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GRAPH ENERGY OF THE COPRIME GRAPH ON GENERALIZED QUATERNION GROUP

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ABSTRACT

This paper investigates the Degree Square Sum Energy ($DSSE$)*, Degree Exponent Energy* (DEE)*, and Degree Exponent Sum Energy* ($DESE$) *of the coprime graph on generalized quaternion group* Q_{4n}*. This research is quantitative study using previous study as the literature review to construct the new theorem. These energy methods provide new insights into the spectral properties of graphs by their vertex degree distributions into eigenvalue computations. Using spectral graph theory, the general formulas for the* $DSSE$, DEE, *and* $DESE$ *of* Q_{4n} *are formulated for* $n = 2^k$ *for every positive integer* k*. Furthermore, we explore the implications of these methods in understanding the algebraic and spectral characteristics of* Q_{4n}*. Numerical results are presented for specific cases to validate the previous theorem. This study contributes to the broader analysis of graph energies, offering a framework for studying other algebraic structures.*

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1. INTRODUCTION

Graph theory is used to represent discrete objects and their relationships. In recent years, it has been widely applied to algebraic structures such as groups and rings. One significant approach is the use of topological indices to analyze graph representations of these structures. These indices, which assign numerical values to graphs, provide insights into connectivity, distance, and complexity. The coprime graph, introduced by Ma [1], was later extended to the generalized quaternion group by Nurhabibah, who also studied its numerical invariants [2]. Conversely, the non-coprime graph, which connects elements with a greatest common divisor greater than one, has been analyzed for its spectral and structural properties [3],[4], [5], [6].

The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix . This concept provides a bridge between algebraic graph theory and spectral graph theory, offering insight into the structural and spectral properties of graphs [7]. Over the years, various generalizations and variants of graph energy have been proposed, including degree-based graph energies such as the Degree Square Sum Energy (DSSE) [8], [9] the Degree Exponent Energy (DEE) [10], and the Degree Exponent Sum Energy (DESE) [11]. These indices aim to capture different structural characteristics of graphs by utilizing vertex degree information raised to various powers, thus extending the analysis beyond traditional spectral properties.

In graph theory, the concept of graph energy is an important approach to understanding the spectral properties of graphs, particularly those related to the adjacency matrix and the eigen spectrum of the graph [12],[13]. The intriguing variant in the development of graph energy is DSSE, DEE and DESE. This concept arises as an effort to broaden the scope of graph energy analysis by considering the degrees of the graph's vertices.

DSSE is defined based on the Degree Square Sum Matrix (DSSM) of a graph G, which is a matrix where each element is determined by the square of the sum of the degrees of the connected vertex pairs [14]. The DSSE differs from the traditional adjacency matrix because its elements not only represent the connections between vertices but also weigh these connections based on the squared degrees of the corresponding vertices. Mathematically, the DSSE is calculated as the sum of the absolute values of the eigenvalues derived from the DSSM [15],[16].

DEE is defined as the sum of the absolute values of the eigenvalues of the Degree Exponent Matrix (DEM). By incorporating exponents in the matrix, DEE provides a novel way to analyze the spectral energy distribution of graphs, emphasizing the influence of vertex degrees. This approach offers deeper insights into how the local structure of vertices (via their degrees) impacts the global properties of the graph [17].

DESE extends the concept of graph energy by considering the Degree Exponent Sum (DES), which is obtained from the sum of the exponentiated vertex degrees in a graph. This approach offers several advantages, such as a direct relationship with the topological properties of the graph and potential applications in fields like network theory, bioinformatics, and communication network engineering [18], [19],[20].

This article will formulate the DSSE, DEE, and DESE on the coprime graph of the generalized quaternion group. Research on the energy of quaternion groups in their coprime graphs is intriguing because it combines two significant fields: group theory and spectral graph theory. Quaternion groups possess a unique structure compared to ordinary abelian groups, resulting in coprime graphs with distinct spectral characteristics.

2. RESEARCH METHODS

This research is a study that using a literature review of previous studies to formulate the new theorem. The research begins with a literature review, followed by deriving the general formula for the DSSE, DEE and DESE of the coprime graph on generalized quaternion group, generalized for several cases of n. Subsequently, a conjecture is formulated, and the conjecture is proven. If the conjecture is validated, it is then established as a theorem.

Several references utilized in this study are as follows:

Definition 1.[21] *A generalized quaternion group* Q_{4n} *is a group presented as*:

$$\langle x, y \mid x^{2n} = e, x^n = y^2, y^{-1}xy = x^{-1} \rangle. \quad (1)$$

In this group, e is an identity element, $n \geq 2$, $x^k y = yx^{-k}$ and the order of $x^k y$ is 4 for every positive integer k. For example, $Q_8 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ are the elements of a generalized quaternion group with order eight, where $n = 2$.

Definition 2.[22] *For a finite group G, the coprime graph of G, denoted as Γ_G, is defined as a graph with the vertex set $V(\Gamma_G) = G$. Two distinct vertices x and y are connected by an edge if and only if the orders of x and y, denoted by $|x|$ and $|y|$, are coprime or $(|x|, |y|) = 1$.*

In this paper, the $DSSE$ and the DEE of a coprime graph of the generalized quaternion groups are determined. Therefore, their definitions are provided in the following.

Definition 3.[23] *Let Γ be a graph and η be an eigenvalue of the matrix graph of Γ, then the energy of Γ is defined as*

$$E(\Gamma) = \sum_{i=1}^{n} |\eta_i|, \quad (2)$$

where n is the number of eigenvalues.

Next, the definitions of Degree Square Sum Matrix (DSSM), Degree Exponent Matrix (DEM) and the Degree Exponent Sum Matrix (DESM) are stated.

Definition 4.[24] *Let Γ be a graph, then the Matrix of Degree Square Sum of Γ is $DSS(\Gamma) = [dss_{ij}]$ with:*

$$dss_{ij} = \begin{cases} {d_i}^2 + {d_j}^2, & if\ i \neq j; \\ 0, & otherwise; \end{cases} \quad (3)$$

where the DSSE will be denoted by E_{DSS}.

Definition 5.[14] *Let Γ be a graph, then the Degree Exponent Matrix of Γ is defined as $DE(\Gamma) = [de_{ij}]$ with:*

$$dss_{ij} = \begin{cases} {d_i}^{d_j}, & if\ i \neq j; \\ 0, & otherwise; \end{cases} \quad (4)$$

where the DEE will be denoted by E_{DE}.

Definition 6.[25] *Let Γ be a graph, then the Matrix of Degree Exponent Sum of Γ is $DES(\Gamma) = [des_{ij}]$ with:*

$$dss_{ij} = \begin{cases} {d_i}^{d_j} + {d_j}^{d_j}, & if\ i \neq j; \\ 0, & otherwise; \end{cases} \quad (5)$$

where the DESE will be denoted by E_{DES}.

The subsequent lemmas are presented as a basis for deriving the DEE and DSSE on the coprime graph of generalized quaternion groups.

Lemma 1.[26] *Given a generalized quaternion group Q_{4n} with $n = 2^k$ for every positive integer k, then the coprime graph of Q_{4n} is a complete bipartite graph.*

Lemma 2.[2] *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for every positive integer k, then $deg(e) = 4n - 1$ and $deg(v) = 1$ for every $v \in Q_{4n} \backslash \{e\}$.*

The Lemma 2 above suggests the numerical degree of the vertices in the graph.

Lemma 3.[27] *If $a, b, c, d \in \mathbb{R}$, then the matrix*

$$A = \begin{pmatrix} (\eta + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\eta + b)I_{n_2} - bJ_{n_2} \end{pmatrix}$$

has the determinant

$$|A| = (\eta + a)^{n_1 - 1}(\eta + b)^{n_2 - 1}[(\eta - (n_1 - 1)a)(\eta - (n_2 - 1)b) - n_1 n_2 cd]. \quad (6)$$

3. RESULTS AND DISCUSSION

In this section, we will determine the $DSSE$ and the DEE of coprime graph on generalized quaternion group with order 2^k for every positive integer k.

3.1. Degree Square Sum Energy

The DSSE of the coprime graph of a generalized quaternion group is given in the following theorem.

Theorem 1. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for positive integer k, then the characteristics polynomial of $DSS(\Gamma_{Q_{4n}})$ is:*

$$P_{DSS}\left(\Gamma_{Q_{4n}}\right)(\eta) = (\eta+2)^{4n-2}[(\eta^2 - 4(2n-1)\lambda) - (4n-1)(16n^2-8n+1)^2].$$

Proof. By using Lemma 1 and Lemma 2, the graph is complete bipartite graph with $deg(e) = 4n-1$ and $deg(v) = 1$. Then, $dss_{ij} = (4n-1)^2 + 1^2$ if $i = e$ and $j = v \in V(\Gamma_{Q_{4n}})\backslash\{e\}$, $dss_{ij} = 1^2 + (4n-1)^2$ if $i = v \in V(\Gamma_{Q_{4n}})\{e\}$ and $j = e$, and $dss_{ij} = 1^2 + 1^2$ if $i = v_1$ and $j = v_2$, where $v_1, v_2 \in V(\Gamma_{Q_{4n}})\backslash\{e\}$ and $v_1 \neq v_2$. Degree Square Sum Matrix of $\Gamma_{Q_{4n}}$ is:

$$DSS(\Gamma_{Q_{4n}}) = \begin{pmatrix} 0 & (4n-1)^2+1^2 & \cdots & (4n-1)^2+1^2 \\ 1^2+(4n-1)^2 & 0 & \cdots & 1^2+1^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^2+(4n-1)^2 & 1^2+1^2 & \cdots & 0 \end{pmatrix}$$

$$DSS(\Gamma_{Q_{4n}}) = \begin{pmatrix} 0 & 16n^2-8n+2 & \cdots & 16n^2-8n+2 \\ 16n^2-8n+2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 16n^2-8n+2 & 2 & \cdots & 0 \end{pmatrix} \quad (7)$$

By using Lemma 3, the matrix in Eq. (7) can be written into four block matrices bellow:

$$DSS(\Gamma_{Q_{4n}}) = \begin{pmatrix} 0 & (16n^2-8n+2)J_{1\times(4n-1)} \\ (16n^2-8n+2)J_{(4n-1)\times 1} & 2(J-I)_{(4n-1)\times(4n-1)} \end{pmatrix} \quad (8)$$

To find the characteristic polynomial of $DSS(\Gamma_{Q_{4n}})$, the matrix Eq. (8) can be written as:

$$P_{DSS}(\Gamma_{Q_{4n}})(\lambda) = \begin{vmatrix} \lambda & -(16n^2-8n+2)J_{(4n-1)\times 1} \\ -(16n^2-8n+2)J_{(4n-1)\times 1} & (\lambda+2)I_{(4n-1)} - 2J_{(4n-1)} \end{vmatrix}. \quad (9)$$

The matrix in Eq. (9) is derived from the operation of det $\left(DSS(\Gamma_{Q_{4n}}) - \lambda I\right)$.

According to Lemma 3 with $a = 0, b = 2$ and $c = d = 16n^2 - 8n + 2$, the matrix in Eq. (9) has the determinant as follows:

$$\begin{aligned} P_{DSS}(\Gamma_{Q_{4n}})(\lambda) &= (\lambda+0)^{1-1}(\lambda+2)^{4n-2}[(\lambda-(1-1)0)(\lambda-(4n-2)2) - (4n-1) \\ &\quad (16n^2-8n+2)(16n^2-8n+2)] \\ &= (\lambda+2)^{4n-2}[(\lambda(\lambda-8n+4)) - (4n-1)(16n^2-8n+2)^2] \\ &= (\lambda+2)^{4n-2}[\lambda^2 - 4(2n-1)\lambda - (4n-1)(16n^2-8n+2)^2]. \blacksquare \end{aligned}$$

Theorem 1 establishes a fundamental property of the graph structure under consideration. Building on this result, we now extend the analysis to derive a more specific characterization in Theorem 2.

Theorem 2. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for positive integer k, then the DSSE of Γ is:*

$$E_{DSS}(\Gamma_{Q_{4n}}) = 2\left(4n - 2 + \sqrt{8(2n-1)^2 + 2(4n-1)(16n^2-8n+1)^2}\right).$$

Proof. By setting the characteristic polynomial of Theorem 1 to zero, we will get 3 eigenvalues, $\lambda_1 = -2$ with multiplicity $4n-2$, $\lambda_2 = 2(2n-1) + \sqrt{8(2n-1)^2 + 2(4n-1)(16n^2-8n+1)^2}$ with multiplicity 1, and $\lambda_3 = 2(2n-1) - \sqrt{8(2n-1)^2 + 2(4n-1)(16n^2-8n+1)^2}$ with multiplicity 1.

According to Definition 3, the $DSSE$ of G is obtained by summing all the eigenvalues, as below:

$$
\begin{aligned}
E_{DSS}(\Gamma_{Q_{4n}}) &= (4n-2)|-2| + \left|2(2n-1)+\sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2}\right| + \\
&\quad |2(2n-1)-\sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2}| \\
&= (8n-4)+2(2n-1)+\sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2} + \\
&\quad \sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2} - 2(2n-1) \\
&= 8n-4+2\sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2} \\
&= 2\left(4n-2+\sqrt{8(2n-1)^2+2(4n-1)(16n^2-8n+1)^2}\right). \blacksquare
\end{aligned}
$$

3.2. Degree Exponent Energy

The DEE of the coprime graph on the generalized quaternion group is given in the following theorem.

Theorem 3. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for every positive integer k, then the characteristics polynomial of $DE(\Gamma_{Q_{4n}})$ is:*

$$P_{DE}(\Gamma_{Q_{4n}})(\lambda) = (\lambda+1)^{4n-2}[(\lambda^2-2(2n+1)\lambda)-(16n^2-8n+1)].$$

Proof. Let $n = 2^k$ with for positive integer k. By using Lemma 1 and Lemma 2, the graph is complete bipartite graph with $deg(e) = 4n-1$ and $deg(v) = 1$. Then, $dss_{ij} = (4n-1)^1$ for $i = e$ and $j = v \in V(\Gamma_{Q_{4n}})\backslash\{e\}$, $dss_{ij} = 1^{4n-1}$ for $i = v \in V(\Gamma_{Q_{4n}})\{e\}$ and $j = e$, and $dss_{ij} = 1^1$ for $i = v_1$ and $j = v_2$, where $v_1, v_2 \in V(\Gamma_{Q_{4n}})\backslash\{e\}$ and $v_1 \neq v_2$. Then the Degree Exponent Matrix of $\Gamma_{Q_{4n}}$ is:

$$DE(\Gamma_{Q_{4n}}) = \begin{pmatrix} 0 & (4n-1)^1 & \cdots & (4n-1)^1 \\ 1^{4n-1} & 0 & \cdots & 1^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1^{4n-1} & 1^1 & \cdots & 0 \end{pmatrix}. \tag{10}$$

By using Lemma 3, the matrix Eq. (10) can be written into four block matrices bellow:

$$DE(\Gamma_{Q_{4n}}) = \begin{pmatrix} 0 & (4n-1)J_{1\times(4n-1)} \\ J_{(4n-1)\times 1} & (J-I)_{(4n-1)\times(4n-1)} \end{pmatrix}. \tag{11}$$

By using Lemma 3, to find the characteristic polynomial of $DEE(\Gamma_{Q_{4n}})$ the matrix Eq. (11) can be written as:

$$P_{DE}(\Gamma_{Q_{4n}})(\lambda) = \begin{vmatrix} \lambda & -(4n-1)J_{(4n-1)\times 1} \\ -J_{(4n-1)\times 1} & (\lambda+1)I_{(4n-1)} - J_{(4n-1)} \end{vmatrix}. \tag{12}$$

The matrix Eq. (12) is derived from the operation of det $\left(DE(\Gamma_{Q_{4n}}) - \lambda I\right)$.

Then, the matrix Eq. (12) has the determinant as follows:

$$
\begin{aligned}
P_{DE}(\Gamma_{Q_{4n}})(\lambda) &= (\lambda+0)^{1-1}(\lambda+1)^{4n-2}\left[(\lambda-(1-1)0)(\lambda-(4n-2)) - (4n-1)(4n-1)\right] \\
&= (\lambda+1)^{4n-2}\left[(\lambda(\lambda-4n-2)) - (16n^2-8n+1)\right] \\
&= (\lambda+1)^{4n-2}[(\lambda^2-2(2n+1)\lambda)-(16n^2-8n+1)]. \blacksquare
\end{aligned}
$$

The preceding result provides a crucial insight that lays the groundwork for the next development. We now refine this perspective to uncover a deeper structural property.

Theorem 4. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for every positive integer k, then the DEE of Γ is:*

$$E_{DE}(\Gamma_{Q_{4n}}) = 2\left(2n-1+\sqrt{20n^2-4n+2}\right).$$

Proof. By setting the characteristic equation in Theorem 3 to zero, we can calculate the DEE of G :

$$(\lambda+1)^{4n-2}[(\lambda^2-2(2n+1)\lambda)-(16n^2-8n+1)] = 0.$$

By this characteristic, the eigenvalues of Degree Exponent matrix of G is $\lambda_1 = 1$ with multiplicity $4n-2$. Then for $(\lambda^2 - 2(2n+1)\lambda) - (16n^2 - 8n + 1)$, $\lambda_2 = (2n+1) + \sqrt{20n^2 - 4n + 2}$ and $\lambda_3 = (2n+1) - \sqrt{20n^2 - 4n + 2}$. Then the DEE of G is:

$$\begin{aligned} E_{DE}\left(\Gamma_{Q_{4n}}\right) &= (4n-2)|-1| + \left|(2n-1) + \sqrt{20n^2-4n+2}\right| + \left|(2n-1) - \sqrt{20n^2-4n+2}\right| \\ &= (4n-2) + (2n-1) + \sqrt{20n^2-4n+2} + \sqrt{20n^2-4n+2} - (2n-1) \\ &= 4n - 2 + 2\sqrt{20n^2-4n+2} \\ &= 2\left(2n - 1 + \sqrt{20n^2-4n+2}\right). \blacksquare \end{aligned}$$

3.3. Degree Exponent Sum Energy

The Degree Exponent Sum Energy (DESE) of the coprime graph on the generalized quaternion group is presented in the following theorem.

Theorem 5. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for positive integer k, then the characteristics polynomial of $DES(\Gamma_{Q_{4n}})$ is:*

$$P_{DES}\left(\Gamma_{Q_{4n}}\right)(\lambda) = (\lambda+2)^{4n-2}[\lambda^2 - 4(2n-1)\lambda - (64n^3 - 16n^2)].$$

Proof. By using Lemma 1 and Lemma 2, the graph is complete bipartite graph with $deg(e) = 4n - 1$ and $deg(v) = 1$. Then the Degree Exponent Sum Matrix of $\Gamma_{Q_{4n}}$ is:

$$DES\left(\Gamma_{Q_{4n}}\right) = \begin{pmatrix} 0 & (4n-1)^1 + 1^{(4n-1)} & \cdots & (4n-1)^1 + 1^{(4n-1)} \\ 1^{(4n-1)} + (4n-1)^1 & 0 & \cdots & 1^1 + 1^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1^{(4n-1)} + (4n-1)^1 & 1^1 + 1^1 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4n & \cdots & 4n \\ 4n & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 4n & 2 & \cdots & 0 \end{pmatrix}.$$

By using Lemma 3, this matrix can be written into four block matrices below:

$$DES\left(\Gamma_{Q_{4n}}\right) = \begin{pmatrix} 0 & (4n)J_{1\times(4n-1)} \\ (4n)J_{(4n-1)\times 1} & 2(J-I)_{(4n-1)\times(4n-1)} \end{pmatrix}.$$

To find the characteristic polynomial of $DSS(\Gamma_{Q_{4n}})$, this matrix can be written as:

$$P_{DES}\left(\Gamma_{Q_{4n}}\right)(\lambda) = \begin{vmatrix} \lambda & -(4n)J_{(4n-1)\times 1} \\ -(4n)J_{(4n-1)\times 1} & (\lambda+2)I_{(4n-1)} - 2J_{(4n-1)} \end{vmatrix}.$$

This matrix is derived from the operation of $\det\left(DES\left(\Gamma_{Q_{4n}}\right) - \lambda I\right)$.

According to Lemma 3 with $a = 0, b = 2$ and $c = d = 4n$, this matrix has the determinant as follows:

$$\begin{aligned} P_{DES}\left(\Gamma_{Q_{4n}}\right)(\lambda) &= (\lambda+0)^{1-1}(\lambda+2)^{4n-2}[(\lambda-(1-1)0)(\lambda-(4n-2)2) - (4n-1)(4n)(4n)] \\ &= (\lambda+2)^{4n-2}[(\lambda(\lambda - 8n + 4)) - (4n-1)(4n)^2] \\ &= (\lambda+2)^{4n-2}[\lambda^2 - 4(2n-1)\lambda - (64n^3 - 16n^2)]. \blacksquare \end{aligned}$$

This result provides further insight into the spectral properties of the graph, allowing us to extend the analysis to the next case.

Theorem 6. *Let $\Gamma_{Q_{4n}}$ be the coprime graph of Q_{4n}. If $n = 2^k$ for positive integer k, then the DESE of Γ is:*

$$E_{DES}\left(\Gamma_{Q_{4n}}\right) = 8n - 4 + \sqrt{256n^3 - 64n + 16}.$$

Proof. By setting the characteristic polynomial of Theorem 5 to zero, we will get 3 eigenvalues: $\lambda_1 = -2$ with multiplicity $4n-2$, $\lambda_2 = 2(2n-1) + \frac{\sqrt{256n^3-64n+16}}{2}$ with multiplicity 1, and $\lambda_3 = 2(2n-1) - \frac{\sqrt{256n^3-64n+16}}{2}$ with multiplicity 1.

By summing all the eigenvalues, we get the $DESE$ of G as follows:

$$\begin{aligned} E_{DES}(\Gamma_{Q_{4n}}) &= (4n-2)|-2| + \left|2(2n-1) + \frac{\sqrt{256n^3-64n+16}}{2}\right| + \left|2(2n-1) - \frac{\sqrt{256n^3-64n+16}}{2}\right| \\ &= (8n-4) + 2(2n-1) + 2\frac{\sqrt{256n^3-64n+16}}{2} - 2(2n-1) \\ &= 8n - 4 + \sqrt{256n^3-64n+16}. \blacksquare \end{aligned}$$

This study shows that the Degree Square Sum Energy, Degree Exponent Energy, dan Degree Exponent Sum Energy of the coprime graph on the generalized quaternion group Q_{4n} with $n = 2^k$ are strongly influenced by the order structure of the group elements. The Degree Square Sum Energy highlights the influence of vertices with large degrees, the Degree Exponent Sum Energy accounts for the cumulative role of degree exponents, and the Degree Exponent Energy provides a spectral perspective through the exponential weighting of vertex degrees. Collectively, these measures offer a comprehensive characterization of the coprime graph's spectral behavior and demonstrate the complexity induced by the quaternion group structure

4. CONCLUSION

In this study, the values of the Degree Square Sum Energy, Degree Exponent Energy, and Degree Exponent Sum Energy of the coprime graph on the generalized quaternion group are presented:

1. $E_{DSS}(\Gamma_{Q_{4n}}) = 2\left(4n - 2 + \sqrt{8(2n-1)^2 + 2(4n-1)(16n^2-8n+1)^2}\right)$,
2. $E_{DE}(\Gamma_{Q_{4n}}) = 2(2n - 1 + \sqrt{20n^2-4n+2})$, and
3. $E_{DES}(\Gamma_{Q_{4n}}) = 8n - 4 + \sqrt{256n^3-64n+16}$.

This research can serve as a foundational reference for discovering alternative energy indices on the same graph or for extending the study of graph energies to other algebraic graphs. Such investigations enrich the interplay between graph theory and algebra, offering deeper insights into structural properties emerging from algebraic structures.

Author Contributions

Miftahurrahman: Conceived the main research idea, formulated the theorems, drafted the manuscript, and collected references used in theorem formulation. I Gede Adhitya Wisnu Wardhana: Provided feedback for improving the writing, assisted the first author with calculations related to theorem development, and contributed references for the introduction. Nur Idayu Alimon: Acted as supervisors of the first author, verified the logical correctness of the proofs, improved the structure and language of the manuscript, and provided suggestions for enhancing the overall quality of the research. Nor Haniza Sarmin: Acted as supervisors of the first author, verified the logical correctness of the proofs, improved the structure and language of the manuscript, and provided suggestions for enhancing the overall quality of the research. All authors discussed the results and contributed to the final manuscript.

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Declarations

The authors declare that there are no conflicts of interest related to this study.

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