analyzing molecular stability and forecasting thermodynamic properties. Conversely, the Second Zagreb Index, which is based on edge degrees, provides additional insights into bond distribution and molecular interactions [15].

The Schultz Index integrates both distance and degree data, offering a comprehensive analysis of graph properties. In chemistry, it is applied to model molecular interactions, predict pharmacological activities, and design compounds with specific characteristics. Its adaptability makes it valuable for both theoretical research and practical applications [16].

Topological indices are essential in graph theory, mathematical chemistry, and network analysis by describing the structural attributes of diverse systems. This study investigates the topological indices of unit graphs derived from the ring of integers modulo n, emphasizing graphs of specific orders [17]. The unit graph $G(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ represents the integers modulo n ring, serves as a framework to explore the algebraic and combinatorial features of these rings [18].

To date, there has been no attempt to formulate the Harary Index, Wiener Index, Randić Index, Schultz Index, and Zagreb Indices for Unit Graphs. Therefore, this study aims to provide general formulas for these indices. This focused investigation enhances our understanding of topological indices, bridging theoretical mathematics with practical applications.

2. RESEARCH METHODS

This research uses the deductive method to validate the theorem by drawing specific conclusions from well-established principles. The steps involved are as follows:

1. Clarify Hypotheses and Objectives: Clearly define the theorem and outline any assumptions or initial hypotheses.
2. Gather Fundamental Principles: Compile all relevant axioms, definitions, and previously proven propositions that will be used in the proof.
3. Logical Deduction: Systematically apply deductive reasoning, using these principles in a logical sequence to reach the theorem's conclusion.
4. Case-by-Case Analysis (if applicable): If the theorem applies to multiple scenarios, divide the proof into cases and verify each one independently.
5. Final Summary and Validation: Check each logical step for coherence, confirming that the conclusion logically follows from the initial assumptions to ensure the theorem's validity.

3. RESULTS AND DISCUSSION

Within this part, the researcher examines the topological indices of $G(\mathbb{Z}_n)$ in the context of the integers modulo n ring, focusing on cases where the order is 2^k for some $k \in \mathbb{N}$ and where the order is a prime number. Definitions and theorems are provided as a foundation for proving the primary findings.

The unit graph is a topic within graph theory that models the structure of a ring. In this graph, the vertex set is equal to the elements of the ring. Two distinct vertices are adjacent if and only if their sum is a unit in the ring. This representation provides a valuable tool for studying algebraic properties through graphical structures.

Definition 1 [19]. Let R be a ring with identity. The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R; distinct vertices u and v are adjacent if and only if $u + v$ is a unit of R.

Some studies reveal that unit graphs can take different forms depending on the ring itself. For instance, research by Lestari et al., [19] identified the unit graph structure in the integers modulo n ring for specific orders. Below is the theorem proven in their study.

Theorem 1 [19]. *Let $\mathbb{Z}_n$ be the integers modulo n ring with order 2^k for some $k \in \mathbb{N}$. Then the unit graph $G(\mathbb{Z}_n)$ is a complete bipartite graph.*

This theorem shows that unit graphs in the integers modulo ring with order 2^k for $k \in \mathbb{N}$ form a bipartite graph that is fully connected. Furthermore, the degree of the $v \in V$ $d_v = \frac{n}{2}$.

Theorem 2 [19]. *Let $\mathbb{Z}_n$ be the ring of integers modulo n with an order that is an odd prime number. Then the unit graph $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}$ −partite graph.*

This theorem demonstrates that unit graphs in the ring of integers modulo with odd prime order create a complete $\frac{n+1}{2}$ −partite graph. Consequently, degree of $v \in V$, $d_0 = n - 1$ and $d_v = n - 2$ for $v \in \mathbb{Z}_n \backslash \{0\}$. For an example, for group $\mathbb{Z}_3$, we have the visual of the graph as:

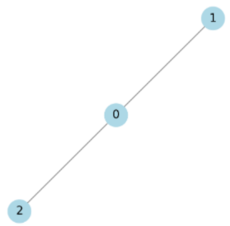

Figure 1. The Unit Graph of the group $\mathbb{Z}_3$

Consider a simple graph G has $V(G)$ represents the vertex set and $E(G)$ represents the edge set. The notation d_u represents the vertex degree, $u \in V(G)$, while $d(u,v)$ represents the vertex pair distence, which is their length of the shortest path. Below are the definitions of the indices of the graph topology discussed in this study.

Definition 2 [20]. The Weiner index of a graph G is denoted by $W(G)$ and defined as:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) \tag{1}$$

Definition 3 [13]. The Randić index of a graph G is denoted by $R(G)$ and defined as:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}} \tag{2}$$

Definition 4 [20]. The Harary index of a graph G is denoted by $H(G)$ and defined as:

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)} \tag{3}$$

Definition 5 [14]. The First Zagreb index of a graph G is denoted by $M_1(G)$ and the Second Zagreb index denoted by $M_2(G)$, defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d_v^2 \tag{4}$$

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v \tag{5}$$

Definition 6 [16]. The Schultz index of a graph G is denoted by $S(G)$ and defined as:

$$S(G) = \sum_{\{u,v\} \subseteq V(G)} (d_u + d_v) d(u,v) \tag{6}$$

The following are the indices of the unit graph's topology in the ring of integers modulo with order $2^k$and odd prime order.

Theorem 3. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is 2^k for some $k \in \mathbb{N}$. Then the Wiener index of $G(\mathbb{Z}_n)$ is $\frac{3}{4}n^2 - n$.*

Proof. According to **Theorem 1**, it has been established that $G(\mathbb{Z}_n)$ forms a bipartite graph that is fully connected. This implies that there are subsets $V_1, V_2 \subset V(G(\mathbb{Z}_n))$, where $\forall u \in V_1$ is adjacent with $\forall v \in V_2$, and within each subset there are no edges exist. Therefore, determining the Wiener index of $G(\mathbb{Z}_n)$ involves two distinct cases, as outlined below.

Case 1.
For $u \in V_1$ and $v \in V_2$, we obtain

$$\sum_{\substack{u \in V_1 \\ v \in V_2}} d(u,v) = \left(\frac{n}{2}\right)^2 (1) = \left(\frac{n}{2}\right)^2 \tag{7}$$

Case 2.
For $u, v \in V_i$ with $i = 1,2$ and $u \neq v$, we obtain

$$\sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} d(u,v) = 2\left[\binom{\frac{n}{2}}{2}(2)\right] = 2\left[\frac{(n)(n-2)}{8}(2)\right] = \frac{(n)(n-2)}{2} \tag{8}$$

referring to **Definition 2** and the two cases discussed earlier, the Wiener index of $G(\mathbb{Z}_n)$ with order 2^k for some $k \in \mathbb{N}$ is

$$W(G(\mathbb{Z}_n)) = \sum_{\substack{u\in V_1\\v\in V_2}} d(u,v) + \sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} d(u,v) = \left(\frac{n}{2}\right)^2 + \frac{(n)(n-2)}{2} = \frac{n^2}{4} + \frac{n^2-2n}{2} = \frac{3n^2-4n}{4}$$
$$= \frac{3}{4}n^2 - n \blacksquare \tag{9}$$

Theorem 4. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is 2^k for some $k \in \mathbb{N}$. Then the Harary index of $G(\mathbb{Z}_n)$ is $\frac{1}{8}(3n^2-2n)$.*

Proof. As stated in **Theorem 1**, $G(\mathbb{Z}_n)$ is a complete graph with two partitions. This indicates the existence of subsets $V_1, V_2 \subset V(G(\mathbb{Z}_n))$, where $\forall u \in V_1$ is adjacent with $\forall v \in V_2$, and within each subset there are no edges exist. As a result, calculating the Harary index of $G(\mathbb{Z}_n)$ involves two distinct cases, as detailed below:

Case 1.
For $u \in V_1$ and $v \in V_2$, we obtain

$$\sum_{\substack{u\in V_1\\v\in V_2}} \frac{1}{d(u,v)} = \left(\frac{n}{2}\right)^2\left(\frac{1}{1}\right) = \left(\frac{n}{2}\right)^2 \tag{10}$$

Case 2.
For $u, v \in V_i$ with $u \neq v$ and $i = 1,2$, we obtain

$$\sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} \frac{1}{d(u,v)} = 2\left[\binom{\frac{n}{2}}{2}\left(\frac{1}{2}\right)\right] = 2\left[\frac{(n)(n-2)}{8}\left(\frac{1}{2}\right)\right] = \frac{(n)(n-2)}{8} \tag{11}$$

According to **Definition 4** and the two cases previously outlined, the Harary index of $G(\mathbb{Z}_n)$ when the order 2^k for some $k \in \mathbb{N}$ is

$$H(G(\mathbb{Z}_n)) = \sum_{\substack{u\in V_1\\v\in V_2}} \frac{1}{d(u,v)} + \sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\u\neq v}} \frac{1}{d(u,v)} = \left(\frac{n}{2}\right)^2 + \frac{(n)(n-2)}{8} = \frac{n^2}{4} + \frac{n^2-2n}{8} = \frac{2n^2+n^2-2n}{8}$$
$$= \frac{1}{8}(3n^2-2n) \blacksquare \tag{12}$$

Theorem 5. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is 2^k for some $k \in \mathbb{N}$. Then the Randić index of $G(\mathbb{Z}_n)$ is $\frac{n}{2}$.*

Proof. As established in **Theorem 1**, $G(Z_n)$ is a complete graph with two partitions. This means there are subsets $V_1, V_2 \subset V(G(Z_n))$, where $\forall u \in V_1$ is adjacent with $\forall v \in V_2$, and within each subset there are no edges. According to **Definition 3**, the Randić index of $G(Z_n)$ is determined by the cardinality of the edge set. Therefore, only one case applies, as outlined below:
For $u \in V_1$ and $v \in V_2$, we obtain

$$R(G(\mathbb{Z}_n)) = \sum_{uv\in E(G(\mathbb{Z}_n))} \frac{1}{\sqrt{d_u d_v}} = \left(\frac{n}{2}\right)^2 \left(\frac{1}{\sqrt{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)}}\right) = \left(\frac{n}{2}\right)^2 \left(\frac{2}{n}\right) = \frac{n}{2} \blacksquare \tag{13}$$

Theorem 6. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is 2^k for some $k \in \mathbb{N}$. Then, the First Zagreb index of $G(\mathbb{Z}_n)$ is $\frac{n^3}{4}$, while the Second Zagreb index of $G(\mathbb{Z}_n)$ is $\left(\frac{n}{2}\right)^4$.*

Proof. $G(Z_n)$ is a bipartite graph that is completely connected, by **Theorem 1**. This means there are subsets $V_1, V_2 \subset V(G(Z_n))$, where where $\forall u \in V_1$ are adjacent with $\forall v \in V_2$, and $|V_1| = |V_2| = \frac{n}{2}$. As a result, degree of $\forall v \in V$ are same, $d_v = \frac{n}{2}$. According to **Definition 5**, the First Zagreb index is based on the vertices degree, while the Second Zagreb index is determined by the degree of the vertices that are adjacent. Therefore, the First Zagreb index is calculated as

$$M_1(G(\mathbb{Z}_n)) = \sum_{v\in V(G(\mathbb{Z}_n))} d_v^2 = n\left(\frac{n}{2}\right)^2 = \frac{n^3}{4} \tag{14}$$

and the Second Zagreb index is calculated as

$$M_2(G(\mathbb{Z}_n)) = \sum_{uv\in E(G(\mathbb{Z}_n))} d_u d_v = \left(\frac{n}{2}\right)^2 \left(\frac{n}{2}\right)^2 = \left(\frac{n}{2}\right)^4 \tag{15}$$

Thus, we have demonstrated that the First Zagreb index of $G(\mathbb{Z}_\mathrm{n})$ is $\frac{\mathrm{n}^3}{4}$ and the Second Zagreb index of $G(\mathbb{Z}_\mathrm{n})$ is $\left(\frac{\mathrm{n}}{2}\right)^4$ $\blacksquare$

Theorem 7. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is 2^k for some $k \in \mathbb{N}$. Then the Schultz index of $G(\mathbb{Z}_n)$ is $n^2(n-1)$.*

Proof. As establihed in **Theorem 1**, it can be concluded that $G(Z_n)$ forms a bipartite graph that is complete. This means that there are subsets $V_1, V_2 \subset V(G(Z_n))$, where each vertex in V_1 is adjecent with each vertex in V_2, and no edges exist within each subset. Therefore, two cases must be considered to calculate the Schultz index of $G(Z_n)$, as described below:

Case 1.
For $u \in V_1$ and $v \in V_2$, we obtain

$$\sum_{\substack{u\in V_1\\ v\in V_2}} (d_u + d_v)d(u,v) = \left(\frac{n}{2}\right)^2 (n)(1) = \left(\frac{n}{2}\right)^2 (n) = \left(\frac{n^3}{2}\right) \tag{16}$$

Case 2.
For $u, v \in V_i$ with $u \neq v$ and $i = 1,2$, we obtain

$$\sum_{i=1}^{2} \sum_{\substack{\{u,v\}\subseteq V_i\\ u\neq v}} (d_u + d_v)d(u,v) = 2\binom{\frac{n}{2}}{2}(n)(2) = 2\left(\frac{(n)(n-2)}{8}\right)(2n) = \frac{n^2(n-2)}{2} \tag{17}$$

In accordance with **Definition 6** and the two preceding cases, the Schultz index of $G(\mathbb{Z}_n)$ when the order is 2^k for some $k \in \mathbb{N}$ is

$$S(G(\mathbb{Z}_n)) = \sum_{\substack{u\in V_1\\ v\in V_2}} (d_u + d_v)d(u,v) + \sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\ u\neq v}} (d_u + d_v)d(u,v) = \frac{n^3}{2} + \frac{n^2(n-2)}{2} = \frac{2(n^3 - n^2)}{2}$$
$$= n^2(n-1) \blacksquare \quad (18)$$

Theorem 8. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ with an order which is an odd prime number. Then the Wiener index of $G(\mathbb{Z}_n)$ is $\frac{1}{2}(n^2-1)$.*

Proof. By **Theorem 2**, $G(Z_n)$ is a $\frac{n+1}{2}$ −partite graph with complete connectivity. This means that there are subsets $V_1, V_2, \dots, V_{\frac{n+1}{2}} \subset V(G(Z_n))$, where every vertex in V_i is connected to all vertices in V_j for $i \neq j$, and no edges exist within any of the subsets. As a result, the Wiener index of $G(Z_n)$ is determined by three distinct cases, as described below:

Case 1.
For $u, v \in \mathbb{Z}_n\backslash\{0\}$ where $u \in V_i$, $v \in V_j$, $i \neq j$, we obtain

$$\sum_{\substack{u\in V_i\\ v\in V_j\\ i\neq j}} d(u,v) = (n-3)\left(\frac{n-1}{2}\right)(1) = \frac{(n-3)(n-1)}{2} \quad (19)$$

Case 2.
For $u, v \in V_i$ with $u \neq v$ and $i = 1,2,\dots,\frac{n-1}{2}$, we obtain

$$\sum_{i=1}^{\frac{n-1}{2}}\sum_{\substack{u,v\in V_i\\ u\neq v}} d(u,v) = \frac{(n-1)}{2}(2) = n-1 \quad (20)$$

Case 3.
For $\{0, v\} \subseteq V(G(\mathbb{Z}_n))$ and $v \in \mathbb{Z}_n\backslash\{0\}$, we obtain

$$\sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))} d(0,v) = (n-1)(1) = n-1 \quad (21)$$

By **Definition 2** and the three cases mentioned before, the Wiener index of $G(\mathbb{Z}_n)$ when the order is an odd prime number is

$$W(G(\mathbb{Z}_n)) = \sum_{\substack{u\in V_i\\ v\in V_j\\ i\neq j}} d(u,v) + \sum_{i=1}^{\frac{n-1}{2}}\sum_{\substack{u,v\in V_i\\ u\neq v}} d(u,v) + \sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))} d(0,v)$$
$$= \frac{(n-3)(n-1)}{2} + (n-1) + (n-1) = \frac{(n-3)(n-1)}{2} + 2(n-1)$$
$$= \frac{(n-3)(n-1) + 4(n-1)}{2} = \frac{1}{2}(n-1)(n+1) = \frac{1}{2}(n^2-1) \blacksquare \quad (22)$$

Theorem 9. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ with an order which is an odd prime number. Then the Harary index of $G(\mathbb{Z}_n)$ is $\frac{(n-1)(2n-1)}{4}$.*

Proof. By **Theorem 2**, $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}$ −partite graph. It means that there are $V_1, V_2, \dots, V_{\frac{n+1}{2}} \subset V(G(\mathbb{Z}_n))$, where $\forall u \in V_i$ is adjacent with $\forall v \in V_j$ with $i \neq j$, and there is no edge within a partition. As a result, the Harary index of $G(Z_n)$ is determined by three cases, as outlined below:

Case 1.
For $u, v \in \mathbb{Z}_n \backslash \{0\}$ where $u \in V_i$, $v \in V_j$, $i \neq j$, we obtain

$$\sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} \frac{1}{d(u,v)} = \frac{(n-3)(n-1)}{2} = (n-3)\left(\frac{n-1}{2}\right)\left(\frac{1}{1}\right) \tag{23}$$

Case 2.
For $u, v \in V_i$ with $i = 1,2,\dots,\frac{n-1}{2}$ and $u \neq v$, we obtain

$$\sum_{i=1}^{\frac{n-1}{2}} \sum_{\substack{u,v \in V_i \\ u \neq v}} \frac{1}{d(u,v)} = \frac{(n-1)}{2}\left(\frac{1}{2}\right) = \frac{(n-1)}{4} \tag{24}$$

Case 3.
For $\{0, v\} \subseteq V(G(\mathbb{Z}_n))$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we obtain

$$\sum_{\{0,v\} \subseteq V(G(\mathbb{Z}_n))} \frac{1}{d(0,v)} = (n-1)(1) = n-1 \tag{25}$$

Referring to **Definition 4** and the three cases mentioned earlier, the Harary index of $G(Z_n)$, when the order is an odd prime number, is

$$\begin{aligned} H(G(\mathbb{Z}_n)) &= \sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} \frac{1}{d(u,v)} + \sum_{i=1}^{\frac{n-1}{2}} \sum_{\substack{u,v \in V_i \\ u \neq v}} \frac{1}{d(u,v)} + \sum_{\{0,v\} \subseteq V(G(\mathbb{Z}_n))} \frac{1}{d(0,v)} \\ &= \frac{(n-1)(n-3)}{2} + \frac{(n-1)}{4} + (n-1) \\ &= \frac{2(n-1)(n-3)}{4} + \frac{(n-1)}{4} + \frac{4(n-1)}{4} \\ &= \frac{2(n-1)(n-3) + 5(n-1)}{4} = \frac{1}{4}(n-1)(2n-1) \blacksquare \end{aligned} \tag{26}$$

Theorem 10. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is an odd prime number. Then the Randić index of $G(\mathbb{Z}_n)$ is* $\frac{1}{2(n-2)}\left[(n-1)(n-3) + 2\sqrt{(n-1)(n-2)}\right]$.

Proof. By **Theorem 2**, $G(\mathbb{Z}_n)$ is a complete $\frac{n+1}{2}$ −partite graph. It means that there are $V_1, V_2, \dots, V_{\frac{n+1}{2}} \subset V(G(\mathbb{Z}_n))$, where each vertex in V_i are adjacent with the each vertex of V_j with $i \neq j$, and there is no edge within a partition. Therefore, there are two cases to calculate the Randić index of $G(Z_n)$, based on the edge set, as outlined below:

Case 1.
For $u, v \in \mathbb{Z}_n \backslash \{0\}$ where $u \in V_i$, $v \in V_j$, $i \neq j$, we obtain

$$\sum_{\substack{u \in V_i \\ v \in V_j \\ i \neq j}} \frac{1}{\sqrt{d_u d_v}} = \frac{(n-3)(n-1)}{2(n-2)} = (n-3)\left(\frac{n-1}{2}\right)\left(\frac{1}{\sqrt{(n-2)^2}}\right) \tag{27}$$

Case 2.
For $\{0, v\} \subseteq V(G(\mathbb{Z}_n))$ and $v \in \mathbb{Z}_n \backslash \{0\}$, we obtain

$$\sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))} \frac{1}{\sqrt{d_0 d_v}} = \frac{(n-1)\sqrt{(n-1)(n-2)}}{(n-1)(n-2)} = (n-1)\left(\frac{1}{\sqrt{(n-1)(n-2)}}\right) = \frac{\sqrt{(n-1)(n-2)}}{(n-2)} \quad (28)$$

According to **Definition 3** and the three cases discussed above, the Randić index of $G(Z_n)$ when the order is an odd prime number is

$$R(G(\mathbb{Z}_n)) = \sum_{\substack{u\in V_i \\ v\in V_j \\ i\neq j}} \frac{1}{\sqrt{d_u d_v}} + \sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))} \frac{1}{\sqrt{d_0 d_v}} = \frac{(n-3)(n-1)}{2(n-2)} + \frac{\sqrt{(n-1)(n-2)}}{(n-2)}$$

$$= \frac{(n-3)(n-1) + 2\sqrt{(n-1)(n-2)}}{2(n-2)}$$

$$= \frac{1}{2(n-2)}\left[(n-3)(n-1) + 2\sqrt{(n-1)(n-2)}\right] \blacksquare \quad (29)$$

Theorem 11. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is an odd prime number. Then the First Zagreb index of $G(\mathbb{Z}_n)$ is $(n-1)[(n-1)+(n-2)^2]$ and the Second Zagreb index of $G(\mathbb{Z}_n)$ is $\frac{1}{2}(n-1)(n-2)[2(n-1)+(n-2)(n-3)]$.*

Proof. Since the order is an odd prime number, $G(Z_n)$ forms a $\frac{n+1}{2}$ –partite graph that is fully connected, where every nonzero vertex $v \in V(G(Z_n))$ has a degree of $n-2$, and the vertex 0 has a degree of $n-1$. Based on **Definition 5**, the First Zagreb index is derived from the degree of each vertex, while the Second Zagreb index is based on the vertex pair degrees. Consequently, the First Zagreb index is determined as

$$M_1(G(\mathbb{Z}_n)) = \sum_{v\in V(G(\mathbb{Z}_n))} d_v^2 = (n-1)^2 + (n-1)(n-2)^2 = (n-1)[(n-1)+(n-2)^2] \quad (30)$$

and the Second Zagreb index is given by

$$M_2(G(\mathbb{Z}_n)) = \sum_{uv\in E(G(\mathbb{Z}_n))} d_u d_v = (n-1)^2(n-2) + (n-3)\frac{(n-1)}{2}(n-2)^2$$

$$= (n-1)(n-2)\left[(n-1) + \frac{(n-2)(n-3)}{2}\right]$$

$$= \frac{1}{2}(n-1)(n-2)[2(n-1)+(n-2)(n-3)] \quad (31)$$

Thus, we have demonstrated that the First Zagreb index of $G(\mathbb{Z}_n)$ is $(n-1)[(n-1)+(n-2)^2]$ and the Second Zagreb index of $G(\mathbb{Z}_n)$ is $\frac{1}{2}(n-1)(n-2)[2(n-1)+(n-2)(n-3)]$. $\blacksquare$

Theorem 12. *Let $G(\mathbb{Z}_n)$ be the unit graph of $\mathbb{Z}_n$ where the order is an odd prime number. Then the Schultz index of $G(\mathbb{Z}_n)$ is $(n-1)[(2n-3)+2(n-2)+(n-2)(n-3)]$.*

Proof. According to **Theorem 2**, $G(\mathbb{Z}_n)$ forms a $\frac{n+1}{2}$ –partite graph that is complete. It means that there are $V_1, V_2, \dots, V_{\frac{n+1}{2}} \subset V(G(\mathbb{Z}_n))$, where each vertex in V_i are adjacent to the each vertex of V_j with $i \neq j$, and there is no edge within a partition. Thus, three cases arise for calculating the Schultz index of $G(Z_n)$, as outlined below:

Case 1.
For $u, v \in \mathbb{Z}_n \setminus \{0\}$ where $u \in V_i$, $v \in V_j$, $i \neq j$, we obtain

$$\sum_{\substack{u\in V_i\\ v\in V_j}}(d_u+d_v)d(u,v)=(n-3)\frac{(n-1)}{2}\big(2(n-2)\big)(1)=(n-1)(n-2)(n-3) \tag{32}$$

Case 2.

For $u,v\in V_i$ with $i=1,2,\dots,\frac{n-1}{2}$ and $u\neq v$, we obtain

$$\sum_{i=1}^{\frac{n-1}{2}}\sum_{\substack{u,v\in V_i\\ u\neq v}}(d_u+d_v)d(u,v)=\frac{(n-1)}{2}\big(2(n-2)\big)(2)=2(n-1)(n-2) \tag{33}$$

Case 3.

For $\{0,v\}\subseteq V(G(\mathbb{Z}_n))$ and $v\in\mathbb{Z}_n\backslash\{0\}$, we obtain

$$\sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))}(d_0+d_v)d(0,v)=(n-1)\big((n-2)+(n-1)\big)(1)=(n-1)(2n-3) \tag{34}$$

Following **Definition 4** and the three cases presented before, the Schultz index of $G(Z_n)$ when the order is an odd prime number is

$$\begin{aligned} S(G(\mathbb{Z}_n)) &= \sum_{\substack{u\in V_i\\ v\in V_j}}(d_u+d_v)d(u,v)+\sum_{i=1}^{2}\sum_{\substack{u,v\in V_i\\ u\neq v}}(d_u+d_v)d(u,v)+\sum_{\{0,v\}\subseteq V(G(\mathbb{Z}_n))}(d_0+d_v)d(0,v)\\ &=(n-1)(n-2)(n-3)+2(n-1)(n-2)+(n-1)(2n-3)\\ &=(n-1)[(n-2)(n-3)+2(n-2)+(2n-3)]\blacksquare \end{aligned} \tag{35}$$

4. CONCLUSION

This study explored how various topological indices behave in unit graphs built from rings of integers modulo certain values. We found that the shape of these graphs—whether bipartite or multipartite—directly affects how the indices describe their structure. Each index gives a unique perspective: some highlight how connected the graph is, others reveal branching patterns or degree interactions. These results show that topological indices can be powerful tools for understanding both algebraic and real-world systems, offering a bridge between abstract math and practical applications in areas like chemistry and network science.

AUTHOR CONTRIBUTIONS

Sahin Two Lestari: Conceptualization, Formal Analysis, Investigation, Methodology, Validation, Writing – Original Draft. Jimboy R. Albaracin: Formal Analysis, Investigation, Visualization, Writing – Review and Editing. I Gede Adhitya Wisnu Wardhana: Funding Acquisition, Project Administration, Resources, Supervision, Validation, Writing – Review and Editing. All authors discussed the results and contributed to the final manuscript.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest to report study

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