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MODEL APPROACH OF AGGREGATE RETURN VOLATILITY: GARCH(1,1)-COPULA VS GARCH(1,1)-BIVARIATE NORMAL

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ABSTRACT

Aggregate risk is an aggregation of single risks that are both independent and interdependent. In this study, aggregate risk is constructed from two interdependent random risk variables. The dependence between two random variables can be determined through the size of dependence and joint distribution properties. However, not all distributions have joint distribution properties; the joint distributions may be unknown, so motivating the use of the Copulas in this study is needed. Sometimes, the Copula model is introduced to construct joint distribution properties. The Copula model in this research is used in financial policies such as investment. In the investment sector, the aggregate risk comes from the sum of the single risks and returns. The model used in aggregate return is the Generalized Autoregressive Conditionally Heteroscedastic (GARCH) model. The data used in this study is the closing price data for Apple and Microsoft stocks from January 01, 2010, to January 01, 2024. The best model selection is the model with the GARCH-Bivariate Normal approach with the smallest MSE value. Model GARCH(1,1)-Bivariate Normal is the best model for the volatility model of aggregate return.

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1. INTRODUCTION

Risk is defined as the potential for an event to occur, which can cause losses. Risks can occur in various fields, including the financial sector. One of the risks in the financial sector is the risk that occurs when investing. In investment, Tandelilin and Marlius [1] stated that risk is the possibility of a difference between the actual return received and the expected return. An investment will always be related to the level of risk. The level of risk is defined as something inherent in every investment alternative. Therefore, a company must have several strategies for making decisions to make investments. These decisions affect the value of the company through their influence on the expected profit level and risk level factors. The level of profit for shareholders is uncertain, so it needs to be considered.

In every investment decision, returns and volatility play an important role. Volatility plays an important role in the field of financial econometrics as a risk indicator [2]. Several studies related to the return model with GARCH and Copula have been conducted because returns can be modeled using the Generalized Autoregressive Conditional Heteroscedastic (GARCH) volatility model. The findings from [3] state the prediction of virtual currency with the GARCH and Stochastic Volatility models, [2] compares the volatility characteristics of returns with the GARCH, GARCH-M, GJR-GARCH, and Log-GARCH models. The impact of cryptocurrency on inflation volatility in India, with the application of the BEKK-GARCH bivariate model, was developed by [4]. Furthermore, [5] compared linear and non-linear GARCH models to estimate volatility in selected emerging countries. [6] researched on exploring the relationship between global economic policies and crude oil futures volatility with GARCH-MIDA two-factor analysis. In practice, the risks that occur do not only involve single risks, but rather aggregate risks, which are an aggregation of risks that may occur. These can be independent or dependent on the risks. In this study, aggregate risk is the aggregate of two random risk variables that are not independent of each other. So, there is dependence between random risk variables. One method that can be used to analyze the dependence between two random variables is the Copula approach.

Several studies to analyze the dependence between two random variables using the Copula approach have been carried out. They conducted an empirical study on four Indices from the Chinese Stock Market, so that the results show that Copula pairs can better characterize the structure of interdependence between assets for portfolio optimization [7]. In addition, [5] explained that investing in Dogecoin significantly reduces the risk due to the significant correlation between Litecoin, Bitcoin, and Binance, with the standard GARCH(1,1) model being the best model in identifying the dependence between virtual currencies against other currencies. Furthermore, [8] stated that the impact of extreme risks posed by international commodities on maritime markets using the GARCH-Copula-CoVaR approach, [9] carried out dependency modeling and portfolio risk estimation using the GARCH-Copula approach, [10] carried out the dependence and risk structure of the Malaysian foreign exchange rate portfolio using Bayesian GARCH, EVT, and Copula models. According to [11], they show a copula approach to market volatility and technology stock dependency. According to [12], they analyzed the uncertainty of crude oil prices and stock markets in Gulf corporate countries using the Var-GARCH Copula model. Findings from [13] indicate that all time series exhibit fat tail shapes, leverage effects, and capture volatility that tend to cluster. Then, both constant and time-varying Copula models show that conditional dependence is similar in most countries. Therefore, this research aims to examine the concept and definition of Copula, examine aggregate risk models with GARCH, and determine the best model for modeling aggregate risk volatility using the GARCH-Copula and the GARCH-Bivariate Normal approach.

2. RESEARCH METHODS

Generally, risk is defined as the potential for an event to occur, whether predictable or unpredictable. Risk in the investment sector is the risk of loss. Risks that occur to investors are not only caused by one cause, but there will be more than one risk, which is called aggregate risk. These risks may have dependencies on each other. This dependency can be determined from the multivariate model. One of the multivariate models that can be used is Copula. The same Copula approach was also used by [14] in their research to analyze the relationship between economic factors that influence the IHSG. Furthermore, [15] explores the analysis of value-at-risk on a blue chip stock portfolio using the Gaussian Copula.

2.1 The Definition of Copula

A copula is a multivariate model that comes from the joint distribution function of random variables with a uniform distribution (0,1). The definition of Copula in **Definition 1** and Sklar's Theorem in **Theorem 1** was introduced by Nelson **[16]**.

Definition 1. The *d-variabel copula is the distribution function of C of the vector U with each component* $\boldsymbol{U_k}$ *for* $\boldsymbol{k = 1, 2, \dots, d}$ *have a Uniform(0,1) distribution.*

$$\boldsymbol{C(u_1, u_2, \dots, u_d) = P(\,U_1 \le u_1, U_2 \le u_2, \dots, U_d \le u_d) \in [0,1]^d} \tag{1}$$

Let $\boldsymbol{(X_1, X_2, \dots, X_d)}$ denotes a vector of random variables with a distribution function of $\boldsymbol{H(x_1, x_2, \dots, x_d) = P(\,X_1 \le x_1, X_2 \le x_2, \dots, X_d \le x_d)}$ and marginal distribution functions $\boldsymbol{F_{X_1}(x_1), \dots, F_{X_d}(x_d)}$. Define $\boldsymbol{X}$ with the distribution function of $\boldsymbol{F_X(x)}$, so $\boldsymbol{F_X(x) \sim Uniform(0,1)}$ if and only if $\boldsymbol{X}$ is a continuous random variable. Vectors $\boldsymbol{(u_1, u_2, \dots, u_d) = \left(F_{X_1}(x_1), \dots, F_{X_d}(x_d)\right)}$ will have a uniform distribution (0,1). The random variable used is the amount of loss of a continuous investment. The multivariate distribution function of these random variables can be written as a Copula.

$$\boldsymbol{H(x_1, x_2, \dots, x_d) = C(u_1, u_2, \dots, u_d)} \tag{2}$$

The copula used in this research consists of two dimensions of two random variables, $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Theorem 1. *(Sklar's Theorem). Let X and Y be random variables with a distribution function of* $\boldsymbol{F_X(x)}$ *and a function of* $\boldsymbol{F_Y(y)}$*. Let* $\boldsymbol{H}$ *is a bivariate distribution function, so there will be C:* $\boldsymbol{[0,1]^2 \to [0,1]}$ *so that for all* $\boldsymbol{x, y}$ *in* $\mathbb{R}$ *as follows*

$$\boldsymbol{H(x, y) = C\left(F_X(x), F_Y(y)\right) = C(u, v)} \tag{3}$$

with $\boldsymbol{u = F_X(x)}$ and $\boldsymbol{v = F_Y(y)}$. Based on the bivariate distribution function of two random variables, the probability function of $\boldsymbol{f_{X,Y}(x, y)}$ is stated as follows

$$\boldsymbol{f_{X,Y}(x, y) = \frac{\partial^2 H(x, y)}{\partial x\, \partial y} = c(u, v) f_X(x) f_Y(y)} \tag{4}$$

with $\boldsymbol{c(u, v)}$ is the probability function for Copula. Copulas can be modeled with bivariate functions, including Copulas from the Elliptical family and the Archimedean family.

2.2 Copula Families

The Copula Family consists of two types, the Elliptical Family and the Archimedean Family. The Copula family used in this research is the Elliptical Copula Family. The Elliptical Copula family consists of the Gaussian Copula and the Student-t Copula. Copula can be used to capture dependency structure and can accommodate symmetric tail distributions. An example is the Normal and Student-t distributions. The following is the Copula function and probability function for the Gaussian Copula and Student-t.

2.2.1 Copula Gaussian

Gaussian Copula (Normal) is a Copula that has a bivariate Normal distribution function, which is defined as follows.

$$C_{Gaussian}(u, v; \rho) = \phi_\rho\left(\phi^{-1}(u), \phi^{-1}(v)\right) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)}\right) ds\, dt \tag{5}$$

with the function of ϕ_ρ is the joint distribution of the standard Normal distribution with a correlation coefficient of ρ and the inverse function of ϕ^{-1} is the normal standard distribution for the Gaussian Copula.

$$\boldsymbol{C_{Gaussian}(u, v; \rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{2\phi^{-1}(u)\phi^{-1}(v)\rho - \rho\left(\phi^{-1}(u)^2 + \phi^{-1}(v)^2\right)}{2(1-\rho^2)}\right)} \tag{6}$$

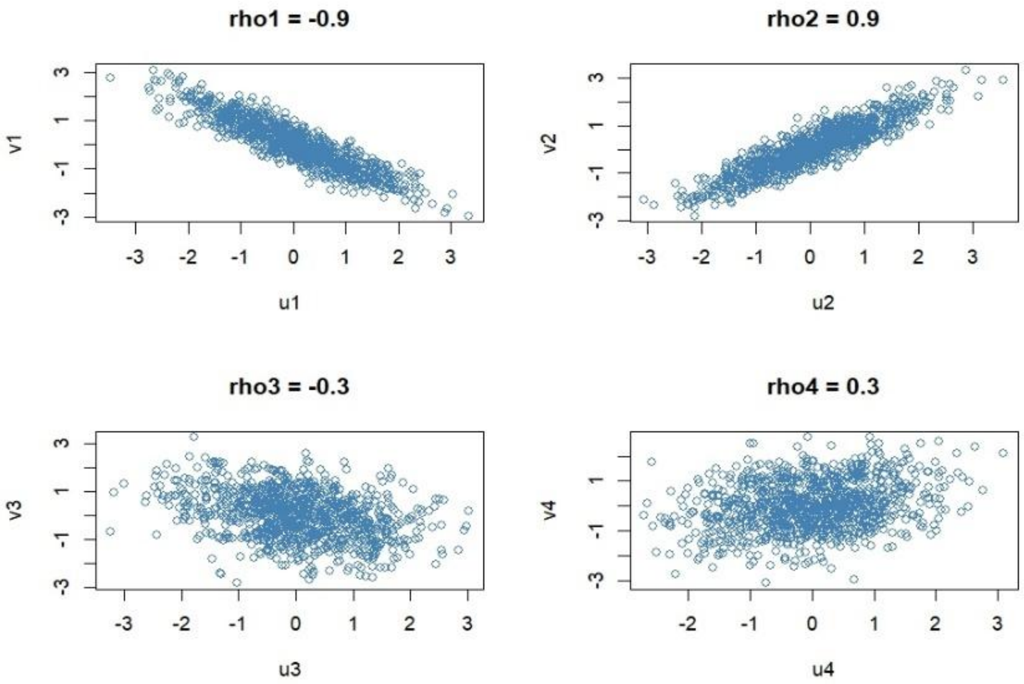


Figure 1. Gaussian Copula Data with Several Parameter Values
Source: Software RStudio

Gaussian copulas are constructed from multivariate normal distributions. Based on **Figure 1**, it can be seen that the parameter values $\boldsymbol{\rho_1 = -0.9}$ and $\boldsymbol{\rho_3 = -0.3}$. The parameter value of $\boldsymbol{\rho}$ is negative, which indicates that the increase in the value of the variable $\boldsymbol{y}$ is not in line with the variable $\boldsymbol{x}$. An increase in the value of one variable causes a decrease in the value of its counterpart. There is a strong negative correlation for the parameter value $\boldsymbol{\rho_1 = -0.9}$, while the correlation is negative for the parameter value $\boldsymbol{\rho_3 = -0.3}$. By contrast with these parameters at $\boldsymbol{\rho_2}$ = 0.9 and $\boldsymbol{\rho_4}$= 0.3. The value of the $\boldsymbol{\rho}$ parameter is positive, which indicates that the increase in the value of the variable $\boldsymbol{y}$ is in line with the variable $\boldsymbol{x}$. If the $\boldsymbol{x}$ value increases, then the $\boldsymbol{y}$ value also increases.

2.2.2 Copula Student-t

Copula t is based on bivariate t which is the same as the Gaussian Copula based on the bivariate Normal. The Copula t formula is defined as

$$C_{Student-t}(u,v;\rho,d) = t_{\rho,d}\left(t_d{}^{-1}(u), t_d{}^{-1}(v)\right) = \int_{-\infty}^{t_d{}^{-1}(u)} \int_{-\infty}^{t_d{}^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2 + t^2 - 2\rho st}{d(1-\rho^2)}\right) ds\, dt \quad (7)$$

with d is the degrees of freedom for the Student-t distribution. The Copula t probability function is

$$C_{Student-t}(u,v;\ \rho,d) = \rho^{\frac{1}{-2}} \frac{\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)^2} \frac{\left(1 + \frac{t_d{}^{-1}(u)^2 + t_d{}^{-1}(v)^2 - 2\,\rho\, t_d{}^{-1}(u) t_d{}^{-1}(v)}{d(1-\rho^2)}\right)}{\left(1 + \frac{t_d{}^{-1}(u)^2}{d}\right)^{-\frac{(d+2)}{2}} \left(1 + \frac{t_d{}^{-1}(v)^2}{d}\right)^{-\frac{(d+2)}{2}}} \quad (8)$$

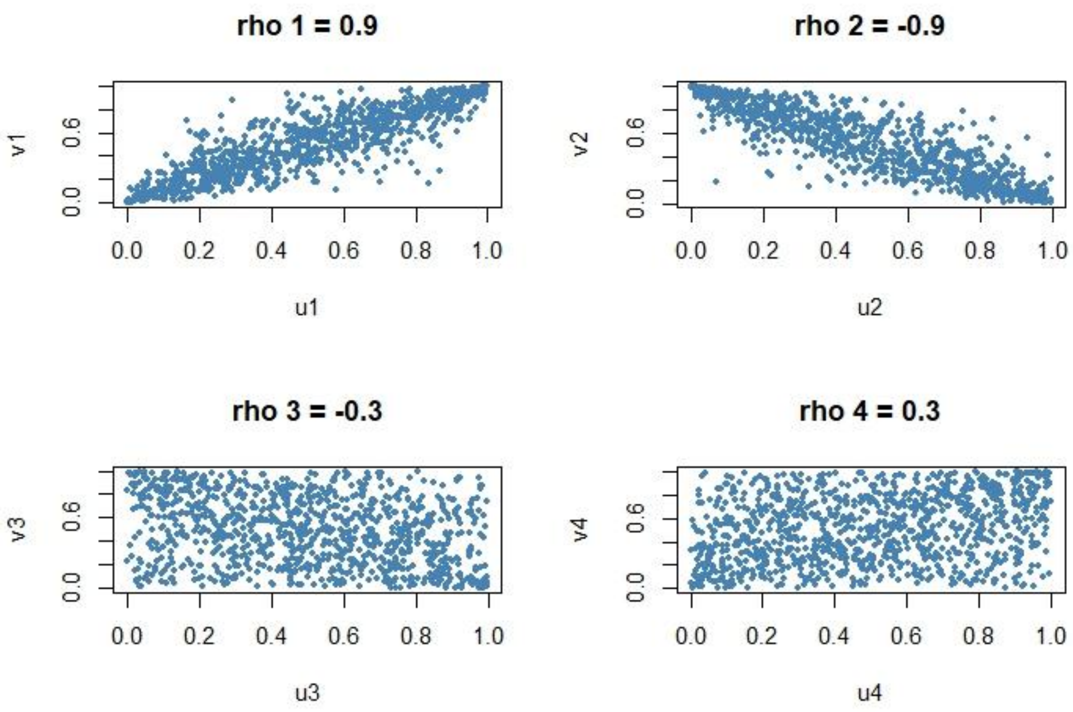


Figure 2. Student t Copula with Several Parameter Values
Source: Software RStudio

Student t copulas are constructed from multivariate Student t distributions. Based on Figure 2, it can be seen that the parameter values $\rho_2 = -0.9$ and $\rho_3 = -0.3$. The parameter value ρ is negative, a negative correlation indicates that the variable of y is higher, so the variable of x is lower.

An increase in the value of one variable causes a decrease in the value of its counterpart. There is a strong negative correlation for the parameter value $\rho_2 = -0.9$, while the correlation is negative for the parameter value $\rho_3 = -0.3$. This should happen at ρ_1 = 0.9 and ρ_4 = 0.3. The value of the ρ parameter is positive, which indicates that the increase in the value of the variable y is in line with the variable x. If the x value increases, then the y value also increases.

2.3 The GARCH(1,1) Aggregate Risk Model

Let X_t denotes risk random variables in time t. The risk considered in this research is the return on share prices. Return is the rate of stock prices. Define P_t is the price of time t and P_{t-1} is the price of time $t-1$. The formula of return is as follows:

$$X_t = -ln\left(\frac{P_t}{P_{t-1}}\right) \tag{9}$$

Mathematically, return has two condition values , which are expressed as follows

$$X_t = \begin{cases} X_t^+; & X_t \le 0 \\ X_t^-; & X_t > 0 \end{cases} \tag{10}$$

The variable X_t^+ represents positive returns (profits), while X_t^+ denotes negative returns (losses). If an investor buys two types of assets (for example, stocks and bonds), then the investment can be called an investment portfolio or asset portfolio. In statistics, an asset portfolio can be denoted as aggregate risk. Investors have a strategy to gain more profits, for example, they buy several stock products. When large profits are obtained, so large losses will also be followed. Losses resulting from the aggregation of several stocks are denoted as aggregate risk. Let X_t and Y_t be the first and second return random variables, respectively, in time t. Random variables of X_t and Y_t can be modeled with the GARCH(1,1) volatility model

$$X_t = \sigma_{t;X}\varepsilon_t \tag{11}$$

with the conditional variance expressed as

$$\sigma_{t;X}^2 = \sigma_{0;X} + \sigma_{1;X}X_{t-1}^2 + \beta_{1;X}\sigma_{t-1;X}^2 \tag{12}$$

and random variables

$$Y_t = \sigma_{t;Y}\varepsilon_t \tag{13}$$

with the conditional variance as follows.

$$\sigma_{t;Y}^2 = \sigma_{0;Y} + \sigma_{1;Y}Y_{t-1}^2 + \beta_{1;Y}\sigma_{t-1;Y}^2 \tag{14}$$

with innovation $\varepsilon_t \sim N(0,1)$. Aggregate risk follows a stochastic process denoted by S_t which is modeled by

$$S_t = X_t + Y_t \tag{15}$$

Random variables of X_t has a Normal distribution with parameter μ_{X_t} and $\sigma_{X_t}^2$ and Y_t has a Normal distribution with parameter μ_{Y_t} and $\sigma_{Y_t}^2$, so X_t and Y_t has a probability distribution function as follows

$$\boldsymbol{f_{X_t,Y_t}(x_t,y_t) = \frac{1}{2\pi\sigma_{X_t}\sigma_{Y_t}\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_t-\mu_{X_t}}{\sigma_{X_t}}\right)^2 - 2\rho\left(\frac{x_t-\mu_{X_t}}{\sigma_{X_t}}\right)\left(\frac{y_t-\mu_{Y_t}}{\sigma_{Y_t}}\right) + \left(\frac{y_t-\mu_{Y_t}}{\sigma_{Y_t}}\right)^2\right]\right\}}; \tag{16}$$

$$\boldsymbol{-\infty < x_t < \infty \text{ and } -\infty < y_t < \infty}$$

The parameter of $\boldsymbol{\rho}$ is a correlation coefficient that shows the linear dependence between risks or assets, which will affect the shape and dispersion of the aggregate distribution of two variables. Based on the joint probability function, the correlation of $\boldsymbol{\rho}$ in aggregate risk is defined as correlations between risks or assets will affect the shape and dispersion of the aggregate distribution. Analytically, determining an explicit formula for this equation is difficult. Therefore, determining the aggregate risk distribution function is carried out using a numerical approach.

2.4 Research Methodology

The complexity of the joint distribution function will use the GARCH-Bivariate Normal Algorithm and the GARCH-Copula Algorithm.

2.4.1 GARCH-Bivariate Normal Algorithm

Determine the explicit formula of this equation by determining the aggregate risk distribution function numerically using the following algorithm. This algorithm is called the GARCH-Bivariate Normal Algorithm. The steps of the GARCH-Bivariate Normal algorithm are as follows.

1. Collecting Apple and Microsoft stock price data downloaded from the Yahoo Finance webpage from January 1st, 2010, to January 1st, 2024. Then calculate the return on Apple and Microsoft stock prices using **Equation (9)**.
2. Determining the estimated parameters of α_0, α_1, and β_1 for X_t and Y_t follow GARCH(1,1) model.
3. Generate n simulated return with the parameters obtained from Step 2, so that we obtained random vectors $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ that follow GARCH(1,1) model.
4. Estimating the parameters of the Normal bivariate distribution based on the data from step 2. Then generate n data from the Normal bivariate data. So, we get x_t^+ and y_t^+ data for time $t = 1,2,\ldots,n$ which is dependent data.
5. Aggregating the two random variables, which are expressed as

$$s_t^+ = x_t^+ + y_t^+ \tag{17}$$

The distribution function of S_t can be determined empirically as follows in **Equation (18).**

$$F_{S_t}^{Empirical}(S_t^+) = \frac{1}{n}\sum_{i=1}^{n} I\{x_t^+ + y_t^+ \leq s_t^+\} \tag{18}$$

Where the indicator function $I(.)$ is defined in **Equation (19)**.

$$I\{x_t^+ + y_t^+ \leq s_t^+\} = \begin{cases} 1\,; & x_t^+ + y_t^+ \leq s_t^+ \\ 0\,; & others \end{cases} \tag{19}$$

2.4.2 GARCH-Copula Algorithm

Determining the aggregate distribution function using the Copula model is determined numerically with the following algorithm.

1. Collecting Apple and Microsoft stock price data downloaded from the Yahoo Finance webpage from January 1st, 2010, to January 1st, 2024. Then calculate the return on Apple and Microsoft stock prices using **Equation (9)**.
2. Determining the estimated parameter of α_0, α_1, and β_1 for X_t^* and Y_t^* follow GARCH(1,1) model.
3. Determining the aggregate distribution function using the Copula model is determined numerically with the following algorithm. So that we get random vectors $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ that follow GARCH(1,1) model.
4. Transforming each return data into its distribution function
$$u_t = F_{X_t}(x_t) \text{ and } v_t = F_{Y_t}(y_t) \tag{20}$$
with $t = 1,2,\dots,n$. Then, estimate the Copula parameters of the pair data $\{u_t, v_t\}$.
5. Based on the parameters obtained by the estimation, generate random data from the Copula to obtain paired data $\{u_t^*, v_t^*\}$.
6. Transforming data u_t^* and v_t^* so we get return data with dependence Copula.
$$x_t^{Copula} = F_{X_t}^{-1}(u_t^*) \tag{21}$$
$$y_t^{Copula} = F_{Y_t}^{-1}(v_t^*) \tag{22}$$
Then the aggregate model of these two random variables can be expressed as follows:
$$s_t^{Copula} = x_t^{Copula} + y_t^{Copula} \tag{23}$$
7. The distribution function of S_t can be obtained empirically with the formula
$$F_{S_t}^{Empirik}\left(S_t^{Copula}\right) = \frac{1}{n}\sum_{i=1}^{n} I\left\{x_t^{Copula} + y_t^{Copula} \leq s_t\right\} \tag{24}$$
with
$$I\left\{x_t^{Copula} + y_t^{Copula} \leq s_t^{Copula}\right\} = \begin{cases} 1; & x_t^{Copula} + y_t^{Copula} \leq s_t^{Copula} \\ & 0; others \end{cases} \tag{25}$$

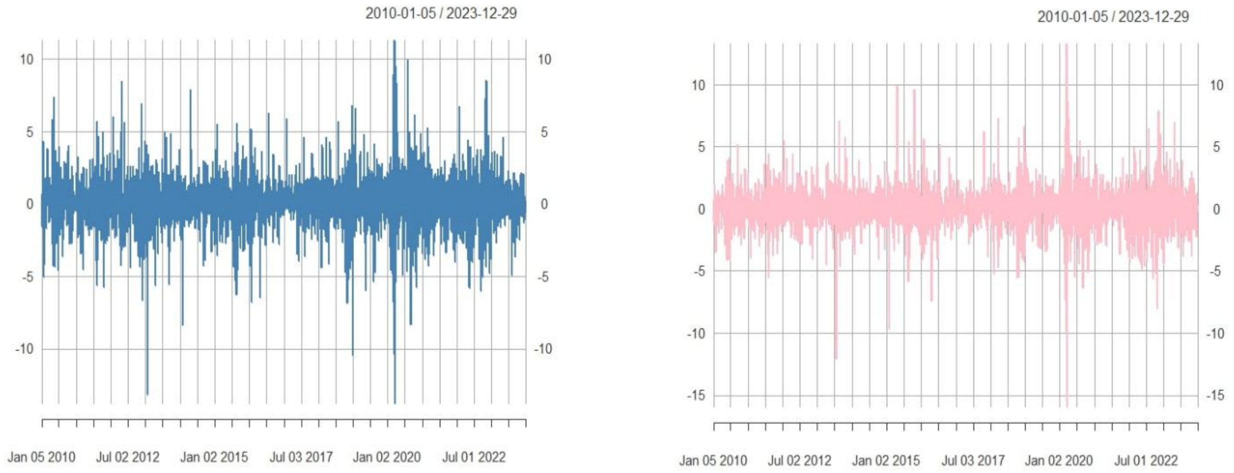
Based on the construction of the GARCH(1,1) aggregate model, there are three types of models considered in the study:

1. GARCH(1,1)-Bivariate Normal Aggregate Model.
2. GARCH(1,1)-Copula Gaussian Aggregate Model.
3. GARCH(1,1)-Copula t Aggregate Model.

3. RESULT AND DISCUSSION

3.1 Descriptive Statistics

The variables used in this research are returns of stock prices from January 1, 2010, to January 1, 2024. Let X_t and Y_t denote Apple and Microsoft return random variables, respectively. For example, the return plot is presented in **Figure 3**.



(a) **(b)**

Figure 3. (a) Apple Return (b) Microsoft Return

Source: Software RStudio

Based on the data plot of the two returns, it can be seen that both returns will be stationary over time. However, there was a significant decline in 2020 for returns. Descriptive statistics on Apple and Microsoft returns are presented in **Table 1** below. The descriptive statistics consist of minimum, maximum, mean, median, 1st quartile, 3rd quartile, kurtosis, and skewness.

Table 1. Descriptive Statistics of Apple and Microsoft Return

Statistic	Apple	Microsoft
Minimum	-0.11316	-0.13293
Maximum	0.13771	0.15945
Mean	-0.00092	-0.00071
Median	-0.00090	-0.00064
1st Quartile	-0.01037	-0.00916
3rd Quartile	0.00758	0.00727
Kurtosis	5.41594	7.99362
Skewness	0.25017	0.18123

Based on **Table 1**, it can be seen that the value of kurtosis for each return for Apple and Microsoft is above 3. The result indicates that the return distribution for Apple and Microsoft is leptokurtic and has a higher peak than the normal distribution. A thick-tailed distribution indicates that a distribution with a probability function has heavier tails than a normal distribution. Research from **[18]** stated that the thick tail distribution has different properties and behavior from the normal distribution and other thin tail distributions. In addition, **[19]** explored several distributions of returns in certain countries, which stated that distributions of returns that do not follow a normal distribution tend to have skewness, kurtosis, and have thick tails. Research from **[20]** stated that the heaviness of tails, that is, the kurtosis of a normal distribution, is 3. If the kurtosis is more than 3, then the data distribution is said to be leptokurtic, and if the kurtosis is less than 3, the distribution is flat. So, the chance of getting values very far from the average (outliers) is greater than in a normal distribution. Suppose the distribution of returns on an asset has a kurtosis of more than 3. In that case, it means that there is a tendency for price fluctuations to occur that are greater than expected with a normal distribution. This indicates a potential risk that is more extreme than a normal distribution. The risk that occurs in returns is more susceptible to very large price movements, both in profits and losses. Skewness is a statistical measure that describes the asymmetry of a data distribution. Skewness of Apple and Microsoft returns is positive. The result indicates that there is potential for large profits with more frequent small losses occurring.

3.2 GARCH(1,1)-Bivariate Normal Aggregate Model

The steps in carrying out the GARCH-Bivariate Normal Algorithm have been explained in the previous section. Firstly, we determine the estimated parameter of α_0, α_1, and β_1 for X_t and Y_t follow GARCH(1,1) model. The parameters of the GARCH(1,1) model are estimated using the software RStudio; the result can be seen in the GARCH(1,1) model in Apple and Microsoft returns in **Table 2**.

Table 2. Estimation Result of GARCH(1,1) Model Parameters

Variable	α_0	α_1	β_1
X_t	0.000016	0.109901	0.840672
Y_t	0.201906	0.131959	0.794346

After estimating the parameters of the GARCH(1,1) model, generate n random return data which follow GARCH(1,1) model. Random vectors of $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ which follow GARCH(1,1) with generated data. For instance, perform the estimation of the parameter in the Bivariate Normal based on data from the second step. The estimated parameter results are as follows:

- **Mean Vector (μ)**

  The mean vector is a vector that consists of the parameters of μ_1 and μ_2 based on the generated data of Bivariate Normal. Here, the parameter of μ_1 is the mean of a random variable of X_t and the parameter of μ_2 is the mean of a random variable of Y_t. The parameter estimation result for the mean vector is as follows

$$\mu = (\mu_1, \mu_2) = (-0.00090, -0.00030)$$

- **Covariance Matrix (Σ)**

  The covariance matrix consists of variances and covariances. Covariance states the relationship between two random variables of X_t and Y_t. The covariance matrix consists of parameters σ_1^2, σ_2^2, and ρ. The parameter of ρ indicates the correlation coefficient between two random variables. The parameter of ρ has value -1 to 1. The covariance matrix is written as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 0.00002 & 0.00003 \\ 0.00003 & 0.00002 \end{pmatrix}$$

After estimating the Bivariate Normal parameters, generate data of n normal bivariate data. So it is obtained x_t^+ and y_t^+ data for $t = 1,2,\dots,n$. Furthermore, Apple and Microsoft return for each row in each type of data is added or aggregated. Data aggregation random variables are denoted, s_t^+, with $s_t^+ = x_t^+ + y_t^+$. Empirically, the graph of the aggregate risk distribution function GARCH(1,1) with this algorithm approach is presented as follows.

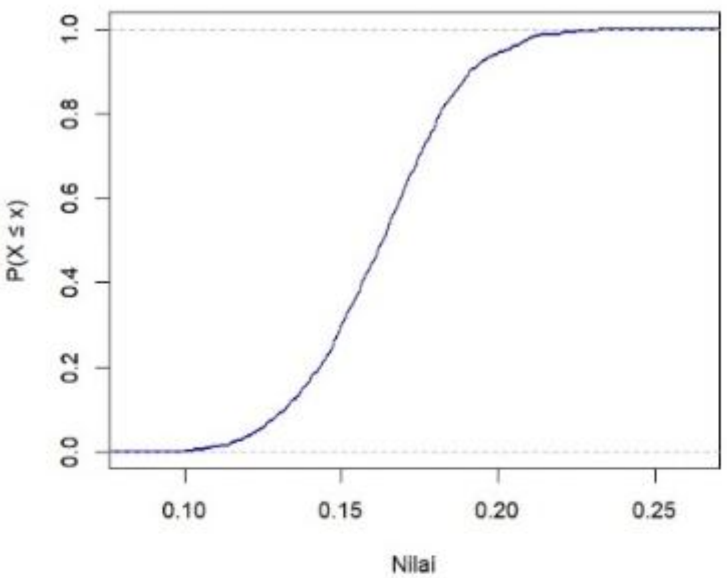


Figure 4. The Distribution Function of GARCH(1,1)-Bivariate Normal
Source: Software RStudio

Based on the distribution function plot of s_t^+ above. It can be seen that the greater the aggregate of returns, the greater the possibility of aggregate returns. This means that if investors have more stock portfolio purchases, the possibility of profit obtained by investors will also increase.

3.3 GARCH(1,1)-Copula Gaussian Aggregate Model.

GARCH(1,1)-Copula-Gaussian Model has the steps described in the previous section. Firstly, we determine the estimation of the parameters of α_0, α_1, and β_1 for X_t and Y_t random variables which follow GARCH(1,1) model. Defined X_t and Y_t random variables assume a Gaussian distribution. The Copula used in this research is the Gaussian Copula. The parameters of model GARCH(1,1) model have been simulated using RStudio. The estimated parameters using GARCH(1,1) in Apple and Microsoft return data are represented in the following **Table 3**.

Table 3. Estimation Result of GARCH(1,1) Model Parameters

Variable	α_0	α_1	β_1
X_t	0.000016	0.109901	0.840672
Y_t	0.201906	0.131959	0.794346

Based on the estimated parameters on the following **Table 3**, we generate n return data. So that we have random vectors $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ which follow GARCH(1,1) model. Furthermore, we conducted the transformation of each return data to a distribution function. Using the estimated parameters of the Copula of data pairs $\{u_t, v_t\}$ with maximum likelihood method. So that we have the estimated of data pairs of Copula-Gaussian is the parameter of $\hat{\rho} = 0.008034$. The graph of the distribution function of aggregate with GARCH(1,1)-Copula-Gaussian ($s_t^{Copula_{Gaussian}}$) are represented in **Figure 5**.

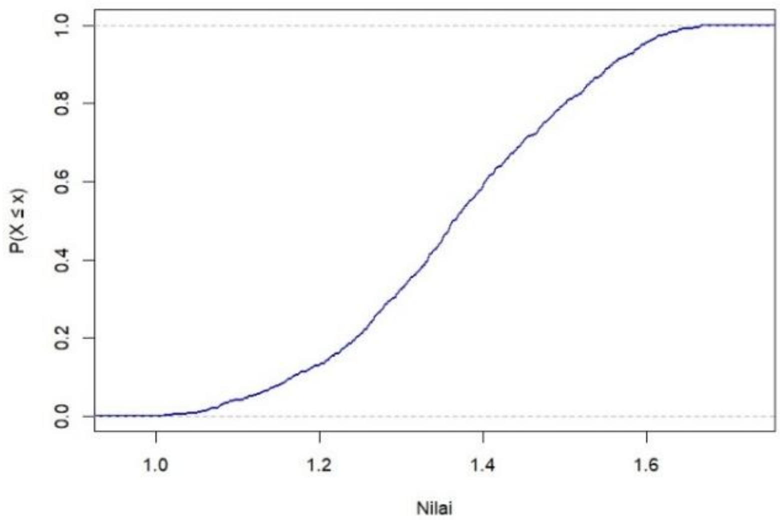


Figure 5. The Distribution Function of GARCH(1,1)-Copula-Gaussian
Source: Software RStudio

3.4 GARCH(1,1)-Copula-Student t Aggregate Model

GARCH(1,1)-Copula-Student t model has similar steps as GARCH(1,1)-Copula-Gaussian. Firstly, we conducted an estimation of the parameters of α_0, α_1, and β_1 for X_t and Y_t random variables which follow GARCH(1,1) model. The marginal distributions of X_t and Y_t follow Student-t, and the dependence structure is modeled by Student-t Copula. Therefore, the other used of the copula in this research is Student t Copula. The parameters in this GARCH(1,1) model have been simulated using RStudio. The estimated parameters of simulation GARCH(1,1) in the return data of Apple and Microsoft can be represented in the following **Table 4**.

Table 4. Estimation Result of GARCH(1,1) Model Parameters

Variable	α_0	α_1	β_1
X_t	0.000016	0.109901	0.840672
Y_t	0.201906	0.131959	0.794346

Using the estimated parameters in **Table 4** above, we simulate and generate n data return, so we have random vectors $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ which follow GARCH(1,1) model. Using the estimated parameters of the Copula of a couple of $\{u_t, v_t\}$ with the maximum likelihood method. So that we have the result of the estimated parameters of the pairs data of Copula-Student t is parameter $\hat{\rho} = 0.00271$ and degree of freedom of $\hat{v} = 19.10281$. The distribution function of GARCH(1,1)-Copula-Gaussian $(s_t^{Copula_t})$ can be represented as follows

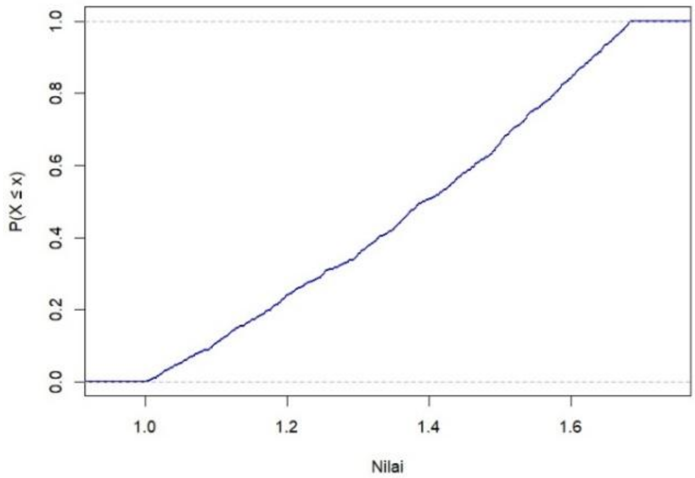


Figure 6. The Distribution Function of GARCH(1,1)-Copula-*t*
Source: Software RStudio

3.5 The Aspects of Selecting the Best Model

The selection of the best model can be obtained from the *Mean Squared Error (*MSE). MSE aims to determine the difference between the actual value and the value produced by the estimator. Let S_t be a stochastic random variable that represents the actual aggregate return. Furthermore, $\widehat{S_t}$ random variables indicate aggregate predicted returns. The formula of MSE can be expressed as the following **Equation (26)**. Based on the results in the table, it is found that the best aggregate risk volatility model is the volatility model with the GARCH-Bivariate Normal algorithm approach.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}\left(S_t - \hat{S}_t\right)^2 \qquad (26)$$

The three models were determined using MSE values, which are stated in **Table 5** below. Based on **Table 5**, the best aggregate risk model is the GARCH(1,1)-Bivariate Normal model with the smallest MSE value among the three models.

Table 5. MSE of Three Models

Model	MSE
GARCH(1,1)-Bivariate Normal	7.01369
GARCH(1,1)-Copula-Gaussian	8.64101
GARCH(1,1)-Copula-Student t	8.61545

4. CONCLUSIONS

Copulas are used to define a framework for multivariate distributions and the modelling of multivariate data. The copula of a multidimensional random vector, or more specifically of its distribution, is a function characterizing the dependence structure, thus the characteristics of its distribution, which do not depend on the margins. However, they can be combined with any set of univariate marginal distributions to form a joint distribution. Thus, copulas are widely used in the construction of univariate models for multivariate data. In this study, the copulas used are the Gaussian Copula, t-Copula, and Bivariate Normal Copula. These copulas will be used in the application to find which one provides the best fit for the chosen data. Gaussian, t-student, and bivariate normal copulas can be useful to generate families of copulas that are based on multivariate Gaussian, t-student, and bivariate normal. Copula GARCH was used to model the dependence structure between Apple and Microsoft returns. The study showed that the GARCH(1,1)-Bivariate Normal model has the smallest MSE, so the GARCH(1,1)-Bivariate Normal model was found to be the most appropriate for examining the dependence between the returns.

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