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MATHEMATICAL MODELLING OF SMOKING BEHAVIOR: TREATMENT AND PREVENTION OPTIMAL CONTROL

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ABSTRACT

Smoking remains a critical global public health challenge, with both traditional tobacco use and the rising prevalence of e-cigarettes contributing to significant morbidity and mortality. This study introduces a novel mathematical model that captures the dynamics of smoking behavior by explicitly integrating two smoker populations: traditional tobacco users and e-cigarette users. The model incorporates optimal control strategies aimed at prevention, via public health campaigns, and cessation, through smoking cessation treatments. The smoking model without control has two basic reproduction numbers for tobacco smokers and e-cigarette smokers, R_t and R_e. The extinction smoker's equilibrium is locally asymptotically stable if $R_t < 1$ and $R_e < 1$. The extinction tobacco smokers equilibrium is locally asymptotically stable if $R_t < R_e$ and $R_e > 1$. The endemic equilibrium tends to be asymptotically stable whenever $R_t > 1$ and $R_e > 1$. Simulations demonstrate that implementing control strategies significantly reduces smoking prevalence, with the combined two strategies achieving the most substantial reduction.

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1. INTRODUCTION

In this modern era, smoking has emerged as a critical and persistent public health challenge, significantly contributing to global morbidity and mortality. Smoking is known to cause blocked arteries and increase the risk of cardiovascular diseases such as heart attacks, strokes, and other vascular complications. Tobacco, the primary component of cigarettes, contains nicotine, a highly addictive substance that fosters dependence upon consumption. The World Health Organization (WHO) identifies tobacco use as one of the principal risk factors for cardiovascular diseases, premature death, and global disability [1]. Alarmingly, recent data indicate that tobacco consumption is responsible for over 8 million deaths annually worldwide [2]. Despite mandatory health warnings on cigarette packaging, smoking prevalence remains high, highlighting the ineffectiveness of current prevention measures.

Compounding this issue is the rising popularity of e-cigarettes, which are often perceived as a safer alternative to traditional tobacco products. As of now, 88 countries lack regulations on the minimum purchasing age for e-cigarettes [3], and only a few require a doctor's prescription for access [4]. Public misconceptions about the safety of e-cigarettes have further fueled their widespread adoption. However, evidence shows that e-cigarettes contain toxic substances that can lead to severe health problems, including cancer, cardiovascular issues, respiratory diseases, brain development impairments, and learning disorders, particularly in adolescents [2]. The dangers of e-cigarettes are comparable to those of traditional tobacco products, yet public awareness of these risks remains critically low.

Efforts to mitigate smoking behavior, such as the WHO Framework Convention on Tobacco Control's Article 11 provisions, mandate clear and graphic health warnings on cigarette packaging [5]. Nevertheless, smoking behaviors remain challenging to curtail due to the strong influence of peer groups, a key factor driving the spread of smoking habits. Behavioral transmission, including smoking, is influenced by complex factors such as media exposure, population dynamics, and evolving social norms [6].

Mathematical modeling offers a powerful approach to analyze and predict the dynamics of smoking behavior. Researchers have utilized differential equation-based models to explore various aspects of smoking spread and control. For instance, [4] employed an SIR (Susceptible-Infected-Recovered) framework to investigate peer influence on smoking initiation, while [7] developed a model incorporating interactions between vulnerable populations and active smokers based on the intensity of their interactions. Additionally, [8] analyzed interactions between traditional smokers and e-cigarette users, and [9] integrated a population of tuberculosis-infected smokers into a smoking dynamics model. Furthermore, [10] and [11] introduced mathematical models focused on control strategies. Additionally, [12] introduced a model distinguishing occasional and temporary quit smokers, coupled with optimal control strategies using the Pontryagin maximum principle. Moreover, [13] proposed a model incorporating media campaigns as a strategy to curtail smoking behavior, providing insights for public health policies and optimal control measures.

Building upon these foundational works, this study introduces a novel mathematical model to capture the dynamics of smoking behavior, explicitly integrating the rapidly growing population of e-cigarette users. Recognizing the significant health risks posed by both traditional and e-cigarettes, this model incorporates optimal control strategies aimed at prevention and cessation. By introducing targeted interventions for non-smokers and active smokers, the study seeks to provide actionable insights for public health policy. This dual focus on prevention and treatment is expected to not only reduce smoking prevalence but also serve as a valuable tool for policymakers in formulating effective strategies to address the evolving landscape of smoking behavior.

2. RESEARCH METHODS

In this section, we present the formulation of the mathematical model both without and with the implementation of control strategies. The first part of the section focuses on the model's baseline structure, which describes the dynamics of the system without any interventions. Then, we introduce control mechanisms in the form of public health campaigns highlighting the dangers of smoking, along with treatment programs aimed at reducing smoking prevalence.

2.1 Model Formulation

The population considered in this study is divided into four compartments: susceptible individuals (S), tobacco smokers (I_t), electronic cigarette smokers (I_e), and individuals who have quit smoking (R). Building upon previous studies, this research introduces a novel mathematical model that explicitly incorporates two distinct populations of smokers: traditional tobacco smokers and e-cigarette smokers. This dual-population approach is crucial, as it acknowledges the unique characteristics, health risks, and behavioral dynamics associated with each group. Unlike prior models that either focus solely on traditional smokers or treat all smoking behaviors uniformly, this model accounts for the interplay between these two populations, including transitions from traditional smoking to e-cigarette use. Transmission diagrams between smokers are shown in **Figure 1**.

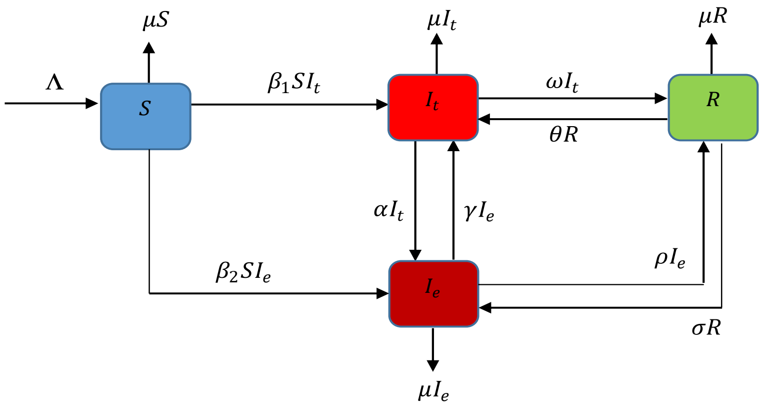


Figure 1. Compartement Diagram of the Smokers

Let $X = (S, I_t, I_e, R)^T$, we consider the following dynamical system:

$$\frac{dS}{dt} = \Lambda - \mu S - \beta_1 S I_t - \beta_2 S I_e, \tag{1}$$

$$\frac{dI_t}{dt} = \beta_1 S I_t + \gamma I_e + \theta R - (\mu + \omega + \alpha) I_t, \tag{2}$$

$$\frac{dI_e}{dt} = \beta_2 S I_e + \alpha I_t + \sigma R - (\mu + \gamma + \rho) I_e, \tag{3}$$

$$\frac{dR}{dt} = \omega I_t + \rho I_e - (\mu + \theta + \sigma) R. \tag{4}$$

With $N = S + I_t + I_e + R$ is the total of human population. In the smoker user model, there are several assumptions used as follows:

(1) The rate of individual recruitment is constant.

(2) Susceptible individuals can become tobacco smokers and electronic smokers.

(3) The population of individuals who quit smoking can return to being smokers.

(4) All parameter values are positive.

Description and dimensions for all compartments are shown in **Table 1**, while **Table 2** provides further detail about the parameter description.

Table 1. Variables Description

Variable	Description
$S(t)$	The population of individuals is vulnerable to smoking at this time t
$I_t(t)$	The Population of individual tobacco smokers at the time t
$I_e\ (t)$	The population of individuals electric cigarette smokers at the time t
$R(t)$	The population of individuals who have stopped using cigarettes at the time t

Table 2. Parameters Description

Parameter	Description
Λ	Rate of recruitment
β_1	Conversion rate of individuals from potential smokers to tobacco smokers compartment
β_2	Conversion rate of individuals from potential smokers to tobacco smokers compartment
α	Transmission rate from smokers moving to electric smokers
γ	Transmission rate from smokers moving to tobacco smokers
ρ	Transmission rate from electric smokers moving to the quitter class
ω	Transmission rate from tobacco smokers moving to the quitter class
θ	Relapse rate of tobacco smokers
σ	Relapse rate of electric smokers
μ	Natural death rate

2.2 Basic Reproduction Number

To reduce the complexity of the model and facilitate the exploration of its dynamics analytically, we assume that the transition parameters from e-cigarette smokers to tobacco smokers can be neglected. Additionally, transitions from individuals who have quit smoking back to either tobacco smokers or e-cigarette smokers are also excluded. The neglect of the transition from e-cigarette smokers to tobacco smokers is justified by the general observation that e-cigarette users rarely revert to tobacco smoking. This is primarily because e-cigarettes are perceived as a healthier or lower-risk alternative to traditional tobacco products. Furthermore, the transition from tobacco smokers to e-cigarette users remains included, reflecting the common trend of traditional smokers switching to e-cigarettes as a means to mitigate health risks or as a step toward reducing nicotine consumption intensity. The exclusion of individuals who have quit smoking back to either tobacco smokers or e-cigarette smokers is supported by the idea that individuals who have quit smoking tend to maintain their non-smoking status. Behavioral studies indicate that relapses are most common shortly after quitting and become increasingly rare among individuals who successfully maintain their non-smoking status over longer periods. By simplifying the model, we establish a reduced system intended solely for analytical analysis, with transmission diagrams shown in Figure 2 and formulated as follows:

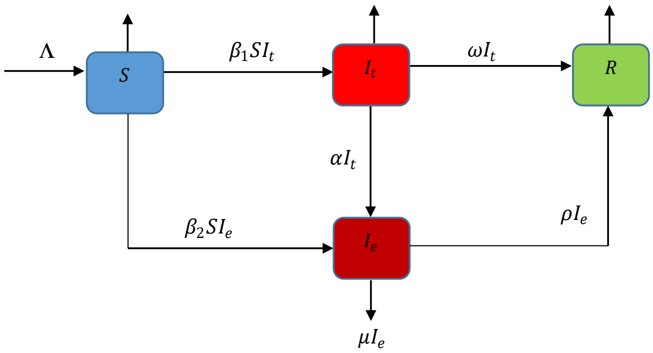


Figure 2. Transmission Diagrams of Reduced System

$$\frac{dS}{dt} = \Lambda - \mu S - \beta_1 S I_t - \beta_2 S I_e, \tag{5}$$

$$\frac{dI_t}{dt} = \beta_1 S I_t - m_1 I_t, \tag{6}$$

$$\frac{dI_e}{dt} = \beta_2 S I_e + \alpha I_t - m_2 I_e, \tag{7}$$

$$\frac{dR}{dt} = \omega I_t + \rho I_e - \mu R. \tag{8}$$

With $m_1 = \mu + \omega + \alpha$, and $m_2 = \mu + \rho$.

The basic reproduction number (R_0) represents the expected number of secondary cases generated by a single infected individual during its infectious period in a completely susceptible population [14]. This parameter serves as a critical threshold in epidemiological modeling. Specifically, if $R_0 > 1$, the infection is expected to spread within the population, potentially leading to an outbreak. Conversely, if $R_0 < 1$, the infection will eventually die out [15].

In this model, the value of R_0 is derived using the Next Generation Matrix (NGM) approach [16], which is constructed from the compartments representing the infected population. The NGM provides a systematic framework to quantify the spread of the infection by capturing transmission dynamics between different compartments. The matrix NGM_r for this model is obtained as follows:

$$NGM_r = \begin{bmatrix} \frac{\beta_1 \Lambda}{\mu m_1} & 0 \\ \frac{\alpha \beta_1 \Lambda}{\mu m_1 m_2} & \frac{\beta_2 \Lambda}{\mu m_2} \end{bmatrix}.$$

The basic reproduction number is defined as the largest eigenvalue of this matrix, which corresponds to the spectral radius; hence, we get:

$$R_0 = \max(R_t, R_e),$$

with $R_t = \frac{\beta_1 \Lambda}{\mu(\mu+\omega+\alpha)}$ and $R_e = \frac{\beta_2 \Lambda}{\mu(\mu+\rho)}$.

2.3 Equilibrium Points

Equilibrium point is a state where the population size remains constant over time [17]. There are three types of equilibria in this model, namely the extinction equilibrium points of smokers, the extinction equilibrium points of tobacco smokers, and the endemic equilibrium.

2.3.1 Extinction Equilibrium Point of Smokers

The extinction equilibrium point of smokers represents a critical state in epidemiological models where the population is entirely free from smokers, serving as a baseline for analyzing disease dynamics [18]. This condition occurs when there is no infection [19]. At this equilibrium, there are no active smokers in the population, meaning that all compartments associated with the disease dynamics are zero. Mathematically, this condition is expressed as:

$$I_t(t) = I_e(t) = 0.$$

Substituting these conditions into **Equations (5)** – **Equation (8)** in the equilibrium condition, the extinction equilibrium point of smokers is derived as:

$$E^0 = (S^0, I_t^0, I_e^0, R^0) = \left(\frac{\Lambda}{\mu}, 0,0,0\right).$$

2.3.2 Extinction Equilibrium Point of Tobacco Smokers

The extinction equilibrium point of tobacco smokers represents a steady-state condition where tobacco smoking behavior is eradicated from the population, but electronic cigarette use persists over time. Unlike the extinction equilibrium of smokers, which occurs when the system is free from any form of smoking, this equilibrium describes a scenario in which the tobacco smoker compartment reaches zero while the electronic cigarette smoker compartment remains non-zero. Mathematically, this condition is expressed as:

$$I_t(t) = 0 \quad \text{and} \quad I_e(t) \neq 0.$$

From **Equation (5)** – **Equation (8)** in the equilibrium condition, the endemic equilibrium point is derived as:

$$E^1 = (S^1, I_t^1, I_e^1, R^1) = \left(\frac{\Lambda}{\mu R_e}, 0, \frac{\mu(R_e-1)}{\beta_2}, \frac{\rho(R_e-1)}{\beta_2}\right).$$

Thus, the equilibrium point for the extinction of tobacco smokers exists when $R_e > 1$.

2.3.3 Endemic Equilibrium Point

The endemic equilibrium represents a steady-state condition in which the infection persists within the population over time **[20]**. Unlike the disease-free equilibrium, this state occurs when all compartments are non-zero, meaning that all populations coexist dynamically. Mathematically, this condition is expressed as:

$$S(t) \neq 0, It(t) \neq 0, I_e(t) \neq 0, R(t) \neq 0.$$

From **Equation (5)** – **Equation (8)** in the equilibrium condition, the endemic equilibrium point is derived as:

$$E^* = (S^*, I_t^*, I_e^*, R^*),$$

where:

$$S^* = \frac{\Lambda}{\mu + \beta_1 I_t^* + \beta_2 I_e^*},$$
$$I_t^* = \frac{\mu m_2[R_t - 1]\left(1 - \frac{R_e}{R_t}\right)}{\beta_1 m_2 - \beta_2(\mu + \omega)},$$
$$I_e^* = \frac{\alpha\mu[R_t - 1]}{\beta_1 m_2 - \beta_2(\mu + \omega)},$$
$$R^* = \frac{\omega I_t^* + \rho I_e^*}{\mu},$$

Thus, the endemic equilibrium point exists when $R_t > 1, \frac{\beta_1 m_2}{\beta_2(\mu+\omega)} > 1$, and $R_e < R_t$

2.4 Optimal Control

In general, an optimal control problem is a problem with the aim of finding a control (t) that can optimize (maximize or minimize) the performance index. The performance index is formulated as follows:

$$J = \varphi\left(x(t_f), t_f + \int_{t_0}^{t_f} L(x(t), u(t), t)dt,\right.$$

with systems

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t), \qquad x(t_0) = x_0,$$

where t_0 and t_f respectively, the initial time of control administration and the final time of administration. Meanwhile φ and L are scalar functions. Control $u(t)$ is optimal control if it can optimize the performance index.

Pontryagin's maximum principle **[21]** establishes the first-order conditions required to identify the optimal solutions for control problems. The procedure for solving optimal control problems using Pontryagin's Maximum Principle is as follows:

For example, given the equation of state:

$$\dot{x} = f(x(t), u(t), t),$$

with $x(t) \in \mathbb{R}^{\mathrm{n}}$ and $u(t) \in \mathbb{R}^n$, and the performance index:

$$J = \varphi\left(x(t_f), t_f + \int_{t_0}^{t_f} L(x(t), u(t), t)dt,\right.$$

with boundary condition values $x(t_0) = x_0$ and t_f given, while $x(t_f) = x_f$ is free.

A sufficient condition for minimizing the performance index J is to convert the state equation and performance index equation into the problem of minimizing the Hamiltonian function. The steps to solve optimal control problems with Pontryagin's Maximum Principle are as follows:

1. The form of the Hamiltonian function is a combination of the function of a problem $(L(x(t), u(t), t)$ and the product of the subject function in the form of a differential equation $(f(x(t), u(t), t))$ with the variable co-state (t). Here is the form of the Hamiltonian function:
$$H(x(t), u(t), \gamma(t), t) = L\ (x(t), u(t), t) + \gamma^T(t) f(x(t), u(t), t).$$
2. Minimize H over all vectors $u(t)$ name $\frac{\partial H}{\partial u} = 0$, so we get:
$$u^*(t) = H^*(x^*(t), \gamma^*(t), t),$$
3. Use the results from Step 2 in the Hamiltonian function that was formed from Step 1, so that the optimal H^* is obtained, namely:
$$H^*(x^*(t), u^*(t), \gamma^*(t), t) = H^*(x^*(t), \gamma^*(t), t).$$
4. Solve the Hamiltonian function that has been formed by:
   a. The state equation is the constraint equation in the model
   $$\dot{x}(t) = \frac{\partial H}{\partial \gamma},$$
   given the initial value $x(t_0) = x_0$.
   b. Co-state equations associated with accumulation constraints of state variables
   $$\dot{\gamma}(t) = -\frac{\partial H}{\partial x},$$
   with the final value $\gamma(t_f) = \left(\frac{\partial \varphi}{\partial x}\right)|t_f$.
5. Substitute the solutions of $x^*(t)$ and $\gamma^*(t)$ from Step 4 into the optimal control $u^*(t)$ obtained from Step 2 to obtain the optimal control.

To mitigate the prevalence of smoking behavior, we extend the mathematical model in **Equation (1)** - **Equation (4)** by incorporating two control strategies: u_1, representing campaigns highlighting the dangers of smoking, and u_2, representing treatments to assist individuals in ceasing smoking behavior. These controls operate under the assumption that susceptible individuals are exposed to environments where smoking behaviors can propagate.

Let $x_i = (S, I_t, I_e, R)^T$ represent the state variables of the system. The dynamical system with control variables is defined as $x_i = \left(f_1(x), f_2(x), f_3(x), f_4(x)\right)^T$, with the following equations:

$$\frac{dS}{dt} = \Lambda - \mu S - (1 - u_1)\beta_1 S I_t - (1 - u_1)\beta_2 S I_e,$$
$$\frac{dI_t}{dt} = (1 - u_1)\beta_1 S I_t + \gamma I_e + \theta R - (\mu + \omega + \alpha + \eta u_2) I_t,$$
$$\frac{dI_e}{dt} = (1 - u_1)\beta_2 S I_e + \alpha I_t + \sigma R - (\mu + \gamma + \rho + \eta u_2) I_e,$$
$$\frac{dR}{dt} = (\eta u_2 + \omega) I_t + (\eta u_2 + \rho) I_e - (\mu + \theta + \sigma) R.$$

The objective is to minimize the number of smokers (I_t and I_e) while balancing the costs associated with the control measures. To achieve this, we define the following objective function:

$$J(u_1, u_2) = \min_{u_1, u_2} \int_0^{t_f} \left[C_1 I_t + C_2 I_e + \frac{1}{2} C_3 u_1^2 + \frac{1}{2} C_4 u_2^2\right] dt,$$

where C_1 and C_2 are weighting parameters for the state variables I_t and I_e, respectively, and C_3 and C_4 are weighting parameters for the control variables u_1 and u_2. The control set is defined as $u^* = \{(u_1, u_2) : 0 \leq u_1, u_2 \leq 1\}$. Pontryagin's maximum principle [22] provides the first-order condition to determine the optimal solutions of the control problems. If the Hamiltonian function fulfills the state equation, the costate equation, and the stationarity condition, it will yield an optimal solution based on Pontryagin's principle [21]. To solve the optimization problem, we introduce the Hamiltonian function:

$$H = C_1 I_t + C_2 I_e + \frac{1}{2} C_3 u_1^2 + \frac{1}{2} C_4 u_2^2 + \sum_{i=1}^{4} \lambda_i x_i,$$

where $\lambda_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ are the adjoint (or costate) variables. The adjoint equations are derived as:

$$\begin{aligned}
\dot{\lambda_1} &= -\frac{\partial H}{\partial S} = (\lambda_1 - \lambda_2)(1 - u_1)\beta_1 I_t + (\lambda_1 - \lambda_3)(1 - u_1)\beta_2 I_e + \lambda_1 \mu, \\
\dot{\lambda_2} &= -\frac{\partial H}{\partial I_t} = -C_1 + (\lambda_1 - \lambda_2)(1 - u_1)\beta_1 S + (\lambda_2 - \lambda_3)\alpha + (\lambda_2 - \lambda_4)(\omega + \eta u_2) + \lambda_2 \mu, \\
\dot{\lambda_3} &= -\frac{\partial H}{\partial I_e} = -C_2 + (\lambda_1 - \lambda_3)(1 - u_1)\beta_2 S + (\lambda_3 - \lambda_2)\gamma + (\lambda_3 - \lambda_4)(\rho + \eta u_2) + \lambda_3 \mu, \\
\dot{\lambda_4} &= -\frac{\partial H}{\partial R} = (\lambda_4 - \lambda_2)\theta - (\lambda_4 - \lambda_3)\sigma + \mu \lambda_4,
\end{aligned}$$

with the transversal condition $\lambda_i(t_{end}) = 0$.

The optimal controls u_1 and u_2 satisfy the stationary condition $\frac{\partial H}{\partial u_i} = 0$, yielding:

$$\begin{aligned}
u_1 &= \frac{(\lambda_2 - \lambda_1)\beta_1 S I_t + (\lambda_3 - \lambda_1)\beta_2 S I_e}{C_3}, \\
u_2 &= \frac{(\lambda_2 - \lambda_4)\eta I_t + (\lambda_3 - \lambda_4)\eta I_e}{C_4}.
\end{aligned}$$

To ensure the controls remain within the feasible range, the solutions are projected as:

$$\begin{aligned}
u_1^* &= \max\left\{0, \min\left(1, \frac{(\lambda_2 - \lambda_1)\beta_1 S I_t + (\lambda_3 - \lambda_1)\beta_2 S I_e}{C_3}\right)\right\}, \\
u_2^* &= \max\left\{0, \min\left(1, \frac{(\lambda_2 - \lambda_1)\eta I_t + (\lambda_3 - \lambda_1)\eta I_e}{C_4}\right)\right\}.
\end{aligned}$$

To substitute the optimal control equation into the Hamiltonian equation to obtain the optimal Hamiltonian H. Determine the state equation in the optimal state:

$$\begin{aligned}
\dot{S} &= \frac{\partial H}{\partial \lambda_1} = \Lambda + -\mu S - (1 - u_1)\beta_1 S I_t - (1 - u_1)\beta_2 S I_e, \\
\dot{I_t} &= \frac{\partial H}{\partial \lambda_3} = (1 - u_1)\beta_1 S I_t + \gamma I_e + \theta R - (\mu + \omega + \alpha + \eta u_2) I_t, \\
\dot{I_e} &= \frac{\partial H}{\partial \lambda_4} = (1 - u_1)\beta_2 S I_e + \alpha I_t + \sigma R - (\mu + \gamma + \rho + \eta u_2) I_e, \\
\dot{R} &= \frac{\partial H}{\partial \lambda_5} = (\eta u_2 + \omega) I_t + (\eta u_2 + \rho) I_e - \mu R.
\end{aligned}$$

Once the state equation is obtained, the next step is to derive the co-state equation. The co-state equation is the result of taking the partial derivative of the Hamiltonian function with respect to the system's variables. The co-state equation is calculated as follows:

$$\begin{aligned}
\dot{\lambda_1} &= -\frac{\partial H}{\partial S} = (\lambda_1 - \lambda_2)\varepsilon M + (\lambda_1 - \lambda_3)(1 - u_1)\beta_1 I_t + (\lambda_1 - \lambda_4)(1 - u_1)\beta_2 I_e + \lambda_1 \mu, \\
\dot{\lambda_2} &= -\frac{\partial H}{\partial I_t} - C_1 + (\lambda_1 - \lambda_3)(1 - u_1)\beta_1 S + (\lambda_3 - \lambda_4)\alpha + (\lambda_3 - \lambda_5)(\omega + \eta u_2) + \lambda_3 \mu - \lambda_6 \tau, \\
\dot{\lambda_3} &= -\frac{\partial H}{\partial I_e} - C_2 + (\lambda_1 - \lambda_4)(1 - u_1)\beta_2 S + (\lambda_4 - \lambda_3)\gamma + (\lambda_4 - \lambda_5)(\rho + \eta u_2) + \lambda_4 \mu - \lambda_6 \tau,
\end{aligned}$$

$$\dot{\lambda_4} = -\frac{\partial H}{\partial R} = (\lambda_5 - \lambda_3)\theta + (\lambda_5 - \lambda_4)\sigma + \lambda_5\mu.$$

3. RESULTS AND DISCUSSION

3.1 Stability Analysis of Equilibrium Point

This section conducts a stability analysis of the three equilibrium points previously identified. The purpose of this stability analysis is to understand the dynamic properties of each equilibrium point and the conditions influencing their persistence. To analyze stability, an analytical method is applied using the eigenvalue approach on the Jacobian matrix for the smoker-free and tobacco-smoker-free equilibrium points. Additionally, a numerical approach is employed to assess the stability of the endemic equilibrium. This approach enables a detailed exploration of the model's dynamics, particularly in scenarios where analytical solutions are challenging to obtain.

3.1.1 Local Stability of the Extinction Equilibrium Point of Smokers

The local stability of the extinction equilibrium point of smokers is obtained by substituting the value of this equilibrium (E^0) into the Jacobian matrix as follows:

$$J(E^0) = \begin{pmatrix} -\mu & -\frac{\beta_1\Lambda}{\mu} & -\frac{\beta_2\Lambda}{\mu} & 0 \\ 0 & \frac{\beta_1\Lambda}{\mu} - m_1 & 0 & 0 \\ 0 & \alpha & \frac{\beta_2\Lambda}{\mu} - m_2 & 0 \\ 0 & \omega & \rho & -\mu \end{pmatrix}.$$

From the matrix $J(E^0)$, we will look for the characteristic equation with $|\lambda I - J(E^0)|$, such that we obtain:

$$(\lambda + \mu)^2 \left(\lambda + m_1 - \frac{\beta_1\Lambda}{\mu}\right)\left(\lambda + m_2 - \frac{\beta_2\Lambda}{\mu}\right) = 0, \tag{9}$$

From **Equation (9)**, we have the eigenvalues $\lambda_1 = \lambda_2 = -\mu$, which are negative, also $\lambda_3 = m_1[R_t - 1]$, which is negative when $R_t < 1$, and $\lambda_4 = m_2[R_e - 1]$, which is negative when $R_e < 1$. Therefore, the extinction equilibrium point of smokers will be locally asymptotically stable if $R_t < 1$ and $R_e < 1$. The foregoing discussion could be summarized in the following theorem.

Theorem 1. *The extinction equilibrium point of smokers* (E^0) *of the system is locally asymptotically stable if* $R_t < 1$ *and* $R_e < 1$.

3.1.2 Local Stability of the Extinction Equilibrium Point of Tobacco Smokers

The local stability of the extinction equilibrium point of tobacco smokers is obtained by substituting the value of this equilibrium (E^1) into the Jacobian matrix as follows:

$$J(E^1) = \begin{pmatrix} -\mu R_e & -\frac{\beta_1\Lambda}{\mu R_e} & -\frac{\beta_2\Lambda}{\mu R_e} & 0 \\ 0 & \frac{\beta_1\Lambda}{\mu R_e} - m_1 & 0 & 0 \\ \mu(R_e - 1) & \alpha & \frac{\beta_2\Lambda}{\mu R_e} - m_2 & 0 \\ 0 & \omega & \rho & -\mu \end{pmatrix}.$$

From the matrix $J(E^1)$, we will look for the characteristic equation with $|\lambda I - J(E^0)|$, such that we obtain:

$$(\lambda+\mu)\left(\lambda+m_1-\frac{\beta_1\Lambda}{\mu R_e}\right)\left(\lambda^2+(\mu R_e)\lambda+\mu m_2(R_e-1)\right)=0, \tag{10}$$

From **Equation (10)**, we have the eigenvalues $\lambda_1=-\mu$, which is negative, $\lambda_2=m_1\left[\frac{R_t}{R_e}-1\right]$, which is negative when $R_t<R_e$, and the remainder are the roots of the following equation:

$$\lambda^2+(\mu R_e)\lambda+\mu m_2(R_e-1) \tag{11}$$

By using the Routh-Hurwitz criterion, the characteristic **Equation (11)** will have roots with negative real parts if and only if $\mu R_e, \mu m_2(R_e-1)>0$. It is clear that $\mu R_e>0$, while $\mu m_2(R_e-1)>0$ if $R_e>1$. Therefore, the extinction equilibrium point of tobacco smokers will be locally asymptotically stable if $R_t<R_e$ and $R_e>1$. The foregoing discussion could be summarized in the following theorem.

Theorem 2. *The extinction equilibrium point of tobacco smokers (E^1) of the system is locally asymptotically stable if $R_t<R_e$ and $R_e>1$.*

3.1.3 Stability of Endemic Equilibrium Point

Due to the analytical complexity of determining the stability of the endemic equilibrium, we investigate its stability through numerical methods. In this simulation, we employ three different initial conditions while keeping the parameter values constant to observe the convergence of the trajectories over an extended period. This simulation uses the parameter set from **Table 3** and initial population values from **Table 4**.

Table 3. Parameters Value for Simulation

Parameter	Value for Endemic Condition	Reference
Λ	20	[13]
β_1	0.0000007	Assumed
β_2	0.00000005	[13]
α	0.05	Assumed
γ	0.05	Assumed
ρ	0.01	[13]
ω	0.01	[13]
θ	0.06	[13]
σ	0.06	[13]
μ	0.00005479	[13]

Table 4. Initial Value for Phase Field Simulation

Initial Value	$S(0)$	$I_t(0)$	$I_e(0)$	$R(0)$	Color
z_1	1,000	200	1,000	20,000	Red
z_2	2,000	100	3,000	10,000	Green
z_3	4,000	300	5,000	30,000	Blue

To provide a more comprehensive understanding of the system, this simulation focuses on the I_t, I_e, and R populations. This three-dimensional representation captures the trajectories of the system as it evolves towards equilibrium. The visualization clearly differentiates between the three primary equilibrium states: the extinction equilibirum point of smokers $(I_t=I_e=R=0)$, extinction equilibrium point of tobacco smokers $(I_t = 0, I_e, R \neq 0)$ and the endemic equilibrium $(I_t, I_e, R \neq 0)$. The results of the simulation are illustrated in **Figure 3**.

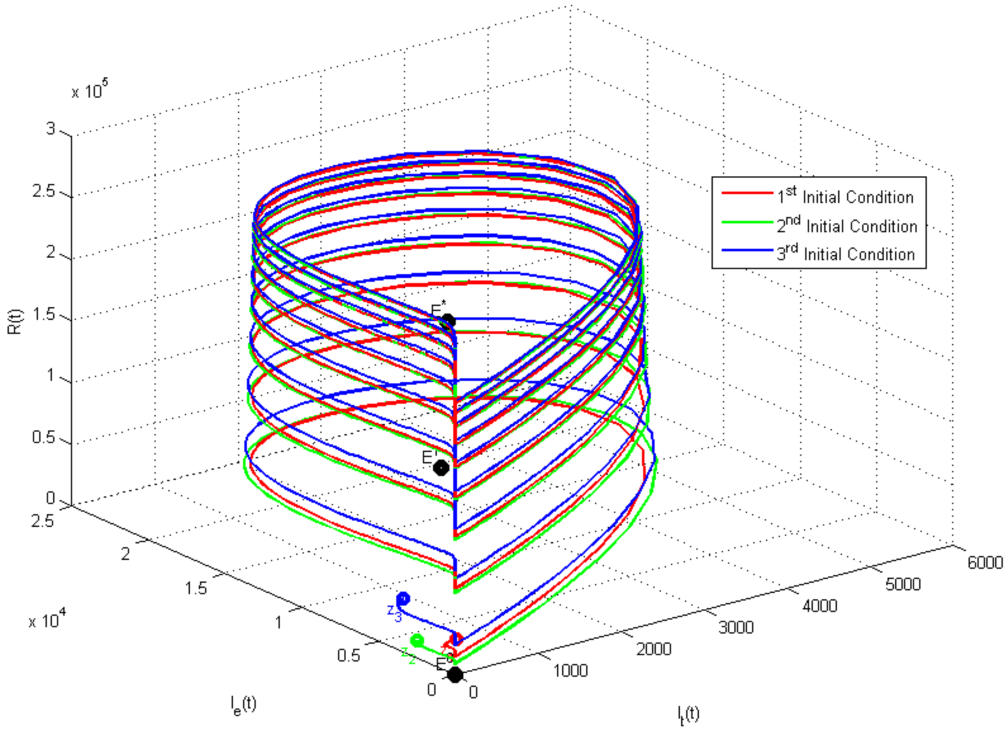


Figure 3. Phase Field Simulation

By substituting the parameter values from **Table 3**, we get $R_t = 4.25$ and $R_e = 1.82$. Since $R_0 = \max\{R_t, R_e\}$, hence the value of R_0 for these parameter values is found to be $R_0 = 4.25 > 1$.

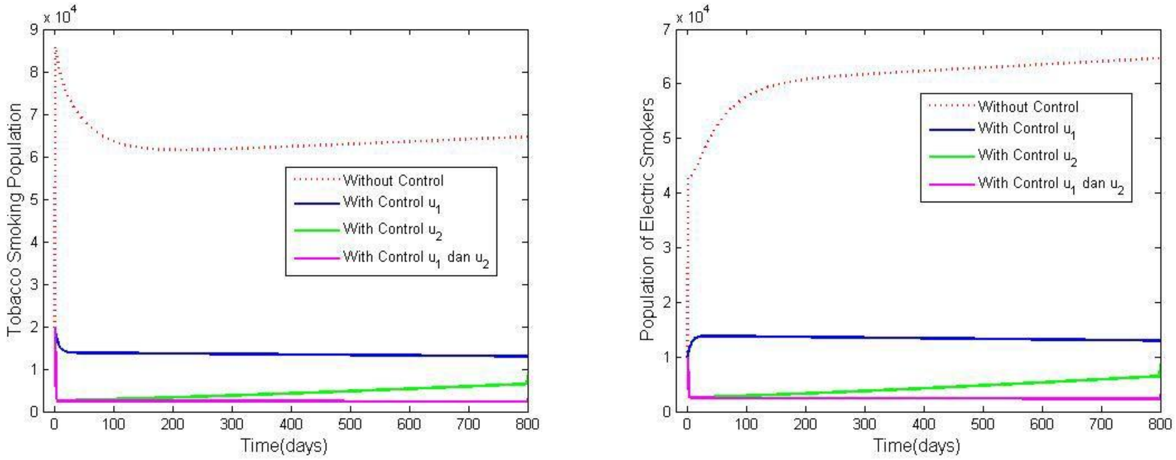
Next, by solving the system of **Equation (5)** - **Equation (8)** with the given parameter values, three equilibrium points are obtained:

$$E^0 = (S; I_t; I_e; R) = (3.65 \times 10^5; 0;\ 0;\ 0)$$
$$E^1 = (S; I_t; I_e; R) = (2.01 \times 10^5; 0;\ 893;\ 1.63 \times 10^5)$$
$$E^* = (S; I_t; I_e; R) = (8.58 \times 10^4; 157;\ 1364;\ 2.78 \times 10^5)$$

E^1 exists because $R_e = 1.82 > 1$. Similarly E^* exists because $R_t = 4.25 > 1$, $\frac{\beta_1 m_2}{\beta_2(\mu+\omega)} = 14.00 > 1$ and $R_e = 1.82 < 4.25 = Rt$. Furthermore, based on **Figure 3**, we observe that when the asymptotic stability conditions for E^0 dan E^1 are not satisfied, and E^* exists, the endemic equilibrium point tends to be asymptotically stable.

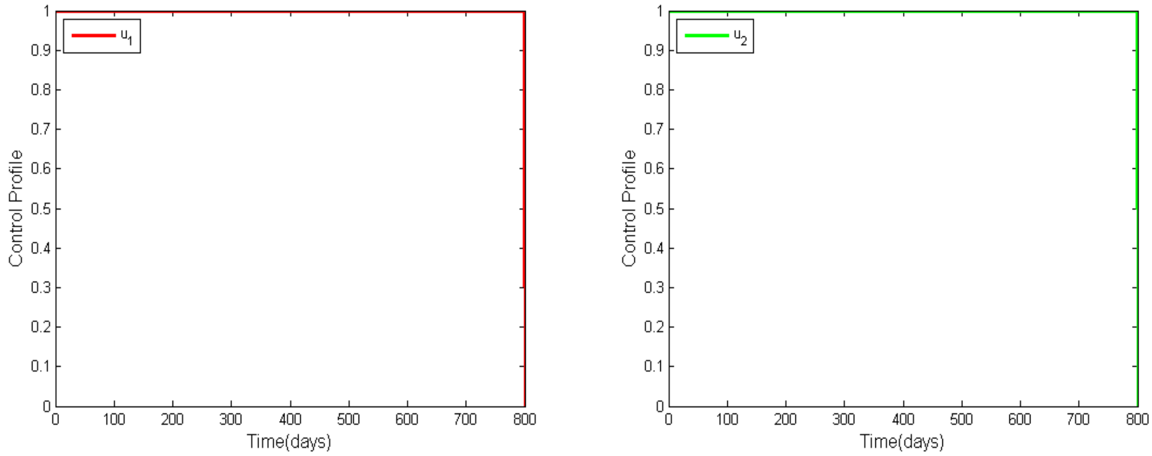
3.2 Optimal Control Simulation

This optimal control problem is numerically solved using the forward-backward sweep method **[23]**. Numerical simulations were conducted to evaluate the effectiveness of two control strategies aimed at reducing smoking prevalence: public campaigns highlighting the dangers of smoking (u_1) and smoking cessation treatments (u_2). Using the parameter values in **Table 3**, except $\beta_1 = 0.00000007$. The results, presented in **Figure 4** and **Figure 5**, illustrate the dynamics of tobacco smokers (I_t) and electronic cigarette smokers (I_e) under four scenarios: without control, with a single control (u_1 or u_2), and with combined controls (u_1,u_2).

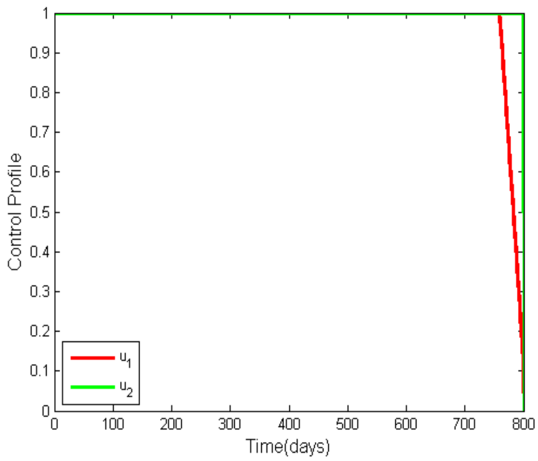


(a) (b)

Figure 4. Optimal Control Simulation of (a) Tobacco Smoker and (b) E-Cigarette Smoker Population



(a) (b)



(c)

Figure 5. Control Profile of (a) Single u_1, (b) Single u_2 and (c) Combination u_1 and u_2

From the simulation results, it is evident that the controls u_1, u_2, and the combination (u_1, u_2) significantly reduce the prevalence of smoking within the population. The combined control strategy was particularly effective, demonstrating the potential to achieve both prevention (through campaigns) and cessation (through treatments).

Table 5. Comparison of Cost Function Values for Three Strategies

Strategy	Cost Function Value (Hundred Rupiah)
Single Control u_1	2.1457×10^7
Single Control u_2	6.9344×10^6
Combination Controls u_1 and u_2	3.8512×10^6

Table 5 summarizes the cost function value for each control strategy. The findings indicate that the combined control strategy achieved the most significant reduction in smoking prevalence. However, the single controls also showed substantial effectiveness, particularly in targeting specific smoking behaviors. These findings emphasize the importance of implementing both preventive and cessation-oriented interventions as a comprehensive approach to reducing smoking prevalence. Public health authorities can leverage these results to design targeted strategies that address the spread of smoking behavior in diverse populations.

4. CONCLUSIONS

This study presents a refined mathematical model to capture the intricate dynamics of smoking behavior, incorporating both traditional tobacco smokers and e-cigarette users. The model employs optimal control strategies: u_1, representing prevention campaigns, and u_2, representing cessation treatments, to assess their effectiveness in reducing smoking prevalence. Stability analysis, with the assumption of neglecting the transition parameters from e-cigarette smokers to tobacco smokers and transitions from individuals who have quit smoking back to either tobacco smokers or e-cigarette smokers, results in three equilibrium points. The extinction equilibrium of smokers is locally asymptotically stable if $R_t < 1$ and $R_e < 1$. The extinction equilibrium of tobacco smokers of the system is locally asymptotically stable if $R_t < R_e$ and $R_e > 1$. The endemic equilibrium point tends to be asymptotically stable when this equilibrium exists. Simulation results demonstrated that while individual strategies u_1 and u_2 significantly reduce smoking rates, their combined implementation achieves the most substantial overall reduction. These findings highlight the critical role of integrated prevention and cessation efforts in controlling smoking behavior. By addressing both traditional and emerging forms of smoking, such as e-cigarette use, this model offers valuable insights for public health policymakers. It underscores the importance of implementing comprehensive strategies to curb smoking prevalence, ultimately contributing to better health outcomes on a global scale.

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