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USING A MONOTONE SEQUENCE OF FUNCTIONS TO DETERMINE THE SHORTEST ARC LENGTH OF CIRCLES CONNECTED ANY TWO POINTS ON SPHERE

Muhammad Kabil Djafar [1*], La Ode Safiuddin[2], Lilis Laome[3], Norma Muhtar[4], Herdi Budiman[5], Edi Cahyono[6], La Gubu[7], Alfian[8], Indra Alamsyah[9], Askar Kohalsum[10]

[1,3,4,5,6,7,8,9,10]Mathematical Department, Faculty of Mathematics and Natural Sciences, Universitas Halu Oleo
[2]Department of Physical Education, Faculty of Teaching and Educational Science, Universitas Halu Oleo
Jalan H.E.A. Mokodompit, Kendari, 92321, Indonesia

*Corresponding author's e-mail: ** **muh.kabiljafar@uho.ac.id**

ABSTRACT

This paper discusses about arc length of circles that connected any two points on a sphere. On a sphere, there are infinitely many circles that connect any two points. Using a monotone sequence of functions, we can show that the shortest arc length of circle that connect any two points on sphere is the circle with its center at the origin.

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1. INTRODUCTION

Sequences are among the most interesting topics in mathematics, with applications spanning various fields such as analysis, algebra, and statistics. Within analysis, they are studied in real analysis, complex analysis, functional analysis, and dynamical systems. The notion of sequences is discussed in references [1], [2], [3], and [4].

In statistics, discussion about sequence can be found in sources [5], [6], [7], and [8]. [5] discusses a weak convergence of the sequence of partial sum processes of residuals (PSPR) when observations obtained from a multivariate spatial regression model (SLRM). The result is then used to reconstruct the rejection region of an asymptotic test of hypothesis based on a type of Cramer-von Mises functional of the PSPR. [6] obtains a generalization of the statistical convergence of asymptotically equivalent sequences via the modulus function, yielding a new non-matrix convergence method that intermediates between ordinary convergence and statistical convergence. [7] defines the notion of I-pointwise convergence and I-uniform convergence of a sequence of functions defined on a probabilistic norm space with respect to the probabilistic norm ν. Meanwhile, [8] generalizes the concept in probability of rough Cesaro and lacunary in statistics by introducing the difference operator $\triangle_{\gamma}^{\alpha}$. In this case, α is a proper fraction and $\gamma = (\gamma_{mnk})$ is any fixed sequence of nonzero real or complex numbers.

In real analysis and complex analysis, discussions about sequences can be can be found in sources [9], [10], and [11]. [9] discusses the so-called generalized Fibonacci sequence, deriving an open domain around the origin of the parameter space where the sequence converges to 0. He analyzed interesting behavior on the boundary of that domain. such as convergence to a non-trivial limit, periodic behavior, or quasi-periodic behavior. By carefully choosing initial conditions, the sequence converges to the open domain[10] discusses the concept of quadratic number fields. He considered the continued fraction expansions, fundamental units, and Yokoi invariants Yokoi invariants in terms of Fibonacci sequences. [11] constructs the existence of a completion for a complex valued S-metric space. The completion is constructed using the quotient space of Cauchy sequence equivalence classes within a complex valued S-metric space.

In functional analysis, discussions about sequences can be found in sources [12], [13], [14], and [15]. [12] constructs a new Orlicz sequence space by replacing a function in the Orlicz with a wider function. [13] introduces the new generalized difference sequences spaces in Banach spaces, which arise from the notion of generalized de la Valle Poussins means and the concept of modulus function. [14] establishes a fixed-point result for multivalued mappings satisfying a contractive condition of Reich type only for the elements in a sequence contained in a closed ball in a complete dislocated metric space. [15] defines some sequence spaces on Hilbert space as a domain of triangle Hilbert matrix and studied some inclusion relations concerning these spaces.

In dynamical systems, discussions about sequences can be found in sources [16] and [17]. [16] discusses distributional chaos in a sequence and topologically weak mixing for nonautonomous discrete dynamical systems. [17] discusses Nakano sequence space of fuzzy numbers, especially in the existence solution of the non-linear dynamical system of the Kannan non-expansive type. [18] further explores fixed points for Kannan contraction and non-expansive mapping.

In algebra, discussions about sequences can be found in sources [18] and [19]. [19] investigates the characteristic of a rough V-coexact sequence in a rough group. They find that the rough V-coexact sequence of the rough group is the generalization of the rough exact sequence of the rough group. On the other hand, [20] constructs a perfect magic cube of order 8n for $n \geq 1$. The entries of the perfect magic cube contain an arithmetic sequence. The difference of the sequence is set to find a specific pattern. The algorithm is then implemented into the programming language to solve large orders.

Finally, [11] also discusses the shortest arc length of circles that connect any two places on the Earth. To prove the shortest arc length, they use a monotone differentiable function. By monotone differentiable function, they show that the shortest arc length of any two places on Earth is the circle with center at the origin. In this paper, we use a monotone sequence of functions to determine the shortest arc length of circles connecting any two places on a sphere.

2. RESEARCH METHODS

We begin by considering a sphere with radius $\boldsymbol{R}$. Without loss of generalization, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two points on the sphere with $\boldsymbol{A} = (\boldsymbol{R}, \boldsymbol{\beta_1}, \boldsymbol{\varphi})$ and $\boldsymbol{B} = (\boldsymbol{R}, \boldsymbol{\beta_2}, \boldsymbol{\varphi})$, $\mathbf{0} \leq \boldsymbol{\varphi} \leq \boldsymbol{\pi}/\mathbf{2}$. For simplification, let $\boldsymbol{R} = \mathbf{1}$, $\boldsymbol{\beta_1} = \mathbf{0}$, and $\boldsymbol{\beta_2} = \boldsymbol{\beta}, \mathbf{0} \leq \boldsymbol{\beta} \leq \boldsymbol{\pi}/\mathbf{2}$. Thus, in Cartesian coordinates, we can express the coordinates

$$A = (\cos\varphi, 0, \sin\varphi), \quad B = (\cos\varphi\cos\beta, \cos\varphi\sin\beta, \sin\varphi) \tag{1}$$

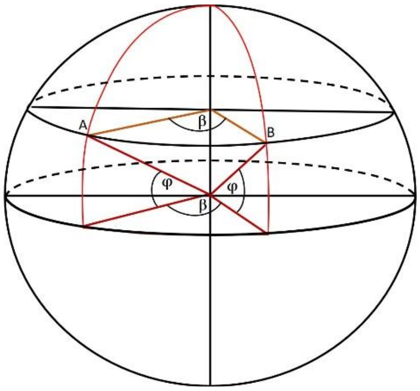

Figure 1. Position of any Two Points *A* and *B* on Sphere

We can construct a sequence of arc length of the circles that connected $\boldsymbol{A}$ and $\boldsymbol{B}$,

Let us define

$$\alpha_n := \left(1 - \frac{1}{n}\right)\varphi \tag{2}$$

Then we have $\mathbf{0} \leq \boldsymbol{\alpha_n} \leq \boldsymbol{\varphi}$. For $\boldsymbol{n} = \mathbf{1}$ we have $\boldsymbol{\alpha_n} = \mathbf{0}$. For $\boldsymbol{n} \to \infty$, $\boldsymbol{\alpha_n} \to \boldsymbol{\varphi}$. Using this fact, we can construct infinitely many circles that passes $\boldsymbol{A}$ and $\boldsymbol{B}$.

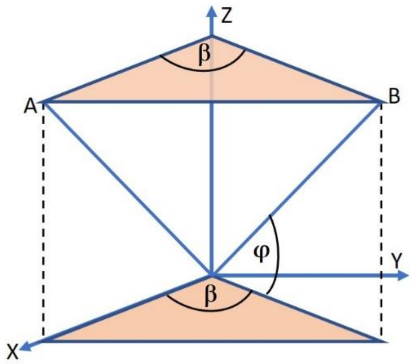

Figure 2. The Polar Coordinates of the Plane $\mathbf{R}^3$

By **Figure 3**, we can express coordinate of the center C_n as

$$O_n = (0, 0, \sin\varphi - \cos\varphi\tan\alpha_n) \tag{3}$$

From **Equation (1)** and **Equation (3)** we get vectors

$$\overrightarrow{O_nA} = \langle cos\,\varphi, 0, cos\,\varphi\, tan\,\alpha_n \rangle, \quad \overrightarrow{O_nB} = \langle cos\,\varphi\, cos\,\beta, cos\,\varphi\, sin\,\beta, cos\,\varphi\, tan\,\alpha_n \rangle \tag{4}$$

with

$$\left\|\overrightarrow{O_nA}\right\| = \left\|\overrightarrow{O_nB}\right\| = cos\,\varphi\, sec\,\alpha_n \tag{5}$$

Using dot product of two vectors, **Equation (4)**, and **Equation (5)**, we have the angle of $\overrightarrow{O_nA}$ and $\overrightarrow{O_nB}$, namely θ_n,

$$\theta_n = cos^{-1}(cos\,\beta\,(cos\,\alpha_n)^2 + (sin\,\alpha_n)^2) \tag{6}$$

We can write **Equation (6)** as

$$\theta_n = cos^{-1}\big(1 - (cos\,\alpha_n)^2(1 - cos\,\beta)\big) \tag{7}$$

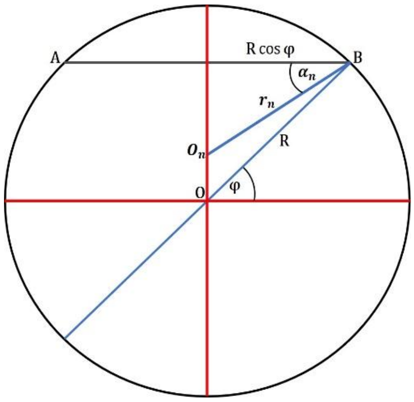

Figure 3. The Polar Coordinates of the Plane R^3

Using **Equation (5)** and **Equation (7)**, arc length of circles C_n, namely L_n, can be written as

$$L_n = \frac{cos\,\varphi}{cos\,\alpha_n} cos^{-1}\left(1-(cos\,\alpha_n)^2(1-cos\,\beta)\right) \tag{8}$$

So, we have sequence of arc length L_n. From **Equation (2)**, $\alpha_n \to \varphi$ for $n \to \infty$. Consequently, $\cos\alpha_n \to \cos\varphi$ when $n \to \infty$. For $\varphi = \pi/2$, $\cos\alpha_n \to \cos\pi/2 = 0$ when $n \to \infty$. This condition still valid for **Equation (8)** with $L_n \to 0$.

3. RESULTS AND DISCUSSION

Let $m, n \in \mathbb{N}$. Using **Equation (8)**, we have sequence of arc lengths L_m and L_n with

$$L_m = \frac{cos\,\varphi}{cos\,\alpha_m} cos^{-1}\left(1-(cos\,\alpha_m)^2(1-cos\,\beta)\right) \tag{9}$$

$$L_n = \frac{cos\,\varphi}{cos\,\alpha_n} cos^{-1}\left(1-(cos\,\alpha_n)^2(1-cos\,\beta)\right) \tag{10}$$

For simplification, we define

$$K_m := \frac{cos\,\varphi}{cos\,\alpha_m}, \qquad K_n := \frac{cos\,\varphi}{cos\,\alpha_n} \tag{11}$$

and

$$M_m \coloneqq cos^{-1}\left(1-(cos\,\alpha_m)^2(1-cos\,\beta)\right), M_n \coloneqq cos^{-1}\left(1-(cos\,\alpha_n)^2(1-cos\,\beta)\right) \tag{12}$$

Lemma 1. *Sequence* $K_n := \frac{cos\,\varphi}{cos\,\alpha_n}$ *is an increasing sequence.*

Proof. For $< n,\ \alpha_m < \alpha_n$. So, we get $\cos\alpha_m > \cos\alpha_n$. Using this result to **Equation (10)** we have

$$\frac{cos\,\varphi}{cos\,\alpha_m} < \frac{cos\,\varphi}{cos\,\alpha_n} \tag{13}$$

Using **Equation (11)** and **Equation (13)** we have, $K_m < K_n$. Then K_n is an increasing sequence.

Lemma 2. *Sequence* $M_n := cos^{-1}\left(1-(cos\,\alpha_n)^2(1-cos\,\beta)\right)$ *is a decreasing sequence.*

Proof. Since $\alpha_m < \alpha_n$, then $\cos\alpha_m > \cos\alpha_n$.

We obtain,

$$\begin{gathered}(cos\,\alpha_m)^2 > (cos\,\alpha_n)^2\\ -(cos\,\alpha_m)^2 < -(cos\,\alpha_n)^2\\ (1-(cos\,\alpha_m)^2) < (1-(cos\,\alpha_n)^2)\\ (1-cos\,\beta)(1-(cos\,\alpha_m)^2) < (1-cos\,\beta)(1-(cos\,\alpha_n)^2)\\ cos^{-1}(1-cos\,\beta)(1-(cos\,\alpha_m)^2) > cos^{-1}(1-cos\,\beta)(1-(cos\,\alpha_n)^2)\end{gathered} \tag{14}$$

Then we have, $M_m > M_n$. So, M_n is a decreasing sequence.

Since $L_n = K_n M_n$, K_n an increasing sequence, and M_n is a decreasing sequence, we can't get a conclusion about this sequence. We have no regularity about this sequence. This sequence is an increasing sequence or a decreasing sequence. So, we try the other way to solve this problem.

Theorem 1. *Sequence* $L_n = \frac{\cos\varphi}{\cos\alpha_n}\cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right)$ *is a decreasing sequence.*

Proof. First, we define

$$f(\beta) := L_m(\beta) - L_n(\beta), m, n \in \mathbb{N}, m < n \tag{15}$$

where

$$L_m(\beta) = \frac{\cos\varphi}{\cos\alpha_m}\cos^{-1}\left(1-(\cos\alpha_m)^2(1-\cos\beta)\right) \tag{16}$$

and

$$L_n(\beta) = \frac{\cos\varphi}{\cos\alpha_n}\cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right) \tag{17}$$

If we differentiate **Equation (15)** respect to β on both sides, we obtain

$$\frac{df(\beta)}{d\beta} = \frac{d\left(L_m(\beta) - L_n(\beta)\right)}{d\beta} \tag{18}$$

Let

$$\theta_m = \cos^{-1}\left(1-(\cos\alpha_m)^2(1-\cos\beta)\right) \tag{19}$$

Then we have

$$\cos\theta_m = \left(1-(\cos\alpha_m)^2(1-\cos\beta)\right) \tag{20}$$

If we differentiate **Equation (20)** respect to β on both sides, we obtain

$$\begin{aligned}\frac{d(\cos\theta_m)}{d\beta} &= \frac{d\left(1-(\cos\alpha_m)^2(1-\cos\beta)\right)}{d\beta}\\ -\sin\theta_m\frac{d\theta_m}{d\beta} &= -(\cos\alpha_m)^2\frac{d(1-\cos\beta)}{d\beta}\\ \frac{d\theta_m}{d\beta} &= \frac{\sin\beta(\cos\alpha_m)^2}{\sin\theta_m}\end{aligned} \tag{21}$$

From **Equation (20)** we obtain

$$\begin{aligned}\sin\theta_m &= \sqrt{1-\left(1-(\cos\alpha_m)^2(1-\cos\beta)\right)^2}\\ \sin\theta_m &= \sqrt{2(1-\cos\beta)(\cos\alpha_m)^2-(1-\cos\beta)^2(\cos\alpha_m)^4}\\ \sin\theta_m &= \sqrt{(1-\cos\beta)(\cos\alpha_m)^2\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}\\ \sin\theta_m &= \cos\alpha_m\sqrt{(1-\cos\beta)\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}\end{aligned} \tag{22}$$

Using **Equation (21)** and **Equation (22)** we obtain

$$\begin{aligned}\frac{d\theta_m}{d\beta} &= \frac{\sin\beta(\cos\alpha_m)^2}{\cos\alpha_m\sqrt{(1-\cos\beta)\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}}\\ \frac{d\theta_m}{d\beta} &= \frac{\sqrt{(\sin\beta)^2}\cos\alpha_m}{\sqrt{(1-\cos\beta)\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}}\\ \frac{d\theta_m}{d\beta} &= \frac{\sqrt{1-(\cos\beta)^2}\cos\alpha_m}{\sqrt{(1-\cos\beta)\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}}\\ \frac{d\theta_m}{d\beta} &= \frac{\sqrt{(1-\cos\beta)(1+\cos\beta)}\cos\alpha_m}{\sqrt{(1-\cos\beta)\left(2-(1-\cos\beta)(\cos\alpha_m)^2\right)}}\end{aligned}$$

$$\frac{d\theta_m}{d\beta} = \frac{\sqrt{(1+\cos\beta)}\cos\alpha_m}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} \tag{23}$$

Let

$$\theta_n = \cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right) \tag{24}$$

Then we obtain

$$\cos\theta_n = \left(1-(\cos\alpha_n)^2(1-\cos\beta)\right) \tag{25}$$

If we differentiate **Equation (25)** respect to β on both sides, we obtain

$$\begin{gathered}\frac{d(\cos\theta_n)}{d\beta} = \frac{d\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right)}{d\beta}\\ -\sin\theta_n\frac{d\theta_n}{d\beta} = -(\cos\alpha_n)^2\frac{d(1-\cos\beta)}{d\beta}\\ \frac{d\theta_n}{d\beta} = \frac{\sin\beta(\cos\alpha_n)^2}{\sin\theta_n}\end{gathered} \tag{26}$$

Using **Equation (25)** we obtain

$$\sin\theta_n = \sqrt{1-\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right)^2} \tag{27}$$

With the similar way as **Equation (22)**, we can simplify **Equation (27)** in the form

$$\sin\theta_n = \cos\alpha_n\sqrt{(1-\cos\beta)(2-(1-\cos\beta)(\cos\alpha_n)^2)} \tag{28}$$

Using **Equation (26)** and **Equation (28)** we obtain

$$\frac{d\theta_n}{d\beta} = \frac{\sin\beta(\cos\alpha_n)^2}{\cos\alpha_n\sqrt{(1-\cos\beta)(2-(1-\cos\beta)(\cos\alpha_n)^2)}} \tag{29}$$

Using the same way as **Equation (23)** we obtain

$$\frac{d\theta_n}{d\beta} = \frac{\sqrt{(1+\cos\beta)}\cos\alpha_n}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}} \tag{30}$$

Using **Equation (15)**, **Equation (19)** and **Equation (13)**, we obtain

$$f(\beta) = \frac{(\cos\varphi)\theta_m}{\cos\alpha_m} - \frac{(\cos\varphi)\theta_n}{\cos\alpha_n} \tag{31}$$

If we differentiate **Equation (31)** on both side respect to β, we obtain

$$\begin{gathered}\frac{df(\beta)}{d\beta} = \frac{\cos\varphi}{\cos\alpha_m}\frac{d\theta_m}{d\beta} - \frac{\cos\varphi}{\cos\alpha_n}\frac{d\theta_n}{d\beta}\\ \frac{df(\beta)}{d\beta} = \cos\varphi\left(\frac{1}{\cos\alpha_m}\frac{d\theta_m}{d\beta} - \frac{1}{\cos\alpha_n}\frac{d\theta_n}{d\beta}\right)\end{gathered} \tag{32}$$

For simplification, lets define

$$P := \frac{1}{\cos\alpha_m}\frac{d\theta_m}{d\beta} - \frac{1}{\cos\alpha_n}\frac{d\theta_n}{d\beta} \tag{33}$$

Using **Equation (23)**, **Equation (29)**, and **Equation (33)**, we obtain

$$\begin{gathered}P = \frac{1}{\cos\alpha_m}\frac{\sqrt{(1+\cos\beta)}\cos\alpha_m}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} - \frac{1}{\cos\alpha_n}\frac{\sqrt{(1+\cos\beta)}\cos\alpha_n}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}}\\ P = \sqrt{(1+\cos\beta)}\left(\frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} - \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}}\right)\end{gathered} \tag{34}$$

For simplification, we define

$$Q := \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} - \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}} \tag{35}$$

For $m < n$, we obtain $\alpha_m < \alpha_n$. Consequently, $\cos\alpha_m > \cos\alpha_n$. Then,

$$\begin{gathered}(\cos\alpha_m)^2 > (\cos\alpha_n)^2 \\ (1-\cos\beta)(\cos\alpha_m)^2 > (1-\cos\beta)(\cos\alpha_n)^2 \\ -(1-\cos\beta)(\cos\alpha_m)^2 < -(1-\cos\beta)(\cos\alpha_n)^2 \\ (2-(1-\cos\beta)(\cos\alpha_m)^2) < (2-(1-\cos\beta)(\cos\alpha_n)^2) \\ \sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)} < \sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)} \\ \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} > \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}}\end{gathered} \tag{36}$$

Finally, using **Equation (35)** and **Equation (36)**, we obtain

$$Q = \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_m)^2)}} - \frac{1}{\sqrt{(2-(1-\cos\beta)(\cos\alpha_n)^2)}} > 0 \tag{37}$$

Since $\sqrt{(1+\cos\beta)} > 0$ for $0 \le \beta \le \pi/2$, using **Equation (34)** and **Equation (37)** we obtain

$$P = \sqrt{(1+\cos\beta)}Q > 0 \tag{38}$$

Using **Equation (32)**, **Equation (33)**, and **Equation (38)** we obtain

$$\frac{df(\beta)}{d\beta} = \cos\varphi\left(\frac{1}{\cos\alpha_m}\frac{d\theta_m}{d\beta} - \frac{1}{\cos\alpha_n}\frac{d\theta_n}{d\beta}\right) > 0 \tag{39}$$

Consequently, $f(\beta)$ is an increasing function. Then, for $\beta_1 < \beta_2$, we have $f(\beta_1) < f(\beta_2)$. Using **Equation (15)**, $\beta_1 = 0$ and $\beta_2 = \beta > 0$, we obtain

$$f(0) = L_m(0) - L_n(0) < L_m(\beta) - L_n(\beta) = f(\beta) \tag{40}$$

Using **Equation (16)** and $\beta = 0$, we obtain

$$L_m(0) = \frac{\cos\varphi}{\cos\alpha_m}\cos^{-1}\left(1-(\cos\alpha_m)^2(1-\cos 0)\right) \tag{41}$$

Since $\cos 0 = 1$, from **Equation (41)** we obtain

$$L_m(0) = \frac{\cos\varphi}{\cos\alpha_m}\cos^{-1}(1) = 0 \tag{42}$$

With the similar way, we obtain

$$L_n(0) = \frac{\cos\varphi}{\cos\alpha_n}\cos^{-1}(1) = 0 \tag{43}$$

Using **Equation (40)**, **Equation (42)**, and **Equation (43)** we obtain

$$0 < L_m(\beta) - L_n(\beta)$$

Then, for any $\beta, 0 \le \beta \le \pi/2$, and $m < n, m, n \in \mathbb{N}$, we obtain

$$L_n(\beta) < L_m(\beta) \tag{44}$$

As a result, L_n is a decreasing sequence.

Furthermore, we will show boundedness of L_n sequence.

Theorem 2. *Sequence* $L_n = \frac{\cos\varphi}{\cos\alpha_n}\cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right)$ *is bounded.*

Proof. Let's define

$$K_n := \frac{\cos\varphi}{\cos\alpha_n} \tag{45}$$

Since $0 \le \alpha_n \le \varphi$, then we have

$$1 \ge \cos\alpha_n \ge \cos\varphi \tag{46}$$

Using this result, we have

$$1 \le \frac{1}{\cos\alpha_n} \le \frac{1}{\cos\varphi} \tag{47}$$

If we multiply each term of **Equation (47)** by $\cos\varphi$ we get

$$\cos\varphi \le \frac{\cos\varphi}{\cos\alpha_n} = K_n \le 1 \tag{48}$$

Consequently, K_n is a bounded sequence. K_n is bounded below by $\cos\varphi$ and bounded above by 1.

Furthermore, let's define

$$M_n := \cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right) \tag{49}$$

sing **Equation (46)** we obtain

$$\begin{gathered}
\cos\varphi \le \cos\alpha_n \le 1 \\
(\cos\varphi)^2 \le (\cos\alpha_n)^2 \le 1 \\
(1-\cos\beta)(\cos\varphi)^2 \le (1-\cos\beta)(\cos\alpha_n)^2 \le (1-\cos\beta) \\
-(1-\cos\beta)(\cos\varphi)^2 \ge -(1-\cos\beta)(\cos\alpha_n)^2 \ge -(1-\cos\beta) \\
1-(1-\cos\beta)(\cos\varphi)^2 \ge 1-(1-\cos\beta)(\cos\alpha_n)^2 \ge 1-(1-\cos\beta) \\
1-(1-\cos\beta)(\cos\varphi)^2 \ge 1-(1-\cos\beta)(\cos\alpha_n)^2 \ge (\cos\beta) \\
\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2) \le \cos^{-1}(1-(1-\cos\beta)(\cos\alpha_n)^2) \le \cos^{-1}(\cos\beta) \\
\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2) \le \cos^{-1}(1-(1-\cos\beta)(\cos\alpha_n)^2) \le \beta
\end{gathered} \tag{50}$$

So, we have M_n is a bounded sequence. M_n bounded below by $\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)$ and bounded above by β.

From **Equation (17)**, **Equation (45)**, and **Equation (49)** we can write L_n as

$$L_n = K_n M_n \tag{51}$$

Using **Equation (48)**, **Equation (50)**, and **Equation (51)** we have

$$\cos\varphi\left(\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)\right) \le L_n \le \beta \tag{52}$$

As a result, L_n is a bounded sequence. L_n is bounded below by $\cos\varphi\left(\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)\right)$ and bounded above by β.

We have shown that L_n is decreasing and bounded. Furthermore, we will show convergence of L_n.

Theorem 3. *Sequence* $L_n = \frac{\cos\varphi}{\cos\alpha_n}\cos^{-1}\left(1-(\cos\alpha_n)^2(1-\cos\beta)\right)$ *is convergent. Its convergence to its infimum.*

Proof. From **Equation (45)** and **Equation (48)** we know that K_n is bounded and increasing. So, K_n is convergent. From definition of α_n, $\alpha_n \to \varphi$ for $n \to \infty$. As a result, $K_n \to 1$ for $n \to \infty$. Furthermore, from **Equation (49)** and **Equation (50)** M_n is bounded and decreasing. Consequently, M_n convergent. From **Equation (49)**, **Equation (50)**, and using the fact that $\alpha_n \to \varphi$ for $n \to \infty$, M_n converge to $\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)$. Finally, using the fact K_n and M_n convergent and $L_n = K_n M_n$, then L_n convergent. Using **Theorem 1** and **Theorem 2**, L_n is decreasing and bounded. So, L_n converge to $\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)$. This is the infimum of L_n. Thus, the shortest arc length of circles passes any two points on sphere is $\cos^{-1}(1-(1-\cos\beta)(\cos\varphi)^2)$, which is the arc length of circle with the center at the origin.

4. CONCLUSIONS

Using a monotone sequence of functions, we show that the shortest arc length of the circles that connect any two points on a sphere is the circle with its center at the origin.

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