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# THE TRIPLE IDENTITY GRAPH OF THE RING $\mathbb{Z}_n$

**Vika Yugi Kurniawan[1*], Chessa Fanny Ekasiwi[2], Santoso Budi Wiyono[3]**

[1,2,3] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret*
*Jln. Ir. Sutami no 36A, Kentingan, Surakarta, 57126, Indonesia*

*Corresponding author's e-mail: * vikayugi@staff.uns.ac.id*

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***ABSTRACT***

*Let $R$ be a commutative ring with identity and $1$ is an identity element of $R$. The triple identity graph of the ring $R$, represented by $TE(R)$, is an undirected simple graph with the vertex set $R-\{0,1\}$. In $TE(R)$, two different vertices $x$ and $y$ is called adjacent if there is an element $z \in R-\{0,1\}$ such that $x.y \neq 1, x.z \neq 1, y.z \neq 1$, and $x.y.z = 1$. The triple identity graph of the ring of integers modulo $n$, represented by $TE(\mathbb{Z}_n)$, is the subject of this study. We obtain several results regarding the properties of the graph $TE(\mathbb{Z}_n)$, which are summarized as follows. The graph $TE(\mathbb{Z}_n)$ is a connected graph if and only if $n$ is prime and $n \geq 5$. If $TE(\mathbb{Z}_n)$ is connected, then $diam(TE(\mathbb{Z}_n)) = 2$ and $gr(TE(\mathbb{Z}_n)) = 3$. Furthermore, $TE(\mathbb{Z}_n)$ is a Hamiltonian graph if $n$ is a prime number and $n \geq 7$.*



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## 1. INTRODUCTION

Algebraic graph theory is a branch of mathematics that has developed rapidly in recent decades, especially in the relationship between algebraic structures and graph theory. Since the introduction of zero divisor graphs by Beck [1] in 1988, some research has been carried out to explore the properties of these graphs and their applications in various fields. A zero-divisor graph is a graph in which the vertices are the set of zero-divisors of a ring, and two distinct vertices are connected if their product is zero. This research has been the starting point for many further researchers, such as Anderson & Livingston [2], Sinha and Kaur [3], Lee and Varmazyar [4], and Nikmehr et al.[5].

Since zero-divisor graphs were introduced, many researchers have explored various aspects of these graphs, including connectivity, diameter, and girth. Sinha and Kaur [3] showed that zero-divisor graphs are always connected and have small diameters and girths, which indicates that this graph structure has interesting properties for further research. In addition, Dhorajia [6] emphasizes the importance of relating graph structure to ideals in rings, which allows the analysis of algebraic properties through the lens of graph theory. This research shows that graphs associated with rings can provide deeper insight into the algebraic and topological properties of the rings. Research by Nazim [7] shows how the Laplacian spectrum of a zero-divisor graph can provide insight into the underlying ring structure. Additionally, research by Reddy et al. [8] explored the vertex and edge connectivity of a zero-divisor graph, which provides important information about the topological properties of the graph. This research shows that spectral analysis can be a powerful tool for understanding the algebraic properties of rings.

The importance of investigating the graph properties of rings, such as connectivity, girth, diameter, and Hamiltonian properties, cannot be ignored because these properties not only provide information about the structure of the graph itself but also about the algebraic structure of the underlying ring. Connectivity measures how well a graph can be connected, while girth, which is the length of the shortest cycle in a graph, provides information about the complexity of the graph's structure [9]. On the other hand, the diameter indicates the maximum distance between two vertices in a graph, which can have implications for the efficiency of algorithms used in practical applications [10]. Research by Samei [11] shows that the diameter and girth of a graph can provide insight into the commutative properties of rings. Hamiltonian properties, which relate to the existence of Hamiltonian cycles in graphs, are also a main focus because they relate to optimization problems that often arise in various applications [12].

The applications of this research are very broad. Algebraic graphs are relevant not only in pure mathematics but also in other fields such as computer science, physics, and even biology. For example, graph concepts can be applied in network analysis, where graph structures are used to model interactions between various entities. Research by Wang et al. [13] showed how graph-related topological indices can be used to understand the physicochemical properties of organic compounds. These graphs also have applications in the fields of network theory and cryptography [14].

Along with the development of algebraic graph theory, many researchers have contributed to broadening the understanding of graphs related to rings. For example, research by Kumar and Prakash[15] shows how zero-divisor graphs can be extended to include complementary graphs, which provides a new perspective in graph analysis. Additionally, research by Nikandish et al.[5] highlighted the importance of cozero-divisor graph coloring, paving the way for further exploration in the context of algebraic graphs. This research shows that by studying cozero-divisor graphs, we can better understand how the elements in a ring interact and how these interactions can influence the properties of the resulting graph. Additionally, research by Toker [16] shows that zero-divisor graphs of Catalan monoids can provide new insights into the algebraic structure of those monoids. This research not only broadens the understanding of the algebraic properties of rings, but also paves the way for the exploration of more complex graphs such as the triple-zero graph of a ring [17], the triple nilpotent graph of a ring [18], and type-II triple unit graph of a ring [19].

One of the important developments in algebraic graph theory was the introduction of idempotent graphs by Akbari et al [20], which is a generalization of zero-divisor graphs. Cahyati [21] explains how algebraic structures and graph theory can interact with each other and how idempotent graphs can be used to study the algebraic properties of rings. This research shows that by studying idempotent graphs, we can better understand how the elements in a ring interact with each other and how these interactions can affect the properties of the resulting graph. Research by Patil and Momale [22] highlights the relationship between idempotent graphs and zero-divisor graphs, as well as how the properties of these graphs can be used to understand more complex ring structures. In the context of idempotent graphs, research by Razaghi and

Sahebi [23] shows how these graphs can be used to explore the algebraic properties of rings, including connectivity, diameter, and girth. This research developed into more complex graphs, such as triple idempotent graphs from rings [24].

Recently, Kurniawan and Ekasiwi [25] introduced the triple identity graph of a ring and provided an algorithm for constructing and investigating the properties of this graph. For a commutative ring $R$, the triple identity graph of $R$ denoted by $TE(R)$ is a graph with the set of vertices $R^* = R - \{0, 1\}$. Two different vertices $x$ and $y$ in $TE(R)$ is called adjacent if there is an element $z \in R - \{0,1\}$ such that $x.y \neq 1, x.z \neq 1, y.z \neq 1$, and $x.y.z = 1$. The properties explored in their research are only conjectures that have not been proven. This paper explores the properties of the triple idempotent graph of the ring $\mathbb{Z}_n$. Important properties such as connectivity, diameter, girth, and Hamiltonian properties are given in the results section. By examining the properties of the triple idempotent graph of the ring $\mathbb{Z}_n$, it is hoped that this article can significantly contribute to understanding the relationship between algebraic structures and graph theory.

## 2. RESEARCH METHODS

This study employs a qualitative approach that combines literature review, deductive axiomatic methods, and mathematical pattern recognition techniques. The methodology begins with a literature review by examining references from books, journals, and scientific articles related to algebraic graph theory, graph theory, and algebraic structures, particularly the characteristics of identity elements in a commutative ring.

The literature review serves as a critical analysis of previous studies to identify research gaps, theoretical perspectives, and methodologies used. It provides a solid foundation for the current study by demonstrating how it builds upon or differs from existing work.

In addition, deductive and axiomatic methods are applied to analyze graph properties based on algebraic structures, while pattern recognition techniques are used to group graphs with similar characteristics to identify specific patterns. These combined methods support the formulation and proof of mathematical theorems relevant to the study.

### 2.1 Research Steps

The following are the steps taken in this study.

1. Study literature related to algebraic graph theory.
2. Construct graphs according to the definition and algorithm given in [25].
3. Grouping graphs based on similar properties to identify certain patterns.
4. Analyze various graph properties, such as degree of vertices, distance between vertices, connectedness, cycles, and Hamiltonian cycles.
5. Using mathematical proof techniques to prove theorems or propositions that relate the algebraic properties of rings and topological properties of graphs.

The research process flowchart is shown in Figure 1.

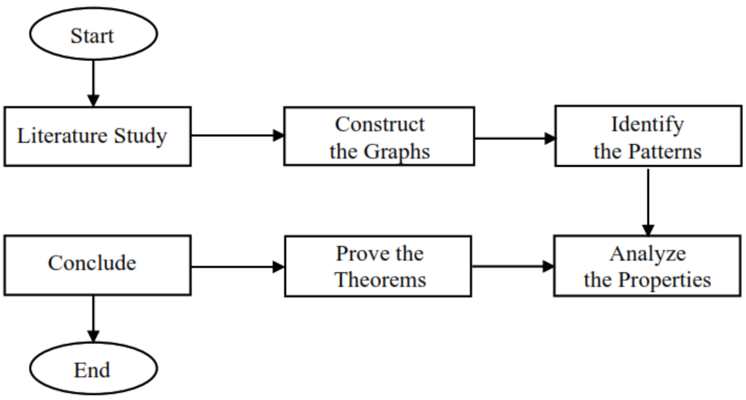



**Figure 1. The Flowchart of the Research Process**

### 2.2 Basic Concepts and Notations

In this section, we give some basic concepts of graphs and the definition of the triple identity graph of a ring. An *empty graph* is a graph $G = (V, E)$ where $E = \emptyset$, meaning there are no edges between any pair of vertices in $V$. For a graph $G$ and a vertex $v \in V(G)$, the *degree* of $v$, denoted deg $(v)$, is the number of edges incident with $v$. The *minimum degree* of a graph $G$, denoted $\delta(G)$, is the least degree among all vertices in $G$. A *path* in a graph is a sequence of distinct vertices such that each consecutive pair of vertices is connected by an edge. A graph $G$ is said to be *connected* if there exists a path between every pair of distinct vertices in $G$. The *distance* between two vertices $u$ and $v$ in a graph $G$, denoted $d(u, v)$, is the length of the shortest path between $u$ and $v$. The *diameter* of a graph $G$, denoted $diam(G)$, is the maximum distance between any two vertices in $G$. A *cycle* is a path in which the first and last vertices are the same. The *girth* of a graph $G$, denoted $gr(G)$, is the length of the shortest cycle in $G$. A *Hamiltonian cycle* is a cycle that passes through every vertex of a graph exactly once. A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The following definition and example of the triple identity graph of a commutative ring are taken from [25].

**Definition 1**. Let $R$ be a commutative ring with identity. The triple identity graph of $R$ denoted by $TE(R)$ is a graph with the set of vertices $R^* = R - \{0, 1\}$. Two different vertices $x$ and $y$ is called adjacent if there is an element $z \in R - \{0,1\}$ such that $x.y \neq 1, x.z \neq 1, y.z \neq 1$, and $x.y.z = 1$.

Since element 0 will always be an isolated vertex in all $TE(R)$, and element 1 will always be adjacent to any vertex that has an inverse in $TE(R)$, they are excluded from the vertex set of $TE(R)$.

**Example 1**. Let $\mathbb{Z}_5$ be a ring of integer modulo 5. Where $\mathbb{Z}_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$.

**Table 1. Adjacency Table of $V(TE(\mathbb{Z}_5))$**

| $x$ | $y$ | $z$ | $x.y$ | $y.z$ | $x.z$ | $x.y.z$ |
|---|---|---|---|---|---|---|
| $\bar{3}$ | $\bar{4}$ | $\bar{3}$ | $\bar{2}$ | $\bar{2}$ | $\bar{4}$ | $\bar{1}$ |
| $\bar{2}$ | $\bar{4}$ | $\bar{2}$ | $\bar{3}$ | $\bar{3}$ | $\bar{4}$ | $\bar{1}$ |

By **Definition 1**, so we have the vertex set of $TE(\mathbb{Z}_5)$ is $V(TE(\mathbb{Z}_5)) = \{\bar{2}, \bar{3}, \bar{4}\}$ and by the rule of adjacency in $TE(R)$ we obtained $E(TE(\mathbb{Z}_5)) = \{(\bar{2}, \bar{4}), (\bar{3}, \bar{4})\}$. The $TE(\mathbb{Z}_5)$ is shown in **Figure 2**.

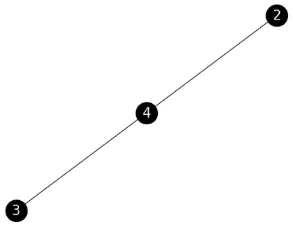



**Figure 2. The Triple Identity Graph of $\mathbb{Z}_5$**

## 3. RESULTS AND DISCUSSION

In this study, the primary object of interest is the ring of integers modulo $n$, denoted $\mathbb{Z}_n$ . Throughout the remainder of this discussion, all references to rings shall be understood to mean $\mathbb{Z}_n$, unless explicitly stated otherwise. At the beginning, still related to the connectedness property, we give a condition where the triple identity graph of the ring $\mathbb{Z}_n$ becomes an empty graph. This also means that the graph is disconnected in this condition.

**Theorem 1**. *If $n = \{3, 4, 6\}$, then $TE(\mathbb{Z}_n)$ is an empty graph.*

**Proof**. By **Definition 1**, the vertex set of $TE(R)$ is $R^* = R - \{0, 1\}$. Hence, $|V(TE(\mathbb{Z}_n))| = n - 2$. Next, it will be proved for 3 cases.

Case 1. For $n = 3$.

Given the ring $\mathbb{Z}_3$. Since $|V(TE(\mathbb{Z}_3))| = 1$, so for $TE(\mathbb{Z}_3)$ there is no edge or $TE(\mathbb{Z}_3)$ is an empty graph.

Case 2. For $n = 4$.

Given the ring $\mathbb{Z}_4$. We get $V(TE(\mathbb{Z}_4)) = \{\bar{2}, \bar{3}\}$. If $x = 2$ and $y = 3$, we get $xy = 2$, then we cannot find the element $z \in V(TE(\mathbb{Z}_4))$ that satisfies the adjacency condition of $TE(\mathbb{Z}_4)$ which is $xyz = 1$. There are two possibilities. The first one is for $z = 2$, we get $xyz = 0 \neq 1$, so it does not satisfy the adjacency condition. The second possibility is for $z = 3$. Because $xyz = 2 \neq 1$, it also does not satisfy the adjacency condition. Therefore, $TE(\mathbb{Z}_4)$ has no edges or $TE(\mathbb{Z}_4)$ is an empty graph.

Case 3. For $n = 6$.

Given the ring $\mathbb{Z}_6$. We get $V(TE(\mathbb{Z}_6)) = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}\}$. The following **Table 2** is presented to show the multiplication between vertices.

**Table 2. Investigation of Adjacency between Vertices in $TE(\mathbb{Z}_6)$**

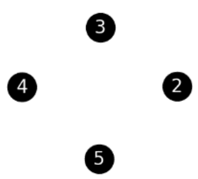
| $x$ | $y$ | $z$ | $x.y$ | $x.z$ | $y.z$ | $x.y.z$ |
|---|---|---|---|---|---|---|
| 2 | 3 | 2 | 0 | 4 | 0 | 0 |
| 2 | 3 | 3 | 0 | 0 | 3 | 0 |
| 2 | 3 | 4 | 0 | 2 | 0 | 0 |
| 2 | 3 | 5 | 0 | 4 | 3 | 0 |
| 2 | 4 | 2 | 2 | 4 | 2 | 4 |
| 2 | 4 | 4 | 2 | 2 | 4 | 2 |
| 2 | 4 | 5 | 2 | 4 | 2 | 4 |
| 2 | 5 | 2 | 4 | 4 | 4 | 2 |
| 2 | 5 | 5 | 4 | 4 | 1 | 2 |
| 3 | 3 | 3 | 0 | 0 | 0 | 0 |
| 3 | 4 | 4 | 0 | 0 | 4 | 0 |
| 3 | 5 | 5 | 0 | 0 | 2 | 0 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 5 | 5 | 3 | 3 | 1 | 3 |
| 4 | 4 | 4 | 2 | 2 | 2 | 2 |
| 4 | 5 | 5 | 2 | 2 | 1 | 4 |

**Table 2** shows that for any two distinct vertices $x$ and $y$, none of them fulfill the adjacency condition of $TE(\mathbb{Z}_n)$. Thus, $TE(\mathbb{Z}_6)$ is an empty graph because it has no edges. Based on the three cases above, it is proven that $TE(\mathbb{Z}_n)$ is an empty graph for $TE(\mathbb{Z}_n)$ with $n = \{3, 4, 6\}$. ■

The construction graph of $TE(\mathbb{Z}_n)$ with $n = \{3, 4, 6\}$ are shown in **Figure 3**.


(a) $TE(\mathbb{Z}_3)$


(b) $TE(\mathbb{Z}_4)$




(c) $TE(\mathbb{Z}_6)$

**Figure 3. Graph $TE(\mathbb{Z}_n)$ with $n = \{3, 4, 6\}$**

The following lemmas will be used to support the proof of Theorem 2. This lemma guarantees that there is a vertex in $TE(\mathbb{Z}_n)$ that is connected to all other vertices if $n$ is a prime number greater than or equal to five.

**Lemma 1.** *If $n$ is prime and $n \geq 5$, then there exists a vertex $u = n - 1 \in V(TE(\mathbb{Z}_n))$ that is adjacent to any other vertices in $TE(\mathbb{Z}_n)$.*

**Proof**. Let $v$ be any vertex in $TE(\mathbb{Z}_n)$ where $n$ is prime and $n \geq 5$. We will prove that $v$ is adjacent to the vertex $u = n - 1 \in V(TE(\mathbb{Z}_n))$. By definition, two distinct vertices are adjacent if there is an element $w \in R - \{0,1\}$ such that $u.v \neq 1, u.w \neq 1, v.w \neq 1,$ and $u.v.w = 1$. Consider

$$(u)(u) = (n-1)(n-1)$$
$$= n^2 - 2n + 1$$
$$\equiv 1(\text{mod } n)$$

Since the inverse of $u = n - 1$ is itself, it follows that $uv \neq 1$ if $v \neq u$. Therefore, the condition $uv \neq 1$ with $u = n - 1$ and $v$ arbitrary is satisfied. Since $n$ is a prime then $\mathbb{Z}_n$ is a field. Every element in $\mathbb{Z}_n$ has an inverse so that $w = (uv)^{-1}$ can be determined that satisfies

$$uvw = uv(uv)^{-1} = 1.$$

Since $uv \neq 1$ and there exists $w$ such that $uvw = 1$, so $u$ and $v$ are adjacent. It is proved that in the ring $\mathbb{Z}_n$ where $n$ is a prime and $n \geq 5$ there exists a vertex $u = n - 1 \in V(TE(\mathbb{Z}_n))$ that is connected to any other vertex. ∎

The following theorem provides necessary and sufficient conditions for the connectedness of the graph $TE(\mathbb{Z}_n)$.

**Theorem 2.** *The graph $TE(\mathbb{Z}_n)$ is a connected graph if and only if $n$ is prime and $n \geq 3$.*

**Proof**. Given $\mathbb{Z}_n$ with $n$ primes and $n \geq 3$, it will be proved that $TE((\mathbb{Z}_n))$ is a connected graph. A graph $G$ is said to be connected if there exists a path between any two vertices of the graph $G$. Since $TE((\mathbb{Z}_3))$ is a graph with only one vertex, it can be said that $TE(\mathbb{Z}_3)$ is a connected graph. Next, by Lemma 1, the $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 5$ must have a vertex that is adjacent to all other vertices, then it is proven that $TE(\mathbb{Z}_n)$ with prime $n$ and $n \geq 5$ is a connected graph.

Furthermore, the converse will be proved by contraposition, which is if $n$ is not prime or $n < 3$, then $TE(\mathbb{Z}_n)$ is not connected. Since $TE(R)$ is a graph with the vertex set $R^* = R - \{0, 1\}$, then $TE(\mathbb{Z}_n)$ with $n < 3$ will not form a graph. Let $n$ be a composite number, then $\mathbb{Z}_n$ contains a non-unit element. Suppose $u$ is a non-unit element in $\mathbb{Z}_n$, then for every $v \in \mathbb{Z}_n$, it holds $uv \neq 1$. Suppose there exists $w \in \mathbb{Z}_n$ that satisfies $uvw = 1$, then $vw$ is the inverse of $u$. This contradicts that $u$ is a non-unit element, so there will be no $w \in \mathbb{Z}_n$ that satisfies $uvw = 1$. So, vertex $u$ cannot be adjacent to any other vertex or it can be called an isolated vertex. Since there is an isolated vertex in $TE(\mathbb{Z}_n)$, then $TE(\mathbb{Z}_n)$ with $n$ is a composite is an unconnected graph. Thus, it is proved that $TE(\mathbb{Z}_n)$ is a connected graph if and only if $n$ is prime and $n \geq 3$. ∎

**Example 2.** Given a ring $\mathbb{Z}_7$. In the $TE(\mathbb{Z}_7)$, we obtain the vertex set $V(TE(\mathbb{Z}_7)) = \{2, 3, 4, 5, 6\}$. Based on Lemma 1, there is a vertex $u = n - 1$ connected to all other vertices. In $TE(\mathbb{Z}_7)$, the vertex is $u = 6$. Since vertex 6 is connected to all other vertices, there will always be a path between any two vertices in $TE(\mathbb{Z}_7)$. Therefore, $TE(\mathbb{Z}_7)$ is a connected graph. The following Figure 4. shows the graph of $TE(\mathbb{Z}_7)$.

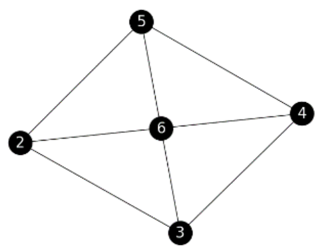



**Figure 4. Graph $TE(\mathbb{Z}_7)$**

Next, we provide an example of a graph $TE(\mathbb{Z}_n)$ where $n$ is not a prime number.

**Example 3.** Given the ring $\mathbb{Z}_8$. In the $TE(\mathbb{Z}_8)$ graph, we get the vertex set $V(TE(\mathbb{Z}_8)) = \{2, 3, 4, 5, 6, 7\}$. Based on **Theorem 2**, for $n$ that is not a prime number, there is a vertex $u$ that is a non-unit element so that $u$ is an isolated vertex. In $TE(\mathbb{Z}_8)$, the vertices that are non-unit elements are $\{2, 4, 6\}$ so that they are isolated vertices. Since there are isolated vertices, $TE(\mathbb{Z}_8)$ is an unconnected graph. The following **Figure 5** shows the graph of $TE(\mathbb{Z}_8)$.

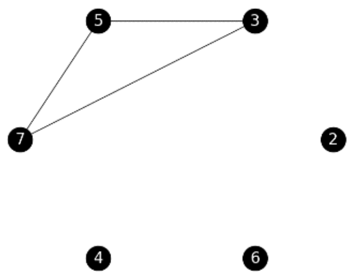



**Figure 5. Graph $TE(\mathbb{Z}_8)$**

Having shown the sufficient and necessary conditions for the connectedness of the triple identity graph, we next explore the properties associated with connected graphs such as girth and diameter. The following theorem shows that if the graph $TE(\mathbb{Z}_n)$ is connected, then its girth is equal to 3.

**Theorem 3.** *If $TE(\mathbb{Z}_n)$ is a connected graph, then $gr(TE(\mathbb{Z}_n)) = 3$.*

**Proof**. If $TE(\mathbb{Z}_n)$ is a connected graph, then at least there are two vertices, $a$ and $b$, are adjacent to each other. Therefore, there exists $c \in V(TE(\mathbb{Z}_n))$ such that $ab \neq 1, ac \neq 1, bc \neq 1$ and $abc = 1$. Thus, $a$, $b$, and $c$ satisfy the adjacency conditions so $a$ is adjacent to $b$ and $c$, $b$ is adjacent to $a$ and $c$, $c$ is adjacent to $a$ and $b$, and the cycle is obtained $a - b - c - a$. Therefore, the girth of the graph $TE(\mathbb{Z}_n)$ is equal to 3 or $gr(TE(\mathbb{Z}_n)) = 3$. ■

The following example shows that a connected triple identity graph has a girth of 3.

**Example 4.** In **Figure 4**, it can be seen that there is a shortest cycle in the $TE(\mathbb{Z}_7)$ graph which consists of at least three vertices, one of which is the cycle $2 - 3 - 6 - 2$. Next, an example with a higher degree of $n$ is given, namely $TE(\mathbb{Z}_{11})$. The following **Figure 6** presents the graph $TE(\mathbb{Z}_{11})$.

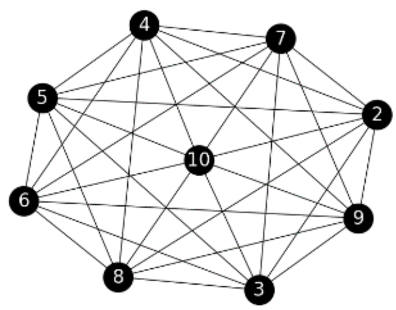



**Figure 6. Graph $TE(\mathbb{Z}_{11})$**

**Figure 6** also shows that there is a shortest cycle in the graph of $TE(\mathbb{Z}_{11})$ which has at least 3 vertices, one of which is the cycle $2 - 3 - 10 - 2$. Based on the definition of girth, it can be concluded that the graph has a girth equal to 3.

The following theorem shows that if the graph $TE(\mathbb{Z}_n)$ is connected, then the diameter of the graph $TE(\mathbb{Z}_n)$ is 2.

**Theorem 4.** *If $TE(\mathbb{Z}_n)$ is a connected graph, then $diam(TE(\mathbb{Z}_n)) = 2$.*

**Proof**. If $TE(\mathbb{Z}_n)$ is a connected graph, then by **Theorem 2**, $n$ is prime and $n \geq 3$. Based on **Lemma 1**, we know that vertex $u = n - 1$ will always be adjacent to all the other vertices in $TE(\mathbb{Z}_n)$. Thus, the distance between any two vertices $a$ and $b$ in $TE(\mathbb{Z}_n)$ contains only two possibilities, $d(a, b) = 1$ or $d(a, b) = 2$. Two distinct vertices will have a distance of one if they are adjacent or directly connected, and will have a distance of two if there is a path through the vertex $u$ or they are not directly connected. Therefore, we obtain $diam(TE(\mathbb{Z}_n)) = 2$. ■

**Example 5.** Given a $TE(\mathbb{Z}_7)$ graph. In **Figure 4** we can see that vertex 6 is connected to all other vertices in $TE(\mathbb{Z}_7)$. Thus, the distance between any two vertices $a$ and $b$ in $TE(\mathbb{Z}_7)$ contains two possibilities, $d(a,b) = 1$ or $d(a,b) = 2$. The distance equals 1 if they are adjacent to each other, such as vertices 2 and 3. The distance equals 2 if there is a path between two different vertices through vertex 6. For example between vertices 2 and 4, we have a path $2-6-4$. Thus, we can conclude that the $TE(\mathbb{Z}_7)$ graph has a diameter equal to 2.

The following theorem states that for $\mathbb{Z}_n$ with $n$ prime and $n \geq 7$, the minimum degree of $TE(\mathbb{Z}_n)$ is $n-4$.

**Theorem 5.** *If $n$ is prime and $n \geq 7$, then $\delta(TE(\mathbb{Z}_n)) = (n-4)$.*

**Proof**. Let $n$ be a prime number with $n \geq 7$, and consider the graph $TE(\mathbb{Z}_n)$. We aim to show that the minimum degree of vertices in this graph is $\delta(TE(\mathbb{Z}_n)) = (n-4)$. Since $n$ is prime, all nonzero elements have a unique multiplicative inverse. The vertex set of $TE(\mathbb{Z}_n)$ is $\mathbb{Z}_n - \{0,1\}$, hence it has $n-2$ vertices. In graph $TE(\mathbb{Z}_n)$, two distinct vertices $a$ and $b$ are adjacent if and only there exists $c \in V(TE(\mathbb{Z}_n))$ such that $ab \neq 1, ac \neq 1, bc \neq 1$ and $abc = 1$.

Now consider an arbitrary vertex $u \in V(TE(\mathbb{Z}_n))$ where $u \neq n-1$. Clearly, $u$ is not adjacent to itself since the graph is simple. Next, let us show that $u$ is also not adjacent to its multiplicative inverse $u^{-1}$. Suppose, for contradiction, that $u$ and $u^{-1}$ are adjacent. This implies $uu^{1} \neq 1$ which is a contradiction. Therefore, $u$ and $u^{-1}$ are not adjacent.

We now show that all other vertices $v \in V\big(TE(\mathbb{Z}_n)\big) - \{u, u^1\}$ are adjacent to $u$. Let $v \in V\big(TE(\mathbb{Z}_n)\big) - \{u, u^1\}$. We construct $c = (uv)^{-1}$. Hence, $uvc = 1$. Now check the other conditions. Since $v \neq u^{-1}$, $uv \neq 1$. Since $v \neq 1$, $uc = v^{-1} \neq 1$. Since $u \neq 1$, $vc = u^{-1} \neq 1$. Hence, all conditions are satisfied and $v$ is adjacent to $u$. Therefore, $u$ has exactly $n-4$ neighbors, and thus $\deg(u) = n-4$.

Furthermore, by **Lemma 1**, vertex $w = (n-1)$ is adjacent to all other vertices in $TE(\mathbb{Z}_n)$. Hence, vertex $w$ has degree $(n-3)$ or $deg(w) = (n-3)$. Thus, $deg(w) > deg(u)$. Therefore, we conclude that the minimum degree in $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ is $n-4$. ■

The following theorem gives the sufficient condition of $TE(\mathbb{Z}_n)$ become a Hamiltonian graph.

**Theorem 6.** *If $n$ is prime and $n \geq 7$, then $TE(\mathbb{Z}_n)$ is a Hamiltonian graph.*

**Proof**. By Dirac's Theorem (**Corollary 6.2** on **[26]**), a sufficient condition for a simple graph $G$ with $k \geq 3$ vertices to be a Hamiltonian graph is if the degree of each vertex in $G$ is at least $\frac{k}{2}$. By **Theorem 5**, it is shown that the minimum degree of $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ is $k-2$, where $k$ is the number of vertices. It will be shown for the number of vertices in $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ that the sufficient condition is satisfied. The number of vertices or $k$ is $n-2$, so it can be written as $k = n-2$. Since the smallest $n$ in $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ is 7, then the smallest $k$ is 5. So,

$$\frac{k}{2} \leq k-2$$
$$k \leq 2k-4$$
$$4 \leq k$$

The condition $k/2 \leq k-2$ is satisfied when $k \geq 4$, while the smallest $k$ in $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ is 5. Therefore, the graph $TE(\mathbb{Z}_n)$ with $n$ prime and $n \geq 7$ satisfies the sufficient condition of the Hamiltonian graph. Thus, we obtain if $n$ is a prime and $n \geq 7$, then $TE(\mathbb{Z}_n)$ is a Hamiltonian graph. ■

**Example 6.** Given a graph $TE(\mathbb{Z}_{11})$, as shown in Figure 4.7. In $TE(\mathbb{Z}_{11})$, the number of vertices is $k = |V(TE(\mathbb{Z}_{11}))| = 9$, so the minimum number of degrees in $TE(\mathbb{Z}_{11})$ is $k-2 = 9-2 = 7$. Furthermore, based on Theorem 2.2.1, a sufficient condition for a simple graph $G$ with $k \geq 3$ vertices to be a Hamiltonian graph is if the degree of each vertex in $G$ is at least $k/2$. It is obtained that $k/2$ in $TE(\mathbb{Z}_{11})$ is $9/2$. Since $7 \geq 9/2$, $TE(\mathbb{Z}_{11})$ satisfies the sufficient condition so that $TE(\mathbb{Z}_{11})$ is a Hamiltonian graph. Next, we show the Hamiltonian cycle contained in $TE(\mathbb{Z}_{11})$, presented in **Figure 7**.

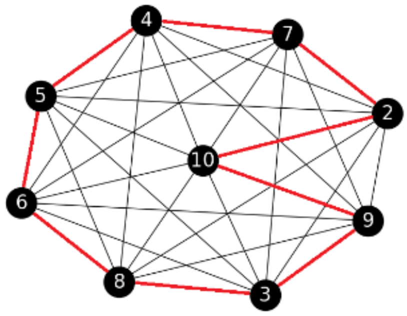



**Figure 7.** **The Hamiltonian Cycle of the Graph $TE(\mathbb{Z}_{11})$**

The Hamiltonian cycles contained in the $TE(\mathbb{Z}_{11})$ graph is $2 - 10 - 9 - 3 - 8 - 6 - 5 - 4 - 7 - 2$.

## 4. CONCLUSION

This research explores the structural properties of the triple idempotent graph of the ring $\mathbb{Z}_n$. Several results have been established. In particular, when $n = \{3, 4, 6\}$, the graph $TE(\mathbb{Z}_n)$ is empty. Furthermore, we show that $TE(\mathbb{Z}_n)$ is a connected graph if and only if $n$ is prime and $n \geq 5$. In such cases, the diameter of the graph is 2 and the girth is 3. We also prove that $TE(\mathbb{Z}_n)$ is Hamiltonian if $n$ is a prime number and $n \geq 7$.

While the current work has focused on several foundational graph-theoretic properties such as connectivity, diameter, girth, and Hamiltonicity, other important characteristics of $TE(\mathbb{Z}_n)$, including chromatic number, domination number, clique number, and planarity, have not yet been investigated. Additionally, although many structural results involve prime moduli, the behavior of the graph for general composite values of n remains a rich area for further exploration.

Future research may consider extending this study by analyzing the triple idempotent graph of a commutative ring $R$. Investigating the spectral properties and topological indices of the triple idempotent graph of a commutative ring could also provide deeper insights into the symmetries and invariants of these graphs.

## AUTHOR CONTRIBUTIONS

Vika Yugi Kurniawan: Conceptualization, Methodology, Resources, Validation, Writing - Review and Editing. Chessa Fanny Ekasiwi: Data Curation, Formal Analysis, Investigation, Software, Writing - Original Draft. Santoso Budi Wiyono: Funding Acquisition, Supervision, Project Administration. All authors discussed the results and contributed to the final manuscript.

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## CONFLICT OF INTEREST

The authors declare no conflicts of interest to report study

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